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Characterization of Function Rings Between $C^*(X)$ and $C(X)$

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Abstract. Let X be a Tychonoff space and $\sum(X)$ the set of all the subrings of $C(X)$ that contain $C^*(X)$. For any $A(X)$ in $\sum(X)$ suppose $\upsilon_A X$ is the largest subspace of βX containing X to which each function in $A(X)$ can be extended continuously. Let us write $A(X) \sim B(X)$ if and only if $\upsilon_A X = \upsilon_B X$, thereby defining an equivalence relation on $\sum(X)$. We have shown that an $A(X)$ in $\sum(X)$ is isomorphic to $C(Y)$ for some space Y if and only if $A(X)$ is the largest member of its equivalence class if and only if there exists a subspace T of βX with the property that $A(X) = \{f \in C(X) : f^*(p)$ is real for each p in $T\}$, f^* being the unique continuous extension of f in $C(X)$ from βX to $\mathbb{R}^*$, the one point compactification of $\mathbb{R}$. As a consequence it follows that if X is a realcompact space in which every C^*-embedded subset is closed, then $C(X)$ is never isomorphic to any $A(X)$ in $\sum(X)$ without being equal to it.

1. Introduction

It is well known in the theory of rings of continuous functions that for a Tychonoff space X, $C^*(X)$ is isomorphic to $C(\beta X)$, where βX is the Stone-Čech compactification of X; in other words every C^* is a function ring in the sense that it is isomorphic to some C. Intimately connected with this fact is the result that the structure space of each of the rings $C(X)$ and $C^*(X)$ is βX. This result has been superseded to a great extent by D. Plank [6], who has proved that the structure space of any ring that lies between $C^*(X)$ and $C(X)$ is also βX. We have

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shown that in case $C^*(X) \neq C(X)$, there exists at least 2^c many such rings, where c is the cardinality of the continuum. It is therefore quite natural to ask which are function rings amongst these. We answer this in terms of a number of conditions, each necessary as well as sufficient for the positive answer to this question. A bit of elaboration is needed to explain these things. Suppose $\sum(X)$ is the set of all rings that lie between $C^*(X)$ and $C(X)$, and $A(X)$ is in $\sum(X)$. Since the structure space of $A(X)$ is βX, the set of all maximal ideals can be written as $\{M_A^p : p \in \beta X\}$. M_A^p is called real if and only if the residue class field $A(X)/M_A^p$ is isomorphic to $\mathbb{R}$, otherwise it is called hyper-real. The set of all those points p in βX for which M_A^p is real, is denoted by $\upsilon_A X$ and is called the A-compactification of X. By the definition of $\upsilon_A X$ it is clear that $X \subset \upsilon_A X \subset \beta X$. In this terminology $\upsilon_C X = \upsilon X$ and $\upsilon_{C^*} X = \beta X$. X is called A-compact if every real maximal ideal in it is fixed. Therefore X becomes A-compact if and only if $X = \upsilon_A X$, and it is established in [2] that $\upsilon_A X$ is the largest subspace of βX containing X to which each function in $A(X)$ can be extended continuously. Furthermore the space $\upsilon_A X = \{p \in \beta X : f^*(p) \in \mathbb{R} \text{ for each } f \in A(X)\}$, where $f^* : \beta X \mapsto \mathbb{R} \equiv \mathbb{R} \bigcup \{\infty\}$ is the unique continuous extension of f in $C(X)$ over βX. It follows that every A-compact space is realcompact. (See [2] for a detailed discussion on all these topics.)

For any $A(X),\ B(X) \in \sum(X)$ we write $A(X) \sim B(X)$ if and only if $\upsilon_A X = \upsilon_B X$. Then '$\sim$' defines an equivalence relation on $\sum(X)$. It was established in [2] that each equivalence class has a largest member, which we record for our ready reference.

Theorem 1.1. *The largest member of the equivalence class* $[A(X)]$ *containing* $A(X)$ *is given by* $\{g|_X : g \in C(\upsilon_A X)\}$.

We establish in section 2 that these largest members can indeed be achieved by considering suitable subsets of βX in the form of the following proposition:

Theorem 1.2. $A(X) \in \sum(X)$ *is the largest member of* $[A(X)]$ *if and only if there exists a subset* T *of* βX *with the property:*

$$A(X) = \{f \in C(X) : f^*(p) \in \mathbb{R} \textit{ for each } p \in T\}.$$

It is clear from Theorem 1.1 that if $A(X)$ is the largest member of its equivalence class then the canonical map: $f \to f^{\upsilon_A}$ establishes an isomorphism from the ring $A(X)$ onto the ring $C(\upsilon_A X)$, and in particular $A(X)$ is identified as a function ring. (Here f^{υ_A} stands for the unique real valued continuous extension of f from X to $\upsilon_A X$.) It is interesting to note that any function ring in the family $\sum(X)$ also shares this property. Indeed in section 2, we prove the following result:

Theorem 1.3. *Let* $A(X) \in \sum(X)$ *be a function ring. Then the map* $f \mapsto f^{\upsilon_A}$ *defines an isomorphism from* $A(X)$ *onto the ring* $C(\upsilon_A X)$.

This Theorem 1.3 has turned out to be crucial towards the following characterization of function rings, which we also establish in section 2.

Theorem 1.4. *$A(X) \in \sum(X)$ is a function ring if and only if it is the largest member of its equivalence class.*

Combining Theorems 1.2, 1.3, 1.4 we have the following comprehensive result almost immediately.

Theorem 1.5. *For any ring $A(X)$ lying between $C^*(X)$ and $C(X)$ the following statements are equivalent.*

(1) *$A(X)$ is a function ring.*

(2) *$A(X)$ is the largest member of its equivalence class.*

(3) *$A(X)$ is isomorphic to the ring $C(\upsilon_A X)$ under the canonical mapping $f \mapsto f^{\upsilon_A}$.*

(4) *There exists a subset T of βX such that*

$$A(X) = \{f \in C(X) : f^*(p) \in \mathbb{R}\, \forall\, p \in T\}.$$

We conclude this introductory section with the statement of our final theorem, which we also prove in section 2. It is well known that $C(X)$ is never isomorphic to $C^*(X)$ without being equal to it [4]. For a large class of spaces X we have improved this result in the following form.

Theorem 1.6. *Suppose X is a non-compact realcompact space in which every C^*-embedded subset is closed (in particular therefore X may be a metrizable space with non measurable cardinal). Then given any $A(X) \in \sum(X)$ with $A(X) \neq C(X)$, $C(X)$ is never isomorphic to $A(X)$.*

It is not known to us whether this theorem remains still valid without the assumption of the closedness of the C^*-embedded subset of X.

2. Function rings: their characterizations

In this section our principal aim is to give proofs of the Theorems 1.2, 1.3, 1.4, 1.6 stated in the introductory section. For any subset T of βX let us set

$$C_T(X) \equiv C_T = \{f \in C(X) : f^*(p) \in \mathbb{R} \text{ for all } p \in T\}.$$

Then it is easy to see that $C_T(X)$ is a subring of $C(X)$ containing $C^*(X)$.

Proof of Theorem 1.2. Let T be a subset of βX and let $B(X)$ be a member of $\sum(X)$ with $\upsilon_B X = \upsilon_{C_T} X$. We choose f in $A(X)$ and p in T arbitrarily. Since $T \subset \upsilon_{C_T} X$ it follows that $p \in \upsilon_{C_T} X$ and therefore $p \in \upsilon_B X$. Accordingly $f^*(p) \in \mathbb{R}$. Thus $f \in C_T$. Hence $A(X) \subset C_T(X)$, consequently $C_T(X)$ is the largest member of its equivalence class. Conversely let $A(X)$ be the largest member of its equivalence class $[A(X)]$.We shall show that $A(X) = C_{\upsilon_A X}(X)$. Since for each f in $A(X)$ and p in $\upsilon_A X$, $f^*(p)$ is real, it is trivial that $A(X) \subseteq C_{\upsilon_A X}(X)$. Conversely let $f \in C_{\upsilon_A X}(X)$ and $p \in \upsilon_A X$. Then $f^*(p)$ is real and therefore $g = f^*|_{\upsilon_A X} \in C(\upsilon_A X)$. Theorem 1.1 tells us that $g|_X \in A$, but since $g|_X = f$ it follows that $f \in A(X)$. Thus $C_{\upsilon_A X}(X) \subseteq A(X)$. Hence $A(X) = C_{\upsilon_A X}(X)$. □

For any point p in $\beta X - \upsilon X$, let us write C_p instead of $C_{\{p\}}$. Now choosing any point q in $\beta X - \upsilon X$ different from p we can find an $l \in C^*(X)$ with $l^\beta(p) = 0$ and $l^\beta(q) = 1$ with l not vanishing anywhere on X (l^β of course stands for the Stone extension of l from X to βX.) On the other hand there also exists an $h \in C^*(X)$, not vanishing anywhere on X for which $h^\beta(q) = 0$. On putting $k = l^2 + h^2$, we have $k \in C^*(X)$, $k^\beta(q) = 0$ and $k^\beta(p) \geqslant 1$. If we take $g = \frac{1}{k}$, then $g^*(q) = \infty$ and $g^*(p) \in \mathbb{R}$. Consequently $g \in C_p$ and $q \notin \upsilon_{C_p} X$. Thus we have for each element p in $\beta X - \upsilon X$, $\upsilon_{C_p} X = \upsilon X \cup \{p\}$. Since for a non-pseudocompact space X, $\beta X - \upsilon X$ contains at least 2^c many points [4], it follows that $\{C_p : p \in \beta X - \upsilon X\}$ is an infinite set containing at least 2^c many members with $C_p \neq C_q$ whenever $p \neq q$. Clearly then if $C(X) \neq C^*(X)$, there exists at least 2^c many different equivalence classes in the family $\sum(X)$.

Proof of Theorem 1.3. Since $A(X)$ is a function ring, there exists a realcompact space Y with an isomorphism t from $A(X)$ onto $C(Y)$. As the property of being a real maximal ideal is an algebraic invariant, for any $p \in \upsilon_A X$, $t(M_A^p)$ is a real maximal ideal in $C(Y)$ and therefore it is fixed due to the realcompactness of Y. Accordingly $\bigcap_{g \in t(M_A^p)} Z_Y(g)$ is a singleton set, where $Z_Y(g)$ stands for the zero set of the function g in the space Y. We define a mapping $\Psi : \upsilon_A X \longrightarrow Y$ by the rule $\Psi(p) = \bigcap_{g \in t(M_A^p)} Z_Y(g)$. Then Ψ is clearly one-to-one. Again for any point y in Y if $M_y = \{h \in C(Y) : h(y) = 0\}$ is the corresponding fixed maximal ideal in $C(Y)$, then there is a unique point p in $\upsilon_A X$ with $M_y = t(M_A^p)$ and so $\Psi(p) = y$. Now $\{S_A(f) : f \in A(X)\}$ being a base for closed subsets of βX, where $S_A(f) = \{p \in \beta X : f \in M_A^p\}$ [6], it is obvious that $\{S_A(f) \bigcap \upsilon_A X : f \in A(X)\}$ is a base for closed subsets of $\upsilon_A X$ and we observe that for any $f \in A(X)$, $\Psi(S_A(f) \bigcap \upsilon_A X) = Z_Y(t(f))$. Hence Ψ carries the basic closed sets in $\upsilon_A X$ onto Y. Suppose $s : C(Y) \longrightarrow C(\upsilon_A X)$ is the isomorphism induced by Ψ, that is, for any g in $C(Y)$, $s(g) = g \circ \Psi$. Since $t : A(X) \longrightarrow C(Y)$ is already an isomorphism, we see that $s \circ t$ becomes an isomorphism from $A(X)$ onto $C(\upsilon_A X)$. Let us choose an $f \in A(X)$. To prove the theorem it is sufficient to prove that $t(f) \circ \Psi = f^{\upsilon_A}$. Since $t(f) \circ \Psi$ is clearly a real-valued continuous function on $\upsilon_A X$ it is enough to show that it is an extension of f. Now if we choose $x \in X$ then for any h in the fixed maximal ideal M_A^x of $A(X)$, we have $t(h)(\Psi(x)) = 0$. Hence it follows that $t(f - \underline{f(x)})(\Psi(x)) = 0$ (where $\underline{f(x)}$ is the constant function on X which takes the value $\overline{f(x)}$ at all points of X), so that $t(f)(\Psi(x)) = f(x)$. □

Proof of Theorem 1.4. If $A(X)$ is the largest member of its equivalence class, then we have already observed in the introductory section that $A(X)$ is a function ring.

Conversely, suppose $A(X)$ is not the largest member of its equivalence class. Then from Theorem 1.1 there exists a $g \in C(\upsilon_A X)$ for which $g|_X$ is not in $A(X)$. Accordingly there cannot exist any $f \in A(X)$ with $f^{\upsilon_A} = g$ and therefore the canonical map $T : A(X) \longrightarrow C(\upsilon_A X)$ defined by $T(f) = f^{\upsilon_A}$ is not an isomorphism on $A(X)$ onto $C(\upsilon_A X)$. Hence by Theorem 1.3, $A(X)$ is not a function ring. □

Proof of Theorem 1.6. If $A(X)$ is not the largest member of its equivalence class, then from Theorem 1.4 it is not a function ring and therefore it can not be isomorphic to $C(X)$. Next suppose that $A(X)$ is the largest member of its equivalence class contained properly in $C(X)$. Hence $A(X)$ does not belong to $[C(X)]$. Now since X is realcompact, $X = \upsilon X$ and therefore $X \subsetneqq \upsilon_A X$. Again the result of Theorem 1.5 tells us that $A(X)$ is not isomorphic to $C(\upsilon_A X)$. Suppose now that $A(X)$ is isomorphic to $C(X)$. Since X and $\upsilon_A X$ are both realcompact, it follows in view of Hewitt's isomorphism theorem [4] that X is homeomorphic to $\upsilon_A X$ under a mapping say, $\alpha : \upsilon_A X \longrightarrow X$. As X is dense in $\upsilon_A X$, it is plain that $\alpha(X)$ is also dense in X and is also contained in X properly; but X is also C^*-embedded in $\upsilon_A X$ from which it follows that $\alpha(X)$ is C^*-embedded in X and therefore closed in X. Altogether we get $\alpha(X) = X$, a contradiction. Hence $A(X)$ is not isomorphic to $C(X)$. □

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