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An Asymmetric Fuglede-Putnam's Theorem and Orthogonality

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Abstract. An asymmetric Fuglede-Putnam theorem for p-hyponormal operators and class $(\mathcal{Y})$ is proved, as a consequence of this result, we obtain that the range of the generalized derivation induced by the above classes of operators is orthogonal to its kernel.

1. Introduction

Let $\mathcal{H}$ be an infinite dimensional complex Hilbert space and $B(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. For a bounded linear operator on a Hilbert space $\mathcal{H}$, we say that T belongs to the class $\mathcal{Y}_\alpha$ for some $\alpha \geq 1$ if there is a positive number K_α such that

$$| TT^* - T^*T |^\alpha \leq {K_\alpha}^2 (T - \lambda I)^*(T - \lambda I) \quad \text{for all } \lambda \in \mathbb{C}.$$

Let $\mathcal{Y} = \bigcup_{\alpha \geq 1} \mathcal{Y}_\alpha$. It's well know that for each α, β such that $1 \leq \alpha \leq \beta$, we have $\mathcal{Y}_\alpha \subseteq \mathcal{Y}_\beta$.

An operator $T \in B(\mathcal{H})$ is said to be p-hyponormal if $(T^*T)^p - (TT^*)^p \geq 0$, $(0 < p \leq 1)$. If $p = 1$, T is called hyponormal and if $p = 1/2$, T is semi-hyponormal. It's know that p-hyponormal operators are q-hyponormal operators for $0 < q \leq p$. Hyponormal operators have been studied by many authors and it is known that hyponormal operators have many interesting properties similar to those of normal operators. Semi-hyponormal operators were first introduced by D. Xia [14], p-hyponormal operators have been studied by A. Aluthge [1], M. Cho [4], [5] and A. Uchiyama [11]. See [11] for properties of the class $(\mathcal{Y})$. The set of all p-hyponormal is denoted by $p - H$.

Given $A, B \in B(\mathcal{H})$, we define the generalized derivation $\delta_{A,B}$: $B(\mathcal{H}) \to B(\mathcal{H})$ by $\delta_{A,B}(X) = AX - XB$.

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J. Anderson and C. Foias [3] proved that if A and B are normal operators, then $R(\delta_{A,B})$ is orthogonal to $\ker(\delta_{A,B})$, where $R(\delta_{A,B})$ and $\ker(\delta_{A,B})$ denotes the range of $\delta_{A,B}$ and the kernel of $\delta_{A,B}$ respectively. The orthogonality here is understood to be in the sense of definition in [2].

The organization of the paper is as follows, in section 2, we recall some results which will be used in the sequel. In section 3, we study the range-kernel orthogonality of certain operators.

2. Preliminaries

In this section, we recall some results which will be used in the sequel.

Definition 2.1. Given $A, B \in B(\mathcal{H})$. We say that the pair (A, B) has $(FP)_{B(\mathcal{H})}$ the Fuglede-Putnam property if $AC = CB$ for some $C \in B(\mathcal{H})$, implies $A^*C = CB^*$.

Theorem 2.2 ([1]). *If $T \in p-H$ and $T = U \mid T \mid$ the polar decomposition of T, then $\mid T \mid^{1/2} U \mid T \mid^{1/2}$ is hyponormal for $1/2 \leq p \leq 1$.*

The next theorem is due to Duggal [6]. This theorem plays important role in our arguments.

Theorem 2.3 ([6]). *Let $A, B \in B(\mathcal{H})$. The following assertions are equivalent:*

(i) *The pair (A, B) has the property $(FP)_{B(\mathcal{H})}$.*

(ii) *If $AC = CB$ for some $C \in B(\mathcal{H})$, then $\overline{R(C)}$ reduces A, $(\ker C)^\perp$ reduces B and $A \mid_{\overline{R(C)}}$ and $B \mid_{(\ker C)^\perp}$ are normal operators.*

Theorem 2.4 ([13]). *If $T \in p-H$ and M be an invariant subspace of T for which $T \mid_M$ is normal, then M reduces T.*

3. Main results

In this section, we prove that the Fuglede-Putnam's Theorem holds when $A \in p-H$ and $B^* \in \mathcal{Y}$.

Theorem 3.1. *If $A \in p-H$ and $B^* \in \mathcal{Y}$, then the pair (A, B) has the property $(FP)_{B(\mathcal{H})}$.*

Proof. (Case 1. $1/2 \leq p \leq 1$). Suppose that $AC = CB$ for some $C \in B(\mathcal{H})$. Let's consider the following decompositions of $\mathcal{H}$

$$\mathcal{H} = (\ker A)^\perp \oplus (\ker A) = (\ker B^*)^\perp \oplus (\ker B^*)$$

$\ker A$ reduces A by [5] and $\ker B^*$ reduces B^* by [12] and so, we can write A, B and C as follows:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix}.$$

From $AC = CB$, it follows that $A_1C_1 = C_1B_1$ and $A_1C_2 = C_3B_1 = 0$. Since A_1 and B_1^* are injective, we have $C_2 = C_3 = 0$. Let's consider the equality

$$A_1C_1 = C_1B_1 \tag{3.1}$$

Let's multiply the two members of (3.1) by $\mid A_1 \mid^{1/2}$ and uses the polar decomposition of $A_1 = V \mid A_1 \mid$, we obtain

$$\mid A_1 \mid^{\frac{1}{2}} V \mid A_1 \mid^{\frac{1}{2}} (\mid A_1 \mid^{\frac{1}{2}} C_1) = (\mid A_1 \mid^{\frac{1}{2}} C_1) B_1 \tag{3.2}$$

Since $B_1 \in \mathcal{Y}$ by [12], there exists an integer $n > 1$ and k_{2^n} such that $B_1 \in (\mathcal{Y}_{2^n})$ i.e.,

$$\mid B_1B_1^* - B_1^*B_1 \mid^{2^n} \le k_{2^n}^2 (B_1 - \lambda)(B_1 - \lambda)^*, \ \forall \lambda \in \mathbb{C}.$$

Then, by [8], for all $x \in R(\mid B_1B_1^* - B_1^*B_1 \mid^{2^n - 1})$, there exist a bounded function $f(\lambda) : \mathbb{C} \to \mathcal{H}$ such that

$$(B - \lambda I) f(\lambda) = x. \tag{3.3}$$

Let's multiply the members of (3.3) by $(\mid A_1 \mid^{\frac{1}{2}} C_1)$ and uses (3.1). Hence

$$\begin{aligned} (\mid A_1 \mid)^{\frac{1}{2}} C_1) x &= (\mid A_1 \mid^{\frac{1}{2}} C_1)(B_1 - \lambda I) f(\lambda) \\ &= (\mid A_1 \mid^{\frac{1}{2}} V \mid A_1 \mid^{\frac{1}{2}} - \lambda I)(\mid A_1 \mid^{\frac{1}{2}} C_1) f(\lambda), \ \forall \lambda \in \mathbb{C}. \end{aligned}$$

Since $\mid A_1 \mid^{\frac{1}{2}} V \mid A_1 \mid^{\frac{1}{2}}$ is hyponormal. If $(\mid A_1 \mid)^{\frac{1}{2}} C_1) f \neq 0$, then $(\mid A_1 \mid^{\frac{1}{2}} C_1) f$ is a bounded entire function by [9]. This yields that the function is constant by Liouvilles's Theorem. This is a contradiction. Hence $(\mid A_1 \mid^{\frac{1}{2}} C_1) x = 0$, for all $x \in R(\mid B_1B_1^* - B_1^*B_1 \mid^{2^n - 1})$, i.e., $(\mid A_1 \mid^{\frac{1}{2}} C_1) \mid B_1B_1^* - B_1^*B_1 \mid^{2^n - 1} \mathcal{H} = \{0\}$. Since $\ker(\mid B_1B_1^* - B_1^*B_1 \mid^{2^n - 1}) = \ker(\mid B_1B_1^* - B_1^*B_1 \mid^2)$, we have

$$(\mid A_1 \mid^{\frac{1}{2}} C_1)(\mid B_1B_1^* - B_1^*B_1 \mid^2) x = 0, \ \forall x \in \mathcal{H}.$$

Since $\mid A_1 \mid$ is one-to-one, we obtain

$$C_1(\mid B_1B_1^* - B_1^*B_1 \mid^2) x = 0, \ \forall x \in \mathcal{H}. \tag{3.4}$$

From equality (3.1), $\overline{R(C_1)}$ and $\ker C_1$ are invariant subspaces of A_1 and B_1 respectively. According to the following decompositions

$$(\ker A_1)^\perp = \overline{R(C_1)} \oplus R(C_1)^\perp, \ (\ker B_1)^\perp = (\ker C_1)^\perp \oplus \ker C_1,$$

we can write A_1, B_1 and C_1 as follows

$$A_1 = \begin{bmatrix} A_{11} & S \\ 0 & T \end{bmatrix}, \quad B_1 = \begin{bmatrix} B_{11} & 0 \\ E & G \end{bmatrix}, \quad \text{and} \quad C_1 = \begin{bmatrix} C_{11} & 0 \\ 0 & 0 \end{bmatrix}.$$

The equality (3.1) would imply that

$$A_{11}C_{11} = C_{11}A_{11}, \tag{3.5}$$

where A_{11} is p-hyponormal by [11]. Since

$$B_1B_1^* - B_1^*B_1 = \begin{bmatrix} B_{11}B_{11}^* - B_{11}^*B_{11} - E^*E & S_1 \\ S_1^* & R_1 \end{bmatrix},$$

we have

$$(B_1B_1^* - B_1^*B_1)^2 = \begin{bmatrix} (B_{11}B_{11}^* - B_{11}^*B_{11} - E^*E)^2 + S_1S_1^* & S_2 \\ S_2^* & R_2 \end{bmatrix}.$$

Hence $C_{11}[(B_{11}B_{11}^* - B_{11}^*B_{11} - E^*E)^2 + S_1S_1^*] = 0$ from equality (3.4). Since C_{11} is one-to-one and $B_{11}B_{11}^* - B_{11}^*B_{11} - E^*E$ is self-adjoint, we obtain

$$B_{11}B_{11}^* - B_{11}^*B_{11} - E^*E = 0. \tag{3.6}$$

Hence B_{11}^* is hyponormal. Let's the two members of (3.5) by $(\mid A_{11} \mid^{\frac{1}{2}})$ and uses the polar decomposition of $A_{11} = U \mid A_{11} \mid$, we obtain

$$\widetilde{A_{11}}(\mid A_{11} \mid^{1/2} C_{11}) = (\mid A_{11} \mid^{1/2} C_{11})B_{11}.$$

Since the Aluthge transform $\widetilde{A_{11}} = \mid A_{11} \mid^{1/2} U \mid A_{11} \mid^{1/2}$ is hyponormal by [1] and B_{11}^* is hyponormal from equality (3.6), The pair $(\widetilde{A_{11}}, B_{11})$ has the $(FP)_{B(\mathcal{H})}$ property by [7] and consequently

$$\widetilde{A_{11}}^*(\mid A_{11} \mid^{1/2} C_{11}) = (\mid A_{11} \mid^{1/2} C_{11})B_{11}^*.$$

Hence $A_{11}|_{\overline{R(|A_{11}^{1/2}C_{11})}}$ and $B_{11}|_{[\ker(|A_{11}|^{1/2})C_{11})]^\perp}$ are normal operators by Theorem 2.3.

Since $\mid A_{11} \mid^{1/2}$ and C_{11} are injective, then $\mid A_{11} \mid^{1/2} C_{11}$ is also injective. Hence $[\ker(\mid A_{11} \mid^{1/2})C_{11})]^\perp = 0^\perp = (\ker C_{11})^\perp$. By similar arguments as before, we have the following equality.

$$\overline{R(\mid A_{11} \mid^{1/2} C_{11})} = [\ker C_{11}^* \mid A_{11} \mid^{1/2}]^\perp = 0^\perp = \overline{R(C_{11})}.$$

Hence $\widetilde{A_{11}}$ is normal. Thus A_{11} is normal by [10]. Therefore $\overline{R(C_{11})}$ reduces A_{11} by theorem 2.4 and $(\ker C_{11})^\perp$ reduces B_{11}^* by [12]. Finally, we obtain:

$A_{11}^*C_{11} = C_{11}B_{11}^*$ and $A_1^*C_1 = C_1B_1^*$, and therefore $A^*C = CB^*$.

(Case 2. $0 < p \leq 1/2$). We put $p' = p + 1/2$, where $1/2 < p' \leq 1$. The rest of the proof is similar to the proof of the first case. □

Corollary 3.2. *A is normal if and only if $A \in p - H$ and $A^* \in \mathcal{Y}$.*

Proof. Put $B = A^*$. □

Theorem 3.3. *If $A \in p-H$ and $B^* \in \mathcal{Y}$, then $R(\delta_{A,B})$ is orthogonal to* $\ker(\delta_{A,B})$.

Proof. The pair (A, B) has the $(FP)_{B(\mathcal{H})}$ property by theorem 3.1. Let $C \in B(\mathcal{H})$ be such $AC = CB$. According to the following decompositions of $\mathcal{H}$.

$$\mathcal{H} = \mathcal{H}_1 = \overline{R(C)} \oplus (\overline{R(C)})^{\perp},\ \mathcal{H} = \mathcal{H}_2 = (\ker C)^{\perp} \oplus \ker C.$$

We can write A, B, C and X

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix},$$

where A_1 and B_1 are normal operators and X is an operator on $\mathcal{H}_1$ to $\mathcal{H}_2$. Since $AC = CA$, we obtain $A_1C_1 = C_1A_1$. Hence

$$AX - XA - C = \begin{bmatrix} A_1X_1 - X_1B_1 - C_1 & A_2X_2 - X_2B_2 \\ A_1X_3 - X_3B_1 & A_2X_4 - X_4B_2 \end{bmatrix}.$$

Since $C_1 \in \ker(\delta_{A_1,B_1})$, A_1 and B_1 are normal, it follows by [3]

$$\|AX - XB - C\| \geq \|A_1X_1 - X_1B_1 - C_1\| \geq \|C_1\| = \|C\|,\ \ \forall X \in B(\mathcal{H})$$

This implies that $R(\delta_{A,B})$ is orthogonal to $\ker(\delta_{A,B})$. □

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