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Making Sense of Fraction — Lessons of Chinese Curriculum

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Teaching of fractions is very challengeable for elementary and middle school teachers. Many teachers feel uncomfortable teaching the subject. How shall we introduce the concept of fractions? Shall we focus more on concept development or computational procedures and skills? What methodology can be used in teaching those important topics? These are the kind of questions many teachers and researchers try to answer. In this article, the authors are to look at this issue from a cross nation perspective, by examining how the Chinese mathematics curriculum and the Chinese teachers deal with this subject.

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ZDM Classification: C30, C40
MSC2000 Classification: 97C30, 97D10

INTRODUCTION

When students begin to study fractions, they step into a new territory of learning mathematics. Their early encounters with fractions do not result in a sense of number size, a sense of quantity (Sowder, 1988). A fraction $\frac{a}{b}$ is defined as an ordered pair of whole numbers a and b ($b \neq 0$) rather than a single whole number that students are familiar with. Many of them feel uneasy about the fact that the magnitude of a fraction does not rely on the magnitude of either a or b, but on both. It takes a considerable length of time for many of students to understand the concept. Unfortunately, many of them never make it. Moreover, subject to different situations, a/b can be interpreted as a number, a relationship (as a ratio), an operation (as division), etc. These interpretations add more difficulties to the learning of the subject. In addition to the concept of fraction, the algorithms by four operations of fractions are entirely different from the four operations with whole numbers. Students find that in calculating fractions they have to handle different cases (*e.g.*, to find common denominator in adding or subtracting fractions), take multiple steps to implement one operation, and process an initial answer to make it acceptable (reducing to the most simple form or converting to mixed numbers). And if that is not enough, the meanings and the results of the operations are extended. For instance, multiplication can no longer simply interpreted as repeated addition and a quotient is not necessarily smaller than the dividend. Many students never truly understand those meanings when they study fractions. Therefore, once they stop practicing on the numbers, their skills diminish rapidly. It is hard to think of letting them solve problems that involve fractions in more complicated context.

There are many studies that document the difficulties of children have in acquiring the fraction concept and these studies make corresponding suggestions to cure the problems. From the cognitive perspective, researchers like Kieren (1975, 1980, 1988) and Behr et al (1983) believe that using a single model to introduce the concept of fraction and its applications impedes the understanding of the concept by children. Accordingly, they argue that it is necessary to teach multiple models of fractions to students. Still others (Leman 1996; Mack 1990, 2001) look at different processes of students' reconceptualization. Their suggestion is that students' pre-knowledge could provide a foundation for the development of their understanding of the concept.

The profound understanding of fundamental mathematics by Chinese teachers as revealed in Ma's study (1999) is very impressive. In particular, their knowledge about fractions is articulate, reflexive, and well connected. Not only did all of them correctly obtain the answer of the fraction division problem

$$1\frac{3}{4} \div \frac{1}{2},$$

they also provided various reasons to explain why "dividing by a number is equivalent to multiplying by its reciprocal." Further, based on their understanding, they constructed a variety of word problems, which were both conceptually correct and pedagogically sound It will be interesting to learn the impact of their knowledge in their practice. What is their general approach in teaching the concept of fraction? What does the Chinese curriculum require from them? Cai (1995a, 1995b) reports that Chinese students outperformed the U. S. students for every fraction problem in his study. The problems used in his study mainly

We want to take a close look at the above questions and issues. The tested the skills on four operations. So, do Chinese teachers put special emphasis on the procedural knowledge since their students are so good at skills? Or, what is the position of conceptual knowledge in the curriculum and textbooks? Many researchers (Behr et al. 1992, 1993; Confrey 1994; Kieren 1988) believe that the multiplicative nature of fraction is critical in understanding the concept and solving fraction problems. Do Chinese teachers pay attention to this phenomenon? What are their tactics in this regard? following discussion is far from the whole picture describing how many Chinese teachers are doing in their classrooms. Rather, it tries to take some snapshots of their practice. Of course, by no means can these snapshots can represent the true practice of a vast number of teachers in a country whose population is over 1.3 billion.

China has developed national curricula for school subjects. There are several versions for mathematics. The main difference among them is the number of subject matters covered. Unified textbooks and teacher reference books embody the vision of the curricula. Teachers usually follow them closely as required. The teachers' attitude towards the books in Ma's study (1999) shows typically how teachers may use these books as benchmarks and resources to make their pedagogical decisions. Because of these circumstances, curriculum, textbooks, and teachers in our discussion may be exchangeable to a certain extent.

INTRODUCING MULTIPLE INTERPRETATIONS OF FRACTION

Knowing multiple representations of a concept does not necessarily mean that an understanding of the concept has been attained. Representation is often visual and intuitive, though not always. To have a deep understanding needs more effort. Kieren (1988) writes that, "knowledge-building experiences must take both the intuitive (context related) and formal (context free) mathematical features into account." It is for us to see what else efforts that Chinese educators make in order to enhance students' learning the subject.

In the national curriculum, the subject of fractions appears twice. In between, decimals, word problems, several polygons, simple equations, area measurement, and divisibility are discussed. Realizing that the concept is abstract to children, the curriculum only tries to give students some rudimentary ideas at the first stage. The concept is illustrated through many graphic examples. Therefore, it is largely visual and descriptive. No formal definition is given. The part-whole interpretation and number line are the two primary models that teachers employ with the part-whole model playing a dominant role. So students see a lot of regular polygons and circles that are divided evenly. Each time an individual figure acts as a whole. In other words, no composite unit is used at this stage. To simulate a number line, a string of rope, often horizontally laid, is divided into equal parts and several of them are taken to signify a fraction.

The second stage starts almost one year later and a definition is provided in the beginning:

> *A fraction is a number, which represents one or several equal parts of a unit, "1." In particular, the fraction that represents one part is called a fraction unit.*

As mentioned earlier, researchers believe that using a single model to introduce the concept of fraction and its applications impedes children's understanding of the concept, though this model is most natural to students because it is linked to counting, which they feel at home. Lamon (2001) argued that this approach as it is currently delivered "was the least valuable road into the system of rational numbers."

Although the language used in the definition by the Chinese sounds close to a part-whole interpretation, there are several notable steps taken during and after introducing this definition. First, the definition emphasizes that a fraction is just a number. This definition seems more appropriate and acceptable for students at this point before they know more about other embedded meanings of a/b. A teacher will let her students see that situations may emerge when the answer of a measurement cannot be represented as a whole number. For example, three people share a pizza equally, how much does each of them get?

So there is a need for another kind of number. The concept of fraction being a number is evident in a teacher's mind when he showed Ma (1999, p. 74–75) his construct of a problem that led to

$$1\frac{3}{4} \div \frac{1}{2}.$$

To him, in a sense, there is no distinction between a whole number division and a division of fractions. All that it counts in that particular context is the meaning of division.

A second measure is to expand the concept of unit before giving the definition. It is a common practice that teachers tell their students that a unit could also be a class of students, a piece of land, a basket of apples, and a project to be accomplished. Graphically, they will be shown the following (See Figure 1), in which a big circle indicates a unit.

Hiebert and Behr (1988) suggested that understanding how to find an appropriate unit largely reflects a person's concept about fraction and is critical when a student tries to solve fraction word problems

The early introduction of composite unit in the textbook seems to be recognition of the importance. The following are two typical problems that students would work on after they learn the definition of fraction.

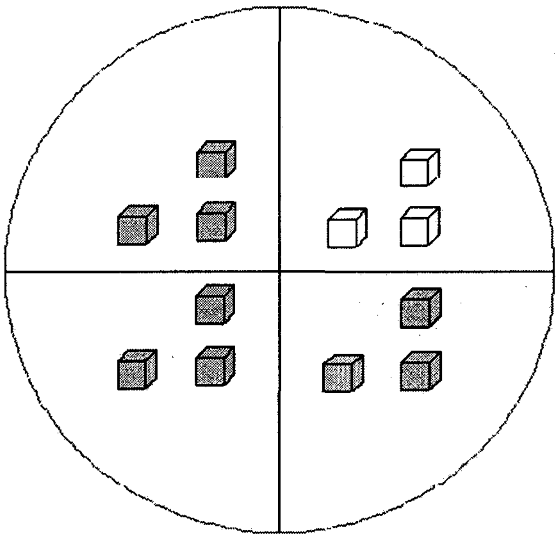

Figure 1. What is the fraction part of the shaded cubes as to the whole?

- What is the fraction part of the colored square as to the whole picture? (See Figure 2)
- A project was finished in 10 days. On average, how much was done everyday? In three days? In seven days?

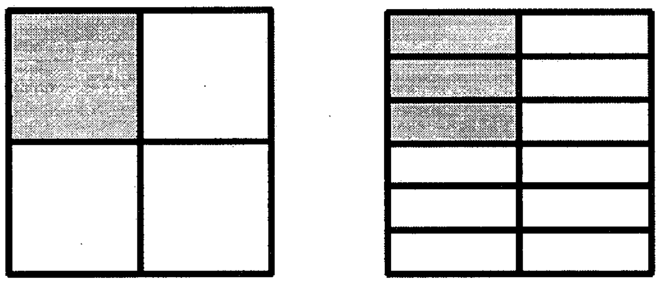

Figure 2. The fraction unit 1/4 on the left itself is three 1/12 on the right

A third measure is the naming of a fraction unit in the definition. Given that definition, 5/9 is five 1/9's just as five is five ones. This makes it easier for students to make sense of the size of a fraction and connect the properties of whole numbers with the properties of fractions. Because of this concept, for some students, fractions may become more manageable. They will make fewer mistakes when they come to operate with fractions because they can estimate the answer. Besides, the fraction unit itself can be a composite unit too. His measure explains the equivalence of fractions clearly (See Figure 3).

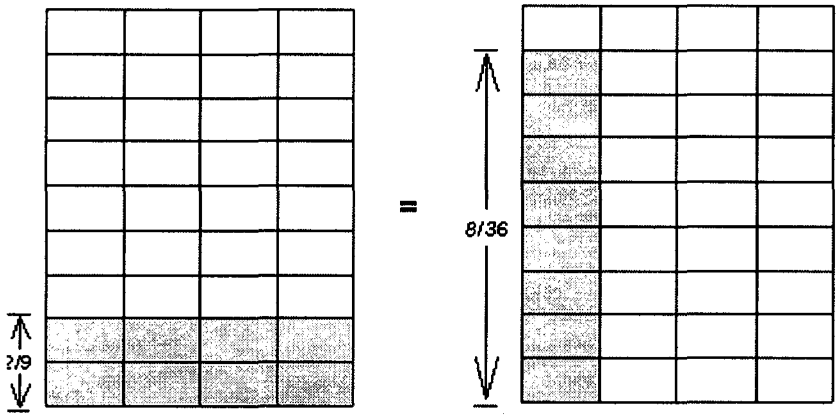

Figure 3. Equivalent fractions

The foregoing measures could prepare students to see more interpretations of fractions, which are important in learning subjects such as rates, proportions, and linear functions. Both (Kieren, 1975) and Behr et al. (1993) propose multiple interpretations or subconstructs of fraction. A step taken by the Chinese curriculum in this aspect is to equate fraction and whole number division. It starts at revisiting the meaning of division. The teacher will first review whole number division and point out that division often means to find "how many (much) for each." Then, she will pose a question such as: "How long is each part if a pipe of one meter is cut into three equal parts?" Obviously, the answer cannot be a whole number. However, students may agree that the meaning of division can be carried over to this situation. As 1/3 is clearly the answer, it may be used to represent the division of $1 \div 3$. Progressively, with the help of a picture like figure 2, students will be asked to consider: "If three pies are divided equally into four servings, how much is each serving?" Students can see that this problem is just an extension of the first one. Again the answer is 3/4 intuitively, and $3 \div 4$ if one understands the problem from division perspective. This measure certainly expands the interpretation of fraction and prepares students for solving problems that would use fractions as a media to represent the result of a division. The following is an exercise given after the discussion:

If 41 kilograms of peanut oil can be extracted from 100 kilograms of peanut, what is the fraction kilogram of peanut oil that one kilograms of peanut can produce?

While division can solve this problem, it is also a proportion problem. Therefore, understanding this operational interpretation of fraction will be beneficial for students to understand the concept of proportion and make connections among fraction, division, ratio, and proportion in the future.

Part of the intention of having the above discussion in the textbook is to provide an alternative way to introduce improper fractions and mixed numbers, which follow the heel of the discussion. Certainly, an improper fraction such as 7/2 can be illustrated through pictures with little confusion and understood as 3 1/2, or 3 + 1/2. However, having equated fraction and division, why 7/2 is 3 1/2 would have another explanation: Since $\frac{7}{2} = 7 \div 2$, the whole number quotient of this operation is 3 and the remainder is 1, it follows that the picture, 7/2, 3 1/2, the division, the concept of quotient and remainder are all connected together. Many students remember that they should convert an improper fraction into a mixed number but fail to give a reason. When numbers such as 7/2 have become legitimate answers at a later time, these students would feel uncomfortable for having them. One possible reason is that they have not developed the notion of fraction being a division as well as a quotient. The conversion is done mechanically and carries little meaning to them.

FOCUSING ON CONCEPTUAL UNDERSTANDING AND DIFFUSING REASONING IN THE TEXT

As indicated in Cai's study, Chinese students (six graders) significantly outperformed their counterparts in the U. S. for fraction problems, which are basically four operation problems. One may wonder if the emphasis of the Chinese curriculum is on computational skills. With this question in our minds, we went through the curriculum and the textbooks closely. Contrary to the speculation, our impression is that the Chinese curriculum goes quite a distance to stress conceptual understanding both explicitly and implicitly and the sections about fractions in the textbooks have a strong reasoning flavor. They stress the importance of grasping the mathematics behind the concept of fraction, the meaning of its operations, and the connection between whole number and fraction. The main goal is to systematically present the meanings and properties of fraction.

Students often build their conceptual understanding when they observe and begin to learn new facts, and they then develop the connections between the new and the old knowledge, and further, integrate them into a new body of knowledge. It is especially true in the case of studying fractions because they must extend their concepts of whole numbers and their operations to include that of fractions. This is not an easy job, however, given the way that fraction is defined and the fact that the main body of studying fractions is about their operations. Students may very well be lost in the process of an operation.

The approach that Chinese curriculum adopts is acknowledging proficiency on operations as an important goal and penetrating the teaching algorithms with plenty of mathematics, or algorithmic reasoning in Chinese literacy, at the same time.

We find that none of the algorithms about the four operations are introduced without explaining where they come from. Every introduction starts with a real-life problem. While the problem itself is relatively easy and usually provides a quite intuitive approach, the book will ask students what operation is needed. A pause at this moment would allow students to negotiate with themselves in order to extend their concept about a whole number operation to include fractions as operands. After the question is answered, an important note, such as "*the meanings of fraction addition is the same as the meaning of whole number addition*" will always be given in the textbook, thereby serving as a summary and a generalization.

In more than one places, the teacher reference book, which echoes the vision of curriculum, asks teachers to avoid using big numbers as either denominator or numerator. Accordingly, at the first stage of learning fractions, the numbers appeared in the text are always less than 10. The idea is to allow students to focus on understanding. In turn, the book says, students will be able to develop desired proficiency later on. It also advises teachers not to confine students' approaches to the methods that they teach. Truly, we find that when teaching how to operate on fractions in the textbook, only general methods are discussed and at no place are "steps" given. In fact, several times the textbook sends students a message that mathematics can be done in different ways. For example, it asks students to think how the basic theorem $\frac{a}{b} = \frac{a \times c}{b \times c}$ could be understood from the meaning of division after providing a figurative approach. In introducing calculations mixed with decimals, fractions, and whole numbers, the book does not specify a definite method. Quite the opposite, "there should not be a general rule given to students," the teachers' reference book states. "Allow students to use whatever that is efficient." In the same way, the textbook tells students that either the arithmetic method that is directly based on the meaning of operations or setting up an equation through analysis of the relationship in solving word problems is acceptable.

Speaking of word problems, the curriculum uses them as a means to help students acquire and strengthen their conceptual understanding. Therefore, these word problems are not used just as applications of fractions but also serve as problematic situations to start a new topic. For instance, to introduce how to multiply a number by a fraction, the book shows these problems:

– If a car travels $\frac{5}{6}$ km in a minute, how many kilometers will it travel in 30 minutes? In an hour?

– One side of a square is $\frac{7}{10}$ m. What is the perimeter?

Problems like these are plentiful in the textbooks. In fact, word problems are the main part of many exercises. They give students more opportunities to recognize that fraction is not something governed by weird rules but a touchable phenomenon and a useful tool in real-life. Gradually, the difficulty level of word problems become quite challenging:

– A factory used 120 tons of coal to produce energy in April. The consumption is 1/9 less than planned. How many tons of coal did the plan allow originally?

One type of problems-the one with composite units-is very visible in the textbooks:

– A house can be built by team A in 12 days and team B in 15 days respectively. If team A has already worked on it for three days and team B joins them, how many days are still needed to finish the construction?

Understanding is the base of reasoning. Reasoning, in turn, will reinforce understanding.

Many Chinese teachers who were interviewed in Ma's study (1999) showed a strong desire to analyze and "prove" what they said. Sometime this desire may even seem a little too enthusiastic at the elementary level. This eagerness was especially evident when these teachers tried to justify several variations of the "invert and multiply" algorithm of fraction division. Whenever an alternative was given, it was almost always accompanied by several reasons provided by these teachers. They were equally passionate when they tried to explain in many different ways why the problems they constructed would lead to

$$1\frac{3}{4} \div \frac{1}{2}.$$

Moreover, the teachers were often making pedagogical decisions at the same time. It is likely that these teachers may also reason a lot with students in teaching fractions.

No study reports that Chinese mathematics teachers in general are behaving like these teachers. However, Ma's study sheds some lights on how Chinese mathematics teachers might teach the subject, and how the curriculum and the textbooks require and support their efforts to do so.

We found that in many places in the textbooks the subjects were presented in a reasoning mode. To deal with different content about fractions, the books employ pattern recognition, making conjecture, inductive reasoning, deductive reasoning, etc. For children at this age, one would not expect them to be ready to learn rigorous proof. But as NCTM (2000) points out:

Reasoning and proof should be a consistent part of students' mathematical experiences in prekindergarten through grade 12. Reasoning mathematically is a habit of mind, and like all habits, it must be developed through consistent use in many contexts. (p. 56)

The Chinese textbooks embrace the idea of using the language of mathematics to express mathematical ideas (NCTM, 2000) throughout the subject of fraction. Words such as because and therefore are frequently used. When reasoning is relatively long, the book incorporates more conjunctional phrases to make the explanation smooth. For example, in introducing that $\frac{a}{b}=\frac{a\times c}{b\times c}$ and $\frac{a}{b}=\frac{a\div d}{b\div d}$ $(b\neq 0, c\neq 0, d\neq 0)$, it uses phrases like "by the same reason," "on the other hand" and "it follows that" to communicate with students. The early use of mathematical language to communicate mathematical ideas not only will benefit students in the future when they encounter more sophisticated mathematics, but also it leaves the impression that mathematics is a subject about reasoning and thinking in students' minds. We see that the textbooks start it in a gradual and natural manner.

FINE TUNING THE PEDAGOGY

When it comes to a unit, a section, or a subject, it is a tradition among Chinese mathematics teachers to identify what the important topics are and what the difficult topics are from learners' perspective, and they make pedagogical decisions accordingly (Ma, 1999). Important topics are thought of as pivotal points in learning a subject and mathematically significant by themselves. Difficult topics on the other hand, are difficult to understand immediately. The two types are not mutually exclusive. A topic could be both. For many Chinese mathematics teachers, the design of a teaching unit is often centered on considering how to clarify these two phenomena and their relationship and strike a balance in terms of time and energy.

A typical example of an important topic versus a challenging topic is multiplication and division with fractions. In learning fraction, the meaning of multiplication with fractions is deemed as an important topic, while the concept of division by fractions is a difficult one. This understanding is a consensus of the teachers who were interviewed by Ma (1999). In fact, researchers (Behr et al., 1992, 1993; Kieren 1988) also noted the importance of the operator interpretation (i.e., multiplication) of fraction.

"One such interpretation is 'operator,' where a fraction such as '3/4' represents a multiplicative size transformation in which a quantity is reduce to three fourths of its original size by both partitioning and duplicating various portions of the quantity" (Mack, 2001).

The teachers whom Ma (1999) interviewed agreed that in order to let students grasp the meaning of division by fractions a teacher must first devote significant time and effort to teaching multiplication with fractions to make sure students understand thoroughly the meaning of this operation. This shared idea warrants a close look at how Chinese mathematics teachers may approach these two topics.

Teaching multiplication with fraction is designed as a three session unit:

(1) a whole number times a fraction,

(2) a fraction times a whole number, and

(3) multiplication of fractions.

The first two are particularly important. It goes like this: For the first session, a teacher will start by showing that an appropriately defined algorithm can simplify repeated addition of fractions, which carries the same meaning of whole number multiplication. For instance, suppose that we want to obtain $4\times\frac{2}{9}$. Using addition, one needs to do: $\frac{2}{9}+\frac{2}{9}+\frac{2}{9}+\frac{2}{9}=\frac{8}{9}$, which is also $\frac{4\times 2}{9}$. So, it seems that $4\times\frac{2}{9}=\frac{4\times 2}{9}$.

However, the model of repeated addition is not enough. If $\frac{2}{9}\times 4$ is given, it does not make sense to say that we repeatedly add four two ninths times. At this point, there are several options. One may use a rectangular model to show that $\frac{2}{9}\times 4=\frac{8}{9}$

The representation leads to multiplication algorithm (Long & DeTemple 2000), or simply tell students that the multiplication of fractions follows commutative law (Musser, Burger, & Peterson). The vehicle that Chinese mathematics teachers use is to pose a word problem. The purpose of adopting this approach is to forge this vital concept in students' minds: the meaning of a fraction times a number is to find the fraction part of this number which, these teachers think, serves as a pivotal piece of knowledge in understanding multiplication with fractions and division with fractions.

The following is what the Chinese mathematics teachers usually do in this circumstance. Students will first see a problem like this one:

There are 48 cookies in one box. How many cookies are there in four boxes? How many are there in $\frac{3}{4}$ of a box?

Since the answer of the first question is 4×48, the teacher will lead students to think of the answer of the second question to be $\frac{3}{4}\times 48$ based on analogy. Is this assumption appropriate? If it is, how should one compute the expression and what is the answer? So next, teacher will use a picture to convert the problem back to the first case.

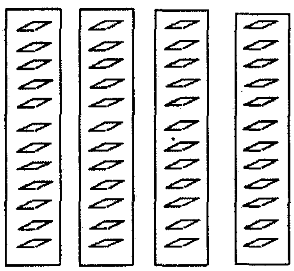

Figure 4. Each rectangle represents 48/4.

Explanation: Partition 48 cookies into four equal groups. There will be $\frac{48}{4}$ cookies in each group. To figure out how many cookies there are in $\frac{3}{4}$ of a box means to repeatedly add $\frac{48}{4}$ three times, or $3\times\frac{48}{4}$, which is already answered in case one: $\frac{3\times48}{4}=\frac{3\times48}{4}=\frac{3}{4}\times48$. Therefore, $\frac{3}{4}\times48=48\times\frac{3}{4}$ and the algorithm is $\frac{3}{4}\times48=\frac{3\times48}{4}$. The assumption is appropriate because it also implies that the commutative law, which is held for this operation.

Through a problem like this, teacher will stress that $\frac{3}{4}$ of 48 is $\frac{3}{4}\times48$, or more generally, to find the fraction part of a number, multiply this number by the fraction. The problem is a typical example used in the curriculum to illustrate multiplicative reasoning (Cai & Sun), which is very important to know. The exercises after this section in the textbook feed students with a lot of word problems again to reinforce this point. With this understanding, the case of fraction times fraction including improper fractions can be interpreted and solved with little trouble.

A solid grasp of the pivotal knowledge by students will help them in several ways. First, it reinforces the concept about fractions, which represents a multiplicative transformation. Second, in the future, students will realize that the concepts of ratio, proportion, and linear function are actually rooted in multiplicative reasoning. For example, in the above example, clearly, if one wants to solve the problem from proportional perspective, he can set up $48:1=x:\frac{3}{4}$, which is another way to say that the answer is $\frac{3}{4}\times48$. The linear function $y=48\cdot x$, on the other hand, stands for a generalization of the problem. Thirdly, when it comes to division, it is now easier to explain why dividing by a number is equivalent to multiplying by its reciprocal.

We all know that there are different methods to show the division algorithm. For example, if we need to find $\frac{3}{7}\div\frac{2}{5}$, assume the answer is x. We ask ourselves what is the relationship that ties these three numbers? Obviously, $\frac{2}{5}\cdot x=\frac{3}{7}$. It follows that $\frac{5}{2}\times\frac{2}{5}\cdot x=\frac{3}{7}\times\frac{5}{2}$, or $x=\frac{3}{7}\times\frac{5}{2}$. However, the Chinese textbook is more concerned about the meaning of the operation. It carefully distinguishes 1) a whole number divided by a fraction, 2) a fraction divided by a whole number, and 3) a fraction divided by a fraction these different cases. While the first case can be understood by directly borrowing the meaning of whole number division, the next two cannot. For instance, how should we make sense out of the operation if a partitive type of division is presented? So the meaning the operation carries in that situation must be explored. Once again, in the Chinese curriculum, the discussion starts with a word problem like the following:

How many parts could Dan polish in an hour if he polishes 60 parts in $\frac{3}{4}$ of an hour?

If $\frac{4}{5}$ inch of a rod weighs $\frac{1}{25}$ kg, how much does one meter of this rod weigh?

The first step, it says, is to translate the problem into an expression. For the first problem, based on "*a fraction times a number means to find the fraction part of this number*" the expression should be:

$$\frac{3}{4} \times (\text{the number of parts could be done in an hour}) = 60\,.$$

Hence, the number of parts could be done in an hour $= 60 \div \frac{3}{4}$. In other words, the solution reads the problem from a multiplication point of view and converts it to a division point of view afterwards. The curriculum believes that this approach reduces the difficulty that children may have to understand the meaning of fraction division and also helps them make clear connections between two operations. To explain why

$$60 \div \tfrac{3}{4} = 60 \times \tfrac{4}{3},$$

it points out that finishing 60 parts in $\frac{3}{4}$ of an hour means that number can be done in three $\frac{1}{4}$ of an hour (using multiplicative reasoning again). It follows that Dan can have $\frac{60}{3}$ parts done in $\frac{1}{4}$ of an hour. Therefore, in one hour,

$$4 \times \tfrac{60}{3} \quad (\tfrac{4}{3} \times 60)$$

parts will be finished.

SUMMARY

Our investigation shows that the Chinese curriculum and the textbooks pay much attention to the conceptual understanding of fractions. The procedural knowledge seems to be of secondary concern. We are not sure if conceptual understanding has a positive impact on students' skills with fractions since the Chinese students performed well on routine problems. In addition, word problems play an important role in helping teachers explain the phenomena, and students reinforce the knowledge. Various levels of reasoning are embedded in the curriculum. The textbooks use and imitate the mathematics language often found at higher levels. The curriculum takes great care of the development of students' knowledge and therefore has an articulated design.

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