

The “Open Approach” to Teaching School Mathematics¹

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The open approach to teaching school mathematics in the United States is an outcome of the collaboration of Japanese and U. S. researchers. We examine the approach by illustrating its three aspects:

- 1) Open process (there is more than one way to arrive at the solution to a problem;
- 2) Open-ended problems (a problem can have several of many correct answers), and
- 3) What the Japanese call “from problem to problem” or problem formulation (students draw on their own thinking to formulate new problems).

Using our understanding of the Japanese open approach to teaching mathematics, we adapt selected methods to teach mathematics more effectively in the United States. Much of this approach is new to U. S. mathematics teachers, in that it has teachers working together in groups on lesson plans, and through a series of discussions and revisions, results in a greatly improved, effective plan. It also has teachers actively observing individual students or groups of students as they work on a problem, and then later comparing and discussing the students’ work.

Keywords: open approach, open process, open-ended problem, from problem to problem.

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INTRODUCTION

From my personal perspective, collaboration among U. S. and Japanese teachers and professors dates back to at least the late 1960s. That was when I first met Professor Shigeru Shimada, the father of the open approach to teaching school mathematics in Japan. On a more comprehensive level, this collaboration has had a variety of outcomes over the years. We have had two bi-national seminars on mathematics education, held at the East-West Center at the University of Hawaii in Honolulu. There have been cross-national research projects and exchanges of visits by delegations of mathematics teachers and mathematics teacher-educators from both countries, as well as informal visits — both short and long-term. Additionally, there have been talks given at professional meetings in both countries, and the proceedings of the conferences have been published and disseminated. Of course, many articles have been published in professional journals. There are also a number of books on the subject, one of which had a very profound impact on me, and led me to develop a different perspective on teaching mathematics (Becker & Shimada 1997; Shimada 1977).

This work with Shigeru Shimada and his colleagues (see Becker & Shimada 1997) is based on research carried out by the Japanese dating back to the 1970s and, in particular, a project that began in 1971 and ended in 1977. That became the first Japanese research report that was translated into English in the U. S., titled “the Open-Ended Approach: A New Proposal for Teaching School Mathematics” (Becker & Shimada 1997). It was published by the National Council of Teachers of Mathematics (NCTM).

REFORMS

My perspective on teaching mathematics is a result of the interactions I’ve had with quite a few mathematics teachers, professors, and researchers in Japan. It is closely related to the NCTM publication, *Principles and Standards for School Mathematics* (2000).

Futurist Joel Barker has said, “Vision without action is merely a dream. Action without vision just passes the time. Vision with action can change the world.” A very important part of the vision for teaching school mathematics that we’ll discuss comes from *Principles and Standards for School Mathematics*, as well as from work in Japanese and German mathematics education.

Reforming school mathematics is an important goal. Calls for reform are urgent, but they’re not new. In many countries, reform writers have prepared authoritative papers,

official reform documents have been published, reform projects have been started, and reform movements have been launched and are underway. In the United States, the impetus for reform culminated in the first three standards documents, which were later integrated and published in 2000 as the *Principles and Standards for School Mathematics*.

In countries undergoing reform in mathematics education we find a common philosophy, which represents a paradigm shift in the way we think about mathematics teaching. Instead of viewing teaching as treatment and learning as effect, students are viewed as learners who are actively involved in their learning of mathematics. An underlying assumption in all of this is that *what* we teach and *how* students experience it are the primary factors that shape students' understanding of and beliefs about mathematics (NCTM 1989).

In a 1991 talk to the Mathematical Association of America, mathematics educator Alan Schoenfeld said, "Students pick up their sense of a domain from their experience with it." We've seen the results of the approach in which we "break the subject into pieces and then make students master it, bit by bit." As an alternative, Schoenfeld suggested that we should "create an instructional environment in which students are, at a level appropriate for them, *doing* mathematics." In other words, we should "engender selected aspects of mathematical culture in the classroom." That, of course, is what the NCTM standards are about, and that is what the work of the Japanese, dating way back to the 1970s, is also about.

OPENNESS

We can characterize the focus of this work of the Japanese as openness in mathematics education. This openness has three aspects, as shown in Figure 1.

With a little reflection, we can see the connection to the NCTM process standards of problem solving and communication. Having students discuss mathematics with each other is very important in this approach. We can also see the relationship with the other process standards that is, reasoning, making connections, and students making and using representations (a standard that was added in the 2000 revision of the standards). This addition was based on research carried out in the U. S., Brazil, and Germany, among other countries.

One of the things that I noticed the first time I had an opportunity to observe classes in Japan was that there tended to be fairly common ways of approaching lessons, as illustrated in Figure 2. I won't say that the teaching was completely uniform, but rather that common threads of practice ran through the ways the lessons unfolded.

NOTE: Some lessons will require more than one period; some may lead to "projects"

that students do and write up. Also, the times indicated on the right are rough estimates.

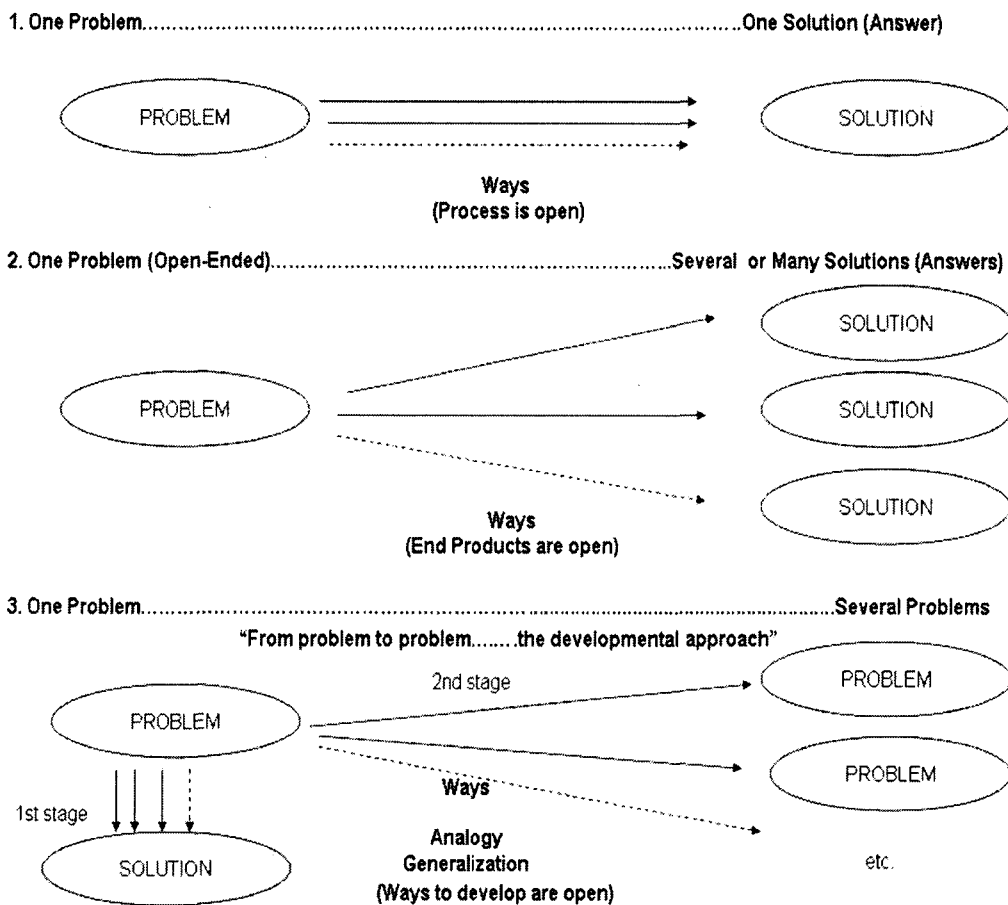


Figure 1. Openness in mathematics education

The lessons typically begin by using some time to introduce the open-ended problem, and making sure that students understand the problem and what is expected of them. The next step is crucial; students solve the problem, working either individually or in small groups. During this process, the students draw on their own natural ways of thinking in finding solutions. While they’re doing that, the teacher purposefully walks around, observes the students’ work, and asks various students to put their work on the blackboard for everyone to see. This is in preparation for the next part of the lesson, which will consist of comparing and discussing the productions (solutions/work) of the students (and not necessarily of the teacher nor the textbook). At the conclusion of the lesson, the teacher summarizes the lesson. The students may then be asked to write down what they learned as a way for the teacher to assess the effectiveness of the lesson.

(Assume a 45-minute class period)	
1. Introduce the open-ended problem	5 minutes
2. Understanding the problem	5 minutes
3. Problem solving by students, working individually or in small groups (putting their work on worksheets)*	20 minutes
4. Comparing and discussing (some students put their solutions on the black-board or OHP)	8 minutes
5. Summing up by the teacher	5 minutes
6. Optional: Ask students to write down what they learned from this lesson	2 minutes
* Using their natural mathematical thinking abilities.	

Figure 2. Organization of Some Lessons Using the Open Approach

USING THE OPEN APPROACH IN THE UNITED STATES

As we adapted some of these ideas from our Japanese colleagues, we found that preparing detailed lesson plans is really a very powerful tool in the open approach. A detailed lesson plan is useful professionally. It helps the teacher to get a good understanding of the problem, respond to students' questions and know the different ways to solve it, as well as to prepare to conduct and facilitate discussion of the students' solutions.

A detailed lesson plan begins with choosing a good problem. One characteristic of a good problem is that all students can have some degree of success with it. Then it involves something that is absolutely new to all teachers with whom we've worked, and that is working in small groups of teachers to write down all of the responses of the students that they can anticipate. Generally when they do that they get a very good list, which means that together they gain some significant insights into the problem and its solutions. This is important, because the heart of this detailed lesson plan is providing an opportunity for students to solve problems using their own natural mathematical ways of thinking.

After the detailed lesson plan is developed, then the teacher, who had major responsibility in developing it, teaches the lesson, and the other teachers observe. After the lesson, the teacher who taught it writes a complete record of the lesson, and then the teachers meet to discuss this lesson record and to improve the lesson plan.

This is quite novel to U. S. teachers, and they are very skeptical about this at first.

They're very reluctant to participate, at least in the beginning, until we have a context in which the teachers trust each other. Then things change, and they begin working together and collaborating.

Now, to my way of thinking, this is an almost ideal way to work towards teacher improvement because the teacher or the teachers who teach that lesson are going to improve in their ability and confidence to teach the lesson. Moreover, this makes a significant contribution to curriculum improvement, in that the lesson plan is improved through a series of discussions and revisions. In our work with teachers at all grade levels we found that usually by the third revision, the detailed lesson plan is in such an effective form that it is almost good enough for a substitute teacher to use. All the details are there, as well as the rationale and the background.

Furthermore, assessing student learning is crucial in this approach. Unfortunately there is not time here to delve deeply into that. But along with the open approach to teaching mathematics, Japanese researchers devised a quite different way of assessing learning, which also fits very nicely with the thinking in the NCTM documents on assessment. Assessment includes analyzing individual students' worksheets, or a group's, and observing the student or groups as they are working on the problem, and later as they are discussing it. As a result, instruction can be adjusted based on these observations, so that the assessment can actually help to improve teaching as the lesson is being taught. The components of assessment are:

Fluency — the number of

- (1) different ways a student finds to solve a problem,
- (2) correct solutions, or
- (3) formulated problems;

Flexibility — the number of different mathematical ideas in a students' work;

Originality — the depth of thinking or insight exhibited the student's work; and

Elegance — the extent to which a student expresses thinking in mathematical notation.

1. Process is Open

Let's consider an example in which the process is open — a problem at the first grade level. Suppose that there are eight butterflies on a bush, and seven more come to join them. How many butterflies are there all together? After the teacher makes it clear what the problem is and what is expected of the students, they work on the problem using their own natural thinking abilities.

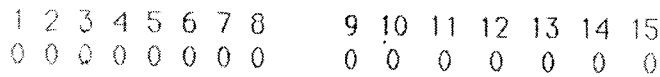
Ahead of time, the teachers have probably thought out all of these ways, and perhaps others, to solve the problem — they anticipate the students' points of view and responses.

Some solution methods are shown in Figure 3.

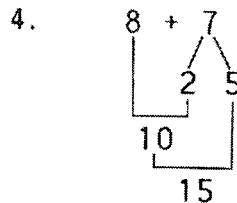
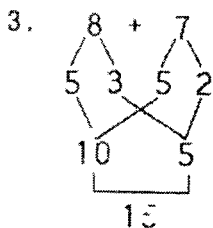
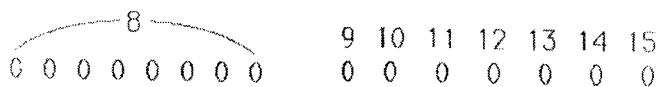
Solution methods may involve counting, decomposing, or doubling numbers. The answer may be found by counting up from 1 or counting on from 8 or 9, as we see in methods 1 and 2, respectively. The third and fourth examples illustrate methods of solving the problem by using decomposition of numbers, which is a different mathematical feature than counting. In method 3, 8 is decomposed into 5 and 3, while 7 is decomposed into 5 and 2; then the two 5's are combined to get 10 and the 3 and 2 are combined to get 5. Finally, the 5 and 10 are combined to get 15. We like method 4 just a little bit more. First, 7 is decomposed into 2 and 5, and that 2 is added to 8 to get 10; finally adding the 5 takes us to the answer, 15. In the last row, we find a third category of solutions: doubling combined with addition or subtraction. There may be other ways that students will actually think of to find the answer.

**Problem: Eight butterflies are on a bush.
Seven more come to join them.
How many are there altogether?**

1. Counting from one



2. Counting from nine



5. $8 + 8 = 16$
 $16 - 1 = 15$

6. $7 + 7 = 14$
 $14 + 1 = 15$

Figure 3. Process if Open

We saw an example of the third category of solution in a kindergarten class at Washington School in Belleville, Illinois. While the demonstration teacher was teaching a lesson, I knelt down next to a little girl and asked, “How much is 7 and 8?” She thought a little bit, and then she said, “15.” So, I said, “How did you get that answer?” She replied, “Well, I doubled 7 and I added 1.” This is a kindergartner, and they were not taught this, but kindergartners and primary school students may have a lot of what is characterized in the literature now as “informal mathematical knowledge.” Another form of informal mathematical knowledge is what students learn in a mathematics lesson without the teacher’s awareness of it. Teachers may not be aware of some of the things that the students are learning.

One aspect of assessment is concerned with assessing the number of different mathematical ideas that the students present. So once the students have solved the problem, we may ask them to think of all the different ways they can solve it. In this case, we’ve already found three categories: counting strategies, decomposition, and a doubling strategy. The larger the number of categories that have at least one response, the better — this shows more flexibility in thinking as the students are presenting more ways of thinking that have different mathematical features.

2. End Products are Open

Some problems may have several or many solutions, such as the one depicted in Figure 4. We begin with a square piece of paper, and we ask the students to divide it into four equal parts. Find different solutions to the problem. First of all, we have to make it clear what the task is for them. We want the four pieces to have equal areas, but we aren’t saying that they have to be congruent. Generally, we use square centimeter paper for a problem like this. Then the students are free to use their own natural thinking abilities to come up with different answers.

The students can find quite a variety of solutions, some of which are illustrated. Let’s look at the right-hand one in the bottom row. Those little arcs at the end of each side of the square show that line segments are equal in length. This particular way of solving the problem is very different from the other methods. Teachers are surprised; they can’t imagine that kids would do that, but they do.

What the students are thinking, of course, is that the center point of the square is fixed, as are the two diagonals of the square, which meet at the center at right angles. Now the diagonals can be rigidly rotated around the center, and wherever we stop, they divide the square into four equal parts. This introduces another mathematical idea, namely rotation. This problem actually gets at rotational symmetry.

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Divide the piece of paper into four equal parts.
How many different ways can you do it?

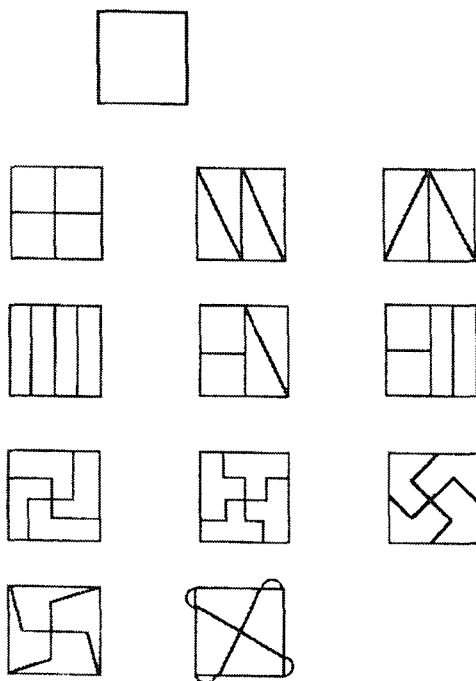


Figure 4. End-Products Are Open

Teachers come up with various proposals to form the different categories that we would use to measure flexibility, but very commonly they will say something like the following: The top two rows of solutions may form one category. Or instead, those solutions may be divided into other categories, based on the collection of shapes into which the square is divided and whether or not the divisions consist of congruent pieces. The two left-hand solutions in the third row form yet another category. This category is related to the one that includes the rightmost solution in the third row and the leftmost solution in the fourth row. Any solutions that involve rotational symmetry could form another category. So we end up with quite a number of categories, each of which differs from the others in some mathematical feature; the variety of categories and the number of solutions in each category contribute to assessment.

3. Formulating Problems - From Problem to Problem

Now let's look at an example in which students formulate problems of their own after

solving a given problem. This problem was used for with intermediate or middle school students. We begin with squares being made using matchsticks, as shown in Figure 5, and we ask how many matchsticks are used when the number of squares is five?

Squares are made using matchsticks as shown in the figure below. When the number of squares is 5, how many matchsticks are used?



Figure 5. Solving a problem and then formulating problems

We find that one of the things that we have to do when we introduce this problem is to be sure that the students understand what “...” means. Incidentally, we found in our U. S.-Japan cross-national research that the Japanese students knew very clearly what it meant, but not all of the U. S. students did.

Once we introduce the problem, we let the students work on it, using their own natural ways of thinking. We found that intermediate or middle school students fairly commonly will find solutions like those shown in Figure 6.

In solution *a*, the students simply made the prescribed number of squares and counted the matchsticks one-by-one. Solution *b* has a mistake — 4 times 5 equals 20 — it will likely come up, but an interesting aspect about the open approach is that the teacher rarely has to correct students — students correct students; when they see a flaw in reasoning they point it out. Solution *c* is based on counting the contributions of alternating squares in different ways. All 4 matchsticks forming the odd-numbered squares are taken into account, and then only the top and bottom matchsticks of the squares in between are counted to get the total, 16. Solution *d* groups 4 groups of 4 matchsticks in a different way. The squares on either end of the row each contribute 4 matchsticks to the total. This leaves the square adjacent to the rightmost square with 3 uncounted sides, which are then grouped with the shared side of the next pair of squares to the left to get another 4 matchsticks. Finally, the top and bottom matchsticks of that pair contribute the final 4 matchsticks of the total, 16. In solution *e*, the 5 matchsticks on the top and the 5 matchsticks on the bottom of the squares are added together to get all of the horizontal

matchsticks. Then the 6 vertical matchsticks are added in to get 16. In solution *f*, the leftmost square contributes 4 matchsticks to the total, and each of the 4 squares to its right contributes 3 matchsticks to the total.

What are some different ways of looking at the problem?

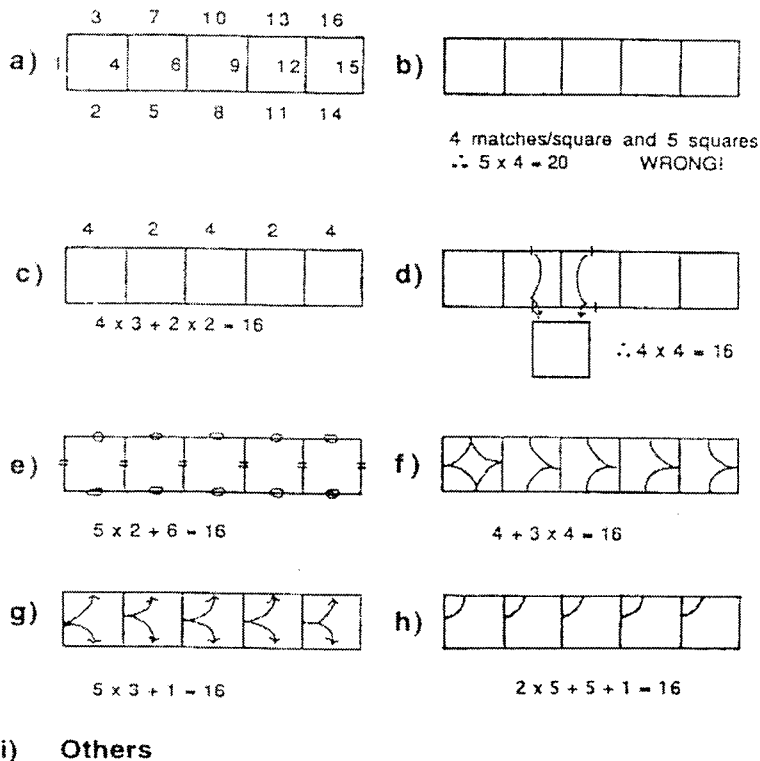


Figure 6. Different ways to find the answer to the matchsticks problem

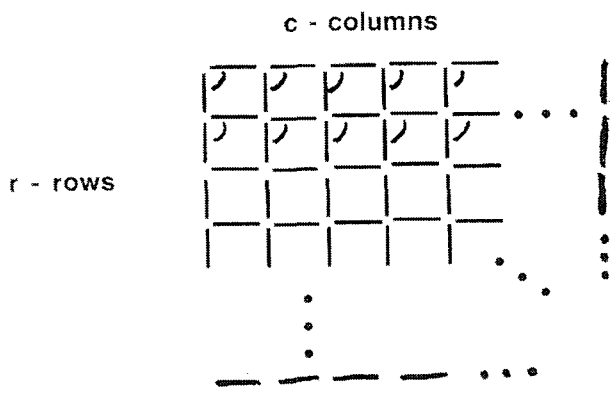
Incidentally, without going into any detail here, another component of the assessment approach used in the Japanese research is “elegance.” However, it means something a little bit different from what we typically take elegance to mean. What it means is the extent to which students represent their thinking in mathematical notation.

When Neal Foland, a mathematician in the Department of Mathematics at Southern Illinois University at Carbondale, and I began using this problem in demonstration lessons, we thought, that at the very least, we would want the lesson to end with all

students seeing that the number of matchsticks could be calculated by multiplying the number of squares by 3 and then adding 1. This method occurs in solution *g*, which is to some extent a generalization of the method used for solution *f*. The method in solution *g* is a very nice way to solve the problem, because it’s easily generalizable to an arbitrary number of squares in a row. We thought it was a very insightful way to solve the problem. When using the open approach to solve the problem, interesting things come up that may not, to a large extent, be anticipated by teachers (or by professors). For example, Jaymee and Khia, middle school students in two different demonstration classes, came up with a better way. In fact, Jaymee said that it really was a much better way. This is the method that we see in solution *h*.

In solution *h*, the 10 matchsticks that form the 5 right angles at the upper left corner of each square are taken into account first. This quantity is added to the contribution from the 5 matchsticks in the bottom row of squares, and the one at the end. Clearly this method works, so we asked Jaymee why she considered this to be a better way. She replied that her way was a more general way; the problem is just a special case of the more general problem, shown in Figure 7, she said.

What is the general rule for finding the number of matchsticks in the figure, as below, where *c* is the number of columns and *r* is the number of rows?

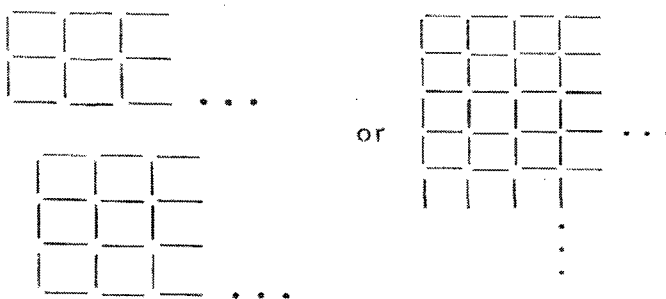


$$(2 \times c + 1) \times r + c = \# \text{ of matchsticks}$$

Figure 7. Matchsticks forming squares of *r* rows and *c* columns

**Now Make Up Problems Of Your Own –
Like The One You Just Solved**

- 1) How many matchsticks if 8 squares?
- 2) “ “ “ “ 20 squares?
- 3) “ “ “ “ 100 squares?
- 4) “ “ “ “ n squares?
- 5) How about this situation?



- 6) Change squares to triangles:



How about pentagons, hexagons, etc.

- 7) Suppose you have 40 matchsticks. How many squares can be made?
- 8) Move from two dimensions to three.

Figure 8. Formulating problems like the problem just solved

As mentioned, there are two stages in this third component of openness in mathematics education. After the students have solved the given problem, they make up new problems of their own, based on the original one. In this case, there are a number of ways that the students could make up new problems by changing conditions, but first they had to understand what the original conditions were.

What were those conditions? We were using matchsticks to make squares; that's two separate conditions. The squares are in the plane with shared sides; that's two more conditions. We began by making a construction out of five squares; that's yet another

condition. Now if we change any one or more of those conditions, we have formulated a new problem. Some new problems are shown in Figure 8, where we have changed one or more of the conditions.

First, we might change the number of squares to 8, 20, 100, or generalize to n squares. We could extend the array of squares in the horizontal direction, the vertical direction, or both to have r rows of c columns. We might also have a problem with triangles instead of squares, and then move on to pentagons, hexagons, and polygons with n sides. This might lead to interesting problems about tiling the plane with polygons. Some students might be able to formulate the converse problem: given a certain number of matchsticks, how many squares can be made? We can also move from two dimensions to three dimensions and examine cubes with shared faces, and then modify this new problem in the ways used above to get more new problems. It gets very interesting, and very challenging, as well.

In our U. S.-Japan cross-national research study, we found that students in the United States at a number of different grade levels — 4, 6, 8, and 11 — were writing some new problems, but these problems had nothing to do with the problem that they had just solved.

This is so new to many U. S. students that it's very hard for them to formulate problems that make sense or that have enough information given. That should not be surprising. They might write what they think are problems, but there's insufficient information given. They might write a complete problem, but a very simple one; for example, they might simply change the number of squares. However, some might write some very nice problems or even highly creative problems. Japanese researchers found that at the different grade levels, students wrote an average of 2.7 valid mathematics problems related to a problem that they had just solved.

4. Making Closed Problem into Open-Ended Problems

We don't typically see “open-ended problems” in textbooks used in the U.S, though it is becoming more common now, where by open-ended we mean similar to those typically found in the open approach to mathematics education. Fairly typically, the kinds of problems that we find could be called “traditional” or “closed” problems. In such problems, there is generally one way of solving the problem, leading to a single answer, and no formulations of new problems. However, there are a variety of ways that we can take traditional problems from our textbooks and other instructional materials and transform them into open-ended problems, sometimes with very little difficulty.

Which is larger: $4/5$ or $3/4$?

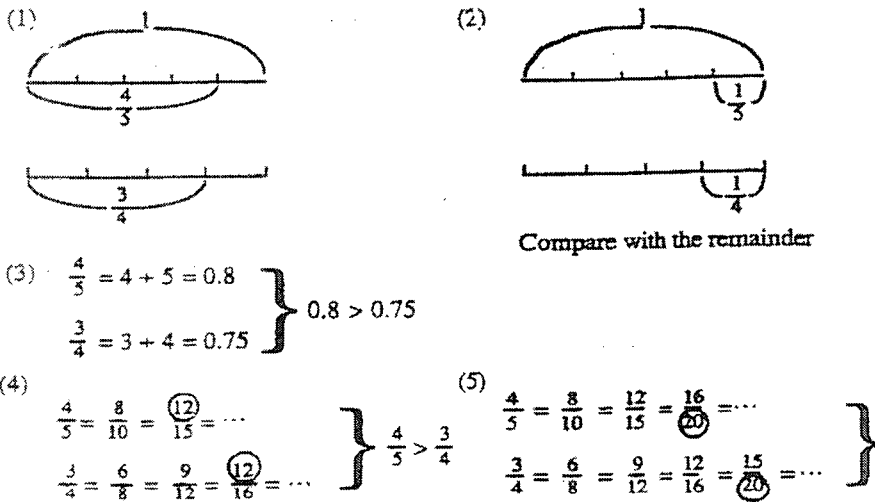


Figure 9. Solutions for “Which is Larger: $4/5$ or $3/4$?”

One example is depicted in Figure 9. We begin with the problem of determining which of the two fractions, $3/4$ or $4/5$, is larger. Generally, all that is taught is a rule by which the larger of two given fractions can be found. Now, if instead of teaching this rule in the traditional way, we simply say, “Which is larger: $3/4$ or $4/5$?”, then we leave it to the students to come up with their own ways to find the answer to the problem. Some possible methods are shown in Figure 9.

In solution 1, two line segments of length 1 are partitioned into pieces. The first is partitioned into 5 equal pieces and the second into 4 equal pieces. The fraction $4/5$ is represented by 4 of the 5 equal pieces and the fraction $3/4$ by 3 of the 4 pieces. Now the students can see which is longer. In method 2, students also partition two line segments into equal pieces, but now they focus on the pieces that remain and decide which is smaller, $1/4$ or $1/5$. In method 3, the fractions $3/4$ and $4/5$ are converted to decimals and then compared. In method 4, the given pair of fractions is converted to an equivalent pair with common numerators, and then the larger fraction can be easily determined. Method 5 is similar to method 4 except that, in this case, the fractions are compared by being converted into an equivalent pair with common denominators. Of course this method is the rationale behind the rule that is generally taught when this problem is presented in a

lesson in the United States.

CONCLUSION

The problems that we have seen and many other problems like them provide a kind of a transition from a traditional approach to what we call the “open approach” to teaching mathematics. There is a great deal of information in Principles and Standards for School Mathematics, but I think it is not the kind of document that a teacher can easily read. It has to be studied. One has to see examples, and the more the better. The open approach is based on Japanese research that dates back to the 1970s, and it helps us to make a transition from the recommendations of reform documents, such as Principles and Standards for School Mathematics, to actually implementing them in a school classroom.

I want to close with more of Alan Schoenfeld’s words. He has a lot of very good ideas about mathematics education. In his 1991 talk to the Mathematical Association of America, he made the bold assertion that: “Mathematics is a living, breathing, and exciting discipline of sense making. Students will come to see it that way if and only if they experience it that way in their classrooms.”

We have found that this process of sense making makes all the difference in working with teachers and students. If they can make sense out of what we’re teaching, then a large part of the battle is won, so to speak. If they can’t, it’s an entirely different thing. Now, in the open approach there are ample opportunities for sense making, because students start from using their own natural thinking abilities. They may also find that the product of the thinking abilities of a different student or students actually may provide more mathematically rich ways of solving a problem.

We cannot offer students such opportunities in the context of the traditional approach to teaching mathematics. In Schoenfeld’s words, virtually all standard classroom instruction should be enhanced by courses in which students grapple with the subject matter in intellectually honest ways. This is exactly what happens when using the open approach to teaching school mathematics. If we can adapt and implement the open approach based on the work of Japanese mathematics educators for use in U. S. classrooms, we will have something of great value to mathematics education. This perspective on teaching mathematics is the ongoing focus of our work in teacher education.

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