Journal of the Korean Statistical Society (2006), **35**: *3, pp 317–329*

ON THE ADMISSIBILITY OF HIERARCHICAL BAYES ESTIMATORS

Byung Hwee Kim[1] and In Hong Chang[2]

Abstract

In the problem of estimating the error variance in the balanced fixed-effects one-way analysis of variance (ANOVA) model, Ghosh (1994) proposed hierarchical Bayes estimators and raised a conjecture for which all of his hierarchical Bayes estimators are admissible. In this paper we prove this conjecture is true by representing one-way ANOVA model to the distributional form of a multiparameter exponential family.

AMS 2000 subject classifications. Primary 62C15; Secondary 62F10.
Keywords. Error variance, multiparameter exponential family, one-way analysis of variance model, relative squared error loss.

1. Introduction

Consider the problem of estimating the error variance σ^2 in the following balanced one-way analysis of variance (ANOVA) model:

$$Y_{ij} = \theta_i + \epsilon_{ij}, \quad i = 1, 2, \ldots, k, \quad j = 1, 2, \ldots, n, \tag{1.1}$$

where the ϵ_{ij}'s are independentely and identically distributed *iid* as $N(0, \sigma^2)$.

The loss to be considered is the relative squared error loss

$$L(d, \sigma^2) = (d\sigma^{-2} - 1)^2. \tag{1.2}$$

Ghosh (1994) developed the hierarchical priors and the corresponding hierarchical Bayes estimators of σ^2 and derived a subclass of these hierarchical Bayes estimators which dominates the best multiple estimator of σ^2, namely

Received October 2005; accepted July 2006.
[1]Department of Mathematics, Hanyang University, Seoul 133-791, Korea (e-mail: bkim@hanyang.ac.kr)
[2]Corresponding author. Department of Computer Science and Statistics, Chosun University, Gwangju 501-759, Korea (e-mail: ihchang@chosun.ac.kr)

$S/\{(n-1)k+2\}$, where S is the error sum of squares. The latter is known to be the best equivariant constant risk minimax estimators of σ^2 under the affine group of transformations. Accordingly, these hierarchical Bayes estimators which dominate $S/\{(n-1)k+2\}$ are also minimax estimators of σ^2. Ghosh (1994) also points out the non-minimaxity of some of the proposed hierarchical Bayes estimators and raised a conjecture for which all of his estimators are admissible.

Datta and Ghosh (1995) proposed a slightly different class of hierarchical Bayes estimators of σ^2, and provided a more detailed discussion of the minimaxity of these estimators. (In particular, the minimaxity of certain subclass of the hierarchical Bayes estimators is proved, while others are shown to be non-minimax).

A closely related problem is to estimate the variance based on a random sample from a normal distribution with unknown mean. Maatta and Casella (1990) traced the history of this estimation problem starting with Stein's (1964) elegant proof of the inadmissibility of the "usual estimator" of the variance. Later results follow from Stein's result in a natural sequence. First, Brown (1968), then Brewster and Zidek (1974) improved on Stein's (1964) result. Strawderman (1974) also exhibited improved estimators of the normal variance using a different technique. Stein's (1964) estimator and Brown's (1968) estimator are inadmissible. Proskin (1985) showed that the Brewster and Zidek estimator is admissible.

The purpose of this paper is to verify that the Ghosh's (1994) conjecture is true. In Section 2 we provide some preliminaries which are crucial in the rest of this paper. In Section 3 we provide a proof concerning the admissibility of Ghosh's (1994) estimators.

2. PRELIMINARIES

In the one-way ANOVA model (1.1) the minimal sufficient statistic is $(\overline{Y}_1, \ldots, \overline{Y}_k, S)$, where

$$\overline{Y}_i = n^{-1} \sum_{j=1}^{n} Y_{ij}, \quad i = 1, 2, \ldots, k$$

and

$$S = \sum_{i=1}^{k} \sum_{j=1}^{n} (Y_{ij} - \overline{Y}_i)^2.$$

Write Y_i for $\overline{Y}_i$, $i = 1, 2, \ldots, k$, and let $\mathbf{Y} = (Y_1, \ldots, Y_k)^T$ and $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_k)^T$.

Consider the following hierarchical Bayesian model:

(I) Conditionally on $\boldsymbol{\theta}$, σ^2 and σ_α^2 , $\mathbf{Y}$ and S are mutually independent with

$$\mathbf{Y} \sim N_k(\boldsymbol{\theta}, \frac{\sigma^2}{n} I_k)$$

and

$$S \sim \sigma^2 \chi_{k(n-1)}, \ n \geq 2;$$

(II) Conditionally on σ^2 and σ_α^2,

$$\boldsymbol{\theta} \sim N_k(\mathbf{0}, \sigma_\alpha^2 I_k);$$

(III) $\pi(\sigma^2, \sigma_\alpha^2) \propto (\sigma^2)^{\frac{a-b}{2}-1} (n\sigma_\alpha^2 + \sigma^2)^{-\frac{4-b}{2}}, \quad 0 \leq a < kn+2, \quad 0 \leq b < k+2.$

Remark 2.1. The model (1.1) is commonly called fixed-effects ANOVA model. Also, note that (I) and (II) represent a balanced one-way random-effects ANOVA model. Box and Tiao (1973, p. 328) noted that there is little difference between two models from a Bayesian viewpoint. They stated, " $\cdots$ the term fixed effect is essentially a sampling theory concept because in this frame work, the effects are regarded as fixed but unknown constants. From the Bayesian viewpoint, all parameters are random variables $\cdots$ ".

Remark 2.2. The prior $\pi(\sigma^2, \sigma_\alpha^2)$ in (III) was first used in Portnoy (1971) with slightly different notations who treated the problem of estmating σ_α^2 under the scale invariant loss. Based on (I) and (II), the prior $\pi(\sigma^2, \sigma_\alpha^2)$ with $a = b = 2$ in (III) is Jeffreys' prior and also the reference prior of Berger and Bernardo (1992) treating σ^2 as the parameter of interest and σ_α^2 as the nuisance parameter.

Based on (I)–(III) with $a = b$ Ghosh (1994) obtained the following hierarchical Bayes estimator of σ^2 under the loss (1.2):

$$\delta_a^{HB}(\mathbf{Y}, S) = \frac{S}{k(n-1)+2}[1 - \phi_a(V)], \tag{2.1}$$

where $\phi_a(V) = \dfrac{2}{kn-a+4} \dfrac{V^{\frac{1}{2}(k-a+2)}(1-V)^{\frac{1}{2}[k(n-1)+2]}}{\int_0^V z^{\frac{1}{2}(k-a)}(1-z)^{\frac{1}{2}[k(n-1)+2]} dz}, \quad 0 \leq a < k+2,$

with $V = T/(T+S)$, $T = n\sum_{i=1}^k Y_i^2$.

Now, from (I), the joint density of $\mathbf{Y}$ and S is

$$f(\mathbf{y}, s|\boldsymbol{\theta}, \sigma^2) = \frac{n^{\frac{k}{2}}}{\Gamma\left(\frac{k(n-1)}{2}\right) 2^{\frac{k(n-1)}{2}}} S^{\frac{k(n-1)}{2}-1} e^{\frac{n}{\sigma^2}\mathbf{y}^T\boldsymbol{\theta} - \frac{1}{2\sigma^2}(n\mathbf{y}^T\mathbf{y}+s)} e^{-\frac{n}{2\sigma^2}\boldsymbol{\theta}^T\boldsymbol{\theta} - \frac{kn}{2} ln\sigma^2}. \tag{2.2}$$

We first make the reparametrization

$$\eta_i = \frac{\theta_i}{\sigma^2}, \quad i = 1, 2, \ldots, k$$

and

$$\eta_{k+1} = -\frac{1}{2\sigma^2}.$$

Put $ny_i = x_i$, $i = 1, 2, \ldots, k$, and $n\mathbf{y}^T\mathbf{y} + s = x_{k+1}$. Then the inverse transformation becomes $y_i = x_i/n$, $i = 1, 2, \ldots, k$, and $s = x_{k+1} - (1/n)\sum_{i=1}^{k} x_i^2$ with Jacobian n^{-k}. Here (2.2) can be written in the form

$$f(\mathbf{x}|\boldsymbol{\eta}) = e^{\boldsymbol{\eta}^T\mathbf{x} - \psi(\boldsymbol{\eta})} \tag{2.3}$$

with respect to a σ- finite measure

$$\mu(d\mathbf{x}) = \frac{n^{\frac{k}{2}}}{\Gamma\left(\frac{k(n-1)}{2}\right)2^{\frac{k(n-1)}{2}}}\left(x_{k+1} - \frac{1}{n}\sum_{i=1}^{k} x_i\right)^{\frac{k(n-1)}{2}-1} d\mathbf{x},$$

where $\mathbf{x} = (x_1, \ldots, x_{k+1})$, $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_{k+1})$, and

$$\psi(\boldsymbol{\eta}) = \frac{n}{2}\left(-\frac{1}{2\eta_{k+1}}\right)\boldsymbol{\eta_1}^T\boldsymbol{\eta_1} + \frac{nk}{2}ln\left(-\frac{1}{2\eta_{k+1}}\right)$$

with $\boldsymbol{\eta_1} = (\eta_1, \ldots, \eta_k)$.

(2.3) has the form of densities belonging to the $(k+1)$-parameter exponential family in the natural form with the natural parameter space

$$\Omega = \{\boldsymbol{\eta} = (\eta_1, \ldots, \eta_{k+1}) : -\infty < \eta_i < \infty, \ i = 1, 2, \ldots, k, \ \textit{and} \ -\infty < \eta_{k+1} < 0\}.$$

Under the same reparametrization the loss in (1.2) becomes

$$L(\eta_{k+1}, d) = 4\eta_{k+1}^2\left(d - \left(-\frac{1}{2\eta_{k+1}}\right)\right)^2. \tag{2.4}$$

Based on (II) and (III) with $a = b$ the joint prior density of $\boldsymbol{\theta}$, σ^2 and σ_α^2 is

$$\pi(\boldsymbol{\theta}, \sigma^2, \sigma_\alpha^2) = (2\pi)^{-\frac{k}{2}}(\sigma_\alpha^2)^{-\frac{k}{2}}e^{-\frac{1}{2\sigma_\alpha^2}\boldsymbol{\theta}^T\boldsymbol{\theta}}(\sigma^2)^{-1}(n\sigma_\alpha^2 + \sigma^2)^{-\frac{4-a}{2}}. \tag{2.5}$$

Using the transformation $\boldsymbol{\theta} = \boldsymbol{\theta}$, $\sigma^2 = \sigma^2$ and $\lambda = \sigma^2/\sigma_\alpha^2$, and Jacobian $J = -\sigma^2/\lambda^2$, we have from (2.5) that the joint density of $\boldsymbol{\theta}$, σ^2 and λ is

$$\begin{aligned}\pi(\boldsymbol{\theta},\sigma^2,\lambda) &= (2\pi)^{-\frac{k}{2}}(\frac{\sigma^2}{\lambda})^{-\frac{k}{2}}e^{-\frac{\lambda}{2\sigma^2}\boldsymbol{\theta}^T\boldsymbol{\theta}}(\sigma^2)^{-1}(\frac{\sigma^2}{\lambda})^{-\frac{4-a}{2}}(n+\lambda)^{-\frac{4-a}{2}}\frac{\sigma^2}{\lambda^2}\\ &= (2\pi)^{-\frac{k}{2}}(\sigma^2)^{-\frac{k-a+4}{2}}\lambda^{\frac{k-a}{2}}(n+\lambda)^{-\frac{4-a}{2}}e^{-\frac{\lambda}{2\sigma^2}\boldsymbol{\theta}^T\boldsymbol{\theta}}\end{aligned}$$

and hence the joint density of $\boldsymbol{\theta}$ and σ^2 is

$$\pi(\boldsymbol{\theta},\sigma^2) = (2\pi)^{-\frac{k}{2}}(\sigma^2)^{-\frac{k-a+4}{2}}\int_0^\infty \lambda^{\frac{k-a}{2}}(n+\lambda)^{-\frac{4-a}{2}}e^{-\frac{\lambda}{2\sigma^2}\boldsymbol{\theta}^T\boldsymbol{\theta}}d\lambda. \tag{2.6}$$

Considering the transformation $\eta_i = \theta_i/\sigma^2$, $i = 1, 2, \ldots, k$, and $\eta_{k+1} = -1/2\sigma^2$ with Jacobian $J = 2(-1/2\eta_{k+1})^{k+2}$ (2.6) becomes

$$\begin{aligned}\pi(\boldsymbol{\eta}) &= 2(2\pi)^{-\frac{k}{2}}\left(-\frac{1}{2\eta_{k+1}}\right)^{-\frac{k-a+4}{2}}\left(-\frac{1}{2\eta_{k+1}}\right)^{k+2}\\ &\qquad\int_0^\infty \lambda^{-\frac{k-a}{2}}(n+\lambda)^{-\frac{4-a}{2}}e^{-\frac{\lambda}{2(-2\eta_{k+1})}\boldsymbol{\eta_1}^T\boldsymbol{\eta_1}}d\lambda\\ &= 2(2\pi)^{-\frac{k}{2}}(-2\eta_{k+1})^{-\frac{k+a}{2}}\int_0^\infty \lambda^{\frac{k-a}{2}}(n+\lambda)^{-\frac{4-a}{2}}e^{\frac{\lambda}{4\eta_{k+1}}\boldsymbol{\eta_1}^T\boldsymbol{\eta_1}}d\lambda. \end{aligned}\tag{2.7}$$

3. ADMISSIBILITY

To prove the admissibility of $\delta_a^{HB}(\mathbf{Y}, S)$ in (2.1) under the loss (2.4) we first consider the problem of estimating a continuous vector function $\boldsymbol{\nabla}\gamma(\boldsymbol{\eta}) = (\nabla_1\gamma(\boldsymbol{\eta}), \ldots, \nabla_{k+1}\gamma(\boldsymbol{\eta}))$ under the loss

$$L_1(\boldsymbol{\eta}, \mathbf{d}) = \sum_{i=1}^{k+1} V_i(\boldsymbol{\eta})(d_i - \nabla_i\gamma(\boldsymbol{\eta}))^2, \tag{3.1}$$

where $V_i(\boldsymbol{\eta}) = 4\eta_i^2$, $i = 1, 2, \ldots, k+1$ and $\gamma(\boldsymbol{\eta}) = -(1/2)ln(-2\eta_{k+1})$. Here $\nabla_i\gamma(\boldsymbol{\eta}) = 0$, $i = 1, 2, \ldots, k$, and $\nabla_{k+1}\gamma(\boldsymbol{\eta}) = -1/2\eta_{k+1} = \sigma^2$.

Write $\pi(\boldsymbol{\eta})$ in (2.7) as

$$\pi(\boldsymbol{\eta}) = g(\boldsymbol{\eta})e^{\psi(\boldsymbol{\eta})+(k+a)\gamma(\boldsymbol{\eta})},$$

where

$$g(\boldsymbol{\eta}) = (2\pi)^{-\frac{k}{2}}\cdot 2\cdot e^{-\psi(\boldsymbol{\eta})}\int_0^\infty \lambda^{\frac{k-2}{2}}(n+\lambda)^{-\frac{4-a}{2}}e^{\frac{\lambda}{4\eta_{k+1}}|\boldsymbol{\eta_1}|^2}d\lambda$$

with

$$\psi(\boldsymbol{\eta}) = \frac{n}{2}\left(-\frac{1}{2\eta_{k+1}}\right)\boldsymbol{\eta_1}^T\boldsymbol{\eta_1} + \frac{nk}{2}ln\left(-\frac{1}{2\eta_{k+1}}\right).$$

Define

$$I_{\mathbf{x}}(h) = \int_\Omega h(\boldsymbol{\eta}) e^{\boldsymbol{\eta}^T \mathbf{x} + (k+a)\gamma(\boldsymbol{\eta})} d\boldsymbol{\eta}$$
$$= \int_\Omega h(\boldsymbol{\eta}) e^{\boldsymbol{\eta}^T \mathbf{x} + \frac{(k+a)}{2} ln(-\frac{1}{2\eta_{k+1}})} d\boldsymbol{\eta}.$$

Consider the estimator $\boldsymbol{\delta}^\pi(\mathbf{X})$ which has the i^{th} coordinate

$$\delta_i^\pi(\mathbf{X}) = -\frac{X_i}{k+a} - \frac{I_{\mathbf{X}}[\nabla_i(V_i(\boldsymbol{\eta})g(\boldsymbol{\eta}))]}{(k+a)I_{\mathbf{X}}[V_i(\boldsymbol{\eta})g(\boldsymbol{\eta})]}, \tag{3.2}$$

$i = 1, 2, \ldots, k+1$. Then it follows from some calculations that

$$\frac{I_{\mathbf{X}}[\nabla_i(V_i(\boldsymbol{\eta})g(\boldsymbol{\eta}))]}{I_{\mathbf{X}}[V_i(\boldsymbol{\eta})g(\boldsymbol{\eta})]} = -X_i, \quad i = 1, 2, \ldots, k$$

and

$$\frac{I_{\mathbf{X}}[\nabla_{k+1}(V_{k+1}(\boldsymbol{\eta})g(\boldsymbol{\eta}))]}{I_{\mathbf{X}}[V_{k+1}(\boldsymbol{\eta})g(\boldsymbol{\eta})]} = -X_{k+1} - (k+a)\delta_a^{HB}(\mathbf{X}), \tag{3.3}$$

where $\delta_a^{HB}(\mathbf{X}) = \delta_a^{HB}(\mathbf{Y}, S)$ is given by (2.1). Combining (3.2) with (3.3) gives

$$\delta_i^\pi(\mathbf{X}) = 0, \quad i = 1, 2, \ldots, k$$

and

$$\delta_{k+1}^\pi(\mathbf{X}) = \frac{X_{k+1}}{-k-a} + \frac{I_{\mathbf{X}}[\nabla_{k+1}(V_{k+1}(\boldsymbol{\eta})g(\boldsymbol{\eta}))]}{(-k-a)I_{\mathbf{X}}[(V_{k+1}(\boldsymbol{\eta})g(\boldsymbol{\eta}))]} = \delta_a^{HB}(\mathbf{Y}, S), \tag{3.4}$$

where $\delta_a^{HB}(\mathbf{Y}, S)$ is as in (2.1). In fact $\boldsymbol{\delta}^\pi(\mathbf{X}) = (\delta_1^\pi(\mathbf{X}), \ldots, \delta_k^\pi(\mathbf{X}), \delta_{k+1}^\pi(\mathbf{X}))^T$ with $\delta_i^\pi(\mathbf{X}) = 0,\ i = 1, 2, \ldots, k$ and $\delta_{k+1}^\pi(\mathbf{X}) = \delta_a^{HB}(\mathbf{Y}, S)$ is a generalized Bayes estimator of $\boldsymbol{\nabla}\gamma(\boldsymbol{\eta}) = (\nabla_1\gamma(\boldsymbol{\eta}), \ldots, \nabla_k\gamma(\boldsymbol{\eta}), \nabla_{k+1}\gamma(\boldsymbol{\eta}))$ with $\nabla_i\gamma(\boldsymbol{\eta}) = 0,\ i = 1, 2, \ldots, k$ and $\nabla_{k+1}\gamma(\boldsymbol{\eta}) = -1/2\eta_{k+1} (= \sigma^2)$ with respect to the prior π given in (2.7) under the loss (3.1).

Note that if $\boldsymbol{\delta}^\pi(\mathbf{X})$ is admissible for estimating $\boldsymbol{\nabla}\gamma(\boldsymbol{\eta})$ under the loss (3.1), then $\boldsymbol{\delta}^\pi(\mathbf{X})$ is admissible for estimating $\boldsymbol{\nabla}\gamma(\boldsymbol{\eta})$ under the loss

$$L_2(\mathbf{d}, \boldsymbol{\eta}) = \sum_{i=1}^{k+1} V_i^*(\boldsymbol{\eta})(d_i - \nabla_i\gamma(\boldsymbol{\eta}))^2, \tag{3.5}$$

and conversely, where

$$V_i^*(\boldsymbol{\eta}) = (-2\eta_{k+1})^{\frac{nk}{2}} e^{\eta_{k+1}} e^{-\frac{n}{2}(-\frac{1}{2\eta_{k+1}})|\boldsymbol{\eta}_1|^2} V_i(\boldsymbol{\eta}), \quad i = 1, 2, \ldots, k+1.$$

We first prove the admissibility of $\boldsymbol{\delta}^\pi(\mathbf{X})$ under the loss (3.5). To do this, we use the following result which was due to Dong and Kim (1993).

LEMMA 3.1. *Let π be as in (3.1). Then $\boldsymbol{\delta}^{\pi}(\mathbf{X})$ in (3.4) is admissible for estimating $\boldsymbol{\nabla}\gamma(\boldsymbol{\eta})$ under the loss (3.5) if, for $i = 1, 2, \ldots, k+1$,*

$$\int_S \frac{V_i^*(\boldsymbol{\eta})\pi(\boldsymbol{\eta})}{\eta_i^2\Lambda^2(\boldsymbol{\eta})ln^2(\Lambda(\boldsymbol{\eta}))}d\boldsymbol{\eta} < \infty, \tag{3.6}$$

$$\int_\Omega [\nabla_i\{lnV_i^*(\boldsymbol{\eta})\pi(\boldsymbol{\eta})\}]^2 V_i^*(\boldsymbol{\eta})\pi(\boldsymbol{\eta})d\boldsymbol{\eta} < \infty \tag{3.7}$$

$$and \quad \int_\Omega V_i^*(\boldsymbol{\eta})\pi(\boldsymbol{\eta})[(k+a)\nabla_i\gamma(\boldsymbol{\eta}) + \nabla_i\psi(\boldsymbol{\eta})]^2 d\boldsymbol{\eta} < \infty, \tag{3.8}$$

where $S = \{\boldsymbol{\eta} \in \Omega : \Lambda(\boldsymbol{\eta}) \geq 2\}$ and $\Lambda^2(\boldsymbol{\eta}) = \sum_{i=1}^{k+1} ln^2(|\eta_i|)$.

The following lemma provides moments of a quadratic form in a multivariate normal distribution. The proof is omitted.

LEMMA 3.2. *Let $\boldsymbol{\eta_1} = (\eta_1, \ldots, \eta_k)^T$ have a k-variate normal distribution with mean vector $\mathbf{0}$ and covariance matrix $-\{2\eta_{k+1}/(n+\lambda)\}I_k$, where $\lambda > 0$ and $\eta_{k+1} < 0$. Then the 1st and 2nd moments of $|\boldsymbol{\eta_1}|^2 = \boldsymbol{\eta_1}^T\boldsymbol{\eta_1}$ are given, respectively, by*

$$E(|\boldsymbol{\eta_1}|^2) = \left(\frac{-2\eta_{k+1}}{n+\lambda}\right)^{\frac{k+2}{2}} \quad and \quad E(|\boldsymbol{\eta_1}|^4) = \left(\frac{-2\eta_{k+1}}{n+\lambda}\right)^{\frac{k+4}{2}}.$$

By using Lemma 3.1 we have the following theorem:

THEOREM 3.1. *$\boldsymbol{\delta}^{\pi}(\mathbf{X})$ in (3.4) is admissible for $\sigma^2 = -1/2\eta_{k+1}$ under the loss (3.5) if $0 \leq a < k+2$.*

PROOF. In the following c is a generic constant depending on k, n and a. We first check (3.6). Now,

$$\int_S \frac{V_1^*(\boldsymbol{\eta})\pi(\boldsymbol{\eta})}{\eta_1{}^2\Lambda^2(\boldsymbol{\eta})ln^2(\Lambda(\boldsymbol{\eta}))}d\boldsymbol{\eta} \leq \frac{1}{2^2 ln^2(2)}\int_\Omega \frac{V_1^*(\boldsymbol{\eta})\pi(\boldsymbol{\eta})}{\eta_1{}^2}d\boldsymbol{\eta}$$

$$= c\left\{\int_{-\infty}^0 (-2\eta_{k+1})^{\frac{kn-a}{2}} e^{\eta_{k+1}}d\eta_{k+1}\right\}\left\{\int_0^\infty \lambda^{\frac{k-a}{2}}(n+\lambda)^{-\frac{k-a+4}{2}}d\lambda\right\}. \tag{3.9}$$

Consider the transformation $u = \lambda/(n+\lambda)$. Then (3.9) becomes, for $a < k+2$,

$$c\left\{\int_{-\infty}^0 (-2\eta_{k+1})^{\frac{kn-a}{2}} e^{\eta_{k+1}}d\eta_{k+1}\right\}\left\{\int_0^1 u^{\frac{k-a}{2}}du\right\} < \infty.$$

Similarly, $\int_S \{V_i^*(\boldsymbol{\eta})\pi(\boldsymbol{\eta})\}/\{\eta_i^2\Lambda^2(\boldsymbol{\eta})ln^2(\Lambda(\boldsymbol{\eta}))\}d\boldsymbol{\eta} < \infty$ for $i = 2, 3, \ldots, k+1$ if $a < k+2$. Hence the condition (3.6) is satisfied.

We next check (3.8). Since $\nabla_i\gamma(\boldsymbol{\eta}) = 0$ for $i = 1, 2, \ldots, k$ and $\nabla_{k+1}\gamma(\boldsymbol{\eta}) = -1/2\eta_{k+1}$, it suffices to show

$$\int_\Omega V_1^*(\boldsymbol{\eta})\pi(\boldsymbol{\eta})(\nabla_1\psi(\boldsymbol{\eta}))^2 d\boldsymbol{\eta} < \infty \tag{3.10}$$

and

$$\int_\Omega V_{k+1}^*(\boldsymbol{\eta})\pi(\boldsymbol{\eta})[(k+a)\nabla_{k+1}\gamma(\boldsymbol{\eta}) + \nabla_{k+1}\psi(\boldsymbol{\eta})]^2 d\boldsymbol{\eta} < \infty. \tag{3.11}$$

Recall that $\nabla_1\psi(\boldsymbol{\eta}) = n(-1/2\eta_{k+1})\eta_1$. Also, note that when $\eta_1 \sim N(0, -2\eta_{k+1}/(n+\lambda))$, $E(\eta_1{}^4) = 3\{-2\eta_{k+1}/(n+\lambda)\}^2$. Hence the left-hand side of (3.10) becomes, for $a < k+2$,

$$\begin{aligned} &c\int_\Omega \eta_1^4(-2\eta_{k+1})^{\frac{k(n-1)-a-4}{2}} e^{\eta_{k+1}} \left(\int_0^\infty \lambda^{\frac{k-a}{2}}(n+\lambda)^{-\frac{4-a}{2}} e^{\frac{n+\lambda}{4\eta_{k+1}}|\boldsymbol{\eta_1}|^2} d\lambda\right) d\boldsymbol{\eta} \\ &= c\left\{\int_{-\infty}^0 (-2\eta_{k+1})^{\frac{kn-a}{2}} e^{\eta_{k+1}} d\eta_{k+1}\right\}\left\{\int_0^1 u^{\frac{k-a}{2}}(1-u)^2 du\right\} < \infty. \end{aligned} \tag{3.12}$$

Since $\nabla_{k+1}\psi(\boldsymbol{\eta}) = (\eta/4\eta_{k+1}^2)|\boldsymbol{\eta_1}|^2 - (nk/2\eta_{k+1})$, the left-hand side of (3.11) becomes

$$\begin{aligned} &c\int_\Omega \eta_{k+1}^2(-2\eta_{k+1})^{\frac{k(n-1)-a}{2}} e^{\eta_{k+1}} \left[-\frac{k+a}{2\eta_{k+1}} + \frac{n}{4\eta_{k+1}^2}|\boldsymbol{\eta_1}|^2 - \frac{nk}{2\eta_{k+1}}\right]^2 \\ &\left(\int_0^\infty \lambda^{\frac{k-a}{2}}(n+\lambda)^{-\frac{4-a}{2}} e^{\frac{n+\lambda}{4\eta_{k+1}}|\boldsymbol{\eta_1}|^2} d\lambda\right) d\boldsymbol{\eta} \\ &= c\int_{-\infty}^0 (-2\eta_{k+1})^{\frac{k(n-1)-a}{2}} e^{\eta_{k+1}} \left\{\int_0^\infty \lambda^{\frac{k-a}{2}}(n+\lambda)^{-\frac{4-a}{2}}\right. \\ &\left.\left(\int_{R^k}\left[\frac{n}{-2\eta_{k+1}}|\boldsymbol{\eta_1}|^2 + (nk+k+a)\right]^2 e^{\frac{n+\lambda}{4\eta_{k+1}}|\boldsymbol{\eta_1}|^2} d\boldsymbol{\eta_1}\right) d\lambda\right\} d\eta_{k+1}. \end{aligned} \tag{3.13}$$

By Lemma 3.2, (3.13) becomes

$$c\left\{\int_{-\infty}^{0}(-2\eta_{k+1})^{\frac{kn-a}{2}}e^{\eta_{k+1}}d\eta_{k+1}\right\}\left\{3kn^2\int_0^\infty \lambda^{\frac{k-a}{2}}(n+\lambda)^{-\frac{k-a+8}{2}}d\lambda\right.$$
$$+2nk(nk+k+a)\int_0^\infty \lambda^{\frac{k-a}{2}}(n+\lambda)^{-\frac{k-a+6}{2}}d\lambda$$
$$\left.+(nk+k+a)^2\int_0^\infty \lambda^{\frac{k-a}{2}}(n+\lambda)^{-\frac{k-a+4}{2}}d\lambda\right\}$$
$$<\infty \quad \text{if} \quad a<k+2.$$

Hence (3.10) and (3.11) are satisfied and therefore (3.8) is satisfied. Finally, we check (3.7). It suffices to check

$$\int_\Omega [\nabla_1\{lnV_1^*(\boldsymbol{\eta})\pi(\boldsymbol{\eta})\}]^2V_1^*(\boldsymbol{\eta})\pi(\boldsymbol{\eta})d\boldsymbol{\eta}<\infty \tag{3.14}$$

and

$$\int_\Omega [\nabla_{k+1}\{lnV_{k+1}^*(\boldsymbol{\eta})\pi(\boldsymbol{\eta})\}]^2V_{k+1}^*(\boldsymbol{\eta})\pi(\boldsymbol{\eta})d\boldsymbol{\eta}<\infty. \tag{3.15}$$

The left-hand side of (3.14) becomes

$$\int_\Omega \frac{[\nabla_1\{V_1^*(\boldsymbol{\eta})\pi(\boldsymbol{\eta})\}]^2}{V_1^*(\boldsymbol{\eta})\pi(\boldsymbol{\eta})}d\boldsymbol{\eta}$$
$$\leq c\left[\int_\Omega(-2\eta_{k+1})^{\frac{k(n-1)-a}{2}}\left\{2-\frac{n\eta_1^2}{(-2\eta_{k+1})}\right\}^2e^{\eta_{k+1}}\right.$$
$$\left(\int_0^\infty \lambda^{\frac{k-a}{2}}(n+\lambda)^{-\frac{4-a}{2}}e^{\frac{n+\lambda}{4\eta_{k+1}}|\boldsymbol{\eta_1}|^2}d\lambda\right)d\boldsymbol{\eta} \quad +\int_\Omega \eta_1^4(-2\eta_{k+1})^{\frac{k(n-1)-a-4}{2}}e^{\eta_{k+1}}$$
$$\left.\cdot\left\{\frac{\left[\int_0^\infty \lambda^{\frac{k-a+2}{2}}(n+\lambda)^{-\frac{4-a}{2}}e^{\frac{n+\lambda}{4\eta_{k+1}}|\boldsymbol{\eta_1}|^2}d\lambda\right]^2}{\int_0^\infty \lambda^{\frac{k-a}{2}}(n+\lambda)^{-\frac{4-a}{2}}e^{\frac{n+\lambda}{4\eta_{k+1}}|\boldsymbol{\eta_1}|^2}d\lambda}\right\}d\boldsymbol{\eta}\right]$$
$$=A_1+A_2, \quad \text{say.} \tag{3.16}$$

Consider the first term, A_1, in (3.16). Then

$$A_1\leq c\int_{-\infty}^{0}(-2\eta_{k+1})^{\frac{kn-a}{2}}e^{\eta_{k+1}}\left\{\int_0^1 u^{\frac{k-a}{2}}du\right\}d\eta_{k+1}$$
$$+c\int_{-\infty}^{0}(-2\eta_{k+1})^{\frac{kn-a}{2}}e^{\eta_{k+1}}\left\{\int_0^1 u^{\frac{k-a}{2}}(1-u)^2du\right\}d\eta_{k+1}$$
$$<\infty \qquad \text{if} \qquad a<k+2.$$

Consider the second term, A_2, in (3.16). By integration by parts,

$$\int_0^\infty \lambda^{\frac{k-a+2}{2}}(n+\lambda)^{-\frac{4-a}{2}} e^{\frac{n+\lambda}{4\eta_{k+1}}|\boldsymbol{\eta_1}|^2} d\lambda$$

$$= \left(\frac{-2\eta_{k+1}}{|\boldsymbol{\eta_1}|^2}\right)(k-a+2)\int_0^\infty \lambda^{\frac{k-a}{2}}(n+\lambda)^{-\frac{4-a}{2}} e^{\frac{n+\lambda}{4\eta_{k+1}}|\boldsymbol{\eta_1}|^2} d\lambda$$

$$-\frac{(-2\eta_{k+1})}{|\boldsymbol{\eta_1}|^2}(4-a)\int_0^\infty \lambda^{\frac{k-a+2}{2}}(n+\lambda)^{-\frac{6-a}{2}} e^{\frac{n+\lambda}{4\eta_{k+1}}|\boldsymbol{\eta_1}|^2} d\lambda$$

and hence

$$\frac{\int_0^\infty \lambda^{\frac{k-a+2}{2}}(n+\lambda)^{-\frac{4-a}{2}} e^{\frac{n+\lambda}{4\eta_{k+1}}|\boldsymbol{\eta_1}|^2} d\lambda}{\int_0^\infty \lambda^{\frac{k-a}{2}}(n+\lambda)^{-\frac{4-a}{2}} e^{\frac{n+\lambda}{4\eta_{k+1}}|\boldsymbol{\eta_1}|^2} d\lambda}$$

$$\leq \begin{cases} \left(\frac{-2\eta_{k+1}}{|\boldsymbol{\eta_1}|^2}\right)(k-a+2) & \text{if } a<4 \\ \left(\frac{-2\eta_{k+1}}{|\boldsymbol{\eta_1}|^2}\right)(k-a+2) + \left(\frac{-2\eta_{k+1}}{|\boldsymbol{\eta_1}|^2}\right)(a-4) & \text{if } a\geq 4 \end{cases}$$

$$= \left(\frac{-2\eta_{k+1}}{|\boldsymbol{\eta_1}|^2}\right)\max\{k-a+2, k-2\}. \tag{3.17}$$

Hence, from (3.17), we have

$$A_2 \leq c\int_\Omega \left(\frac{\eta_1}{|\boldsymbol{\eta_1}|}\right)^4 (-2\eta_{k+1})^{\frac{k(n-1)-a}{2}} e^{\eta_{k+1}}\left(\int_0^\infty \lambda^{\frac{k-a}{2}}(n+\lambda)^{-\frac{4-a}{2}} e^{\frac{n+\lambda}{4\eta_{k+1}}|\boldsymbol{\eta_1}|^2} d\lambda\right) d\boldsymbol{\eta}. \tag{3.18}$$

Since $\eta_1^2/|\boldsymbol{\eta_1}|^2 \leq 1$, we have, from (3.18)

$$A_2 \leq c\left\{\int_{-\infty}^0 (-2\eta_{k+1})^{\frac{kn-a}{2}} e^{\eta_{k+1}} d\eta_{k+1}\right\}\left\{\int_0^\infty \lambda^{\frac{k-a+2}{2}}(n+\lambda)^{-\frac{k-a+6}{2}} d\lambda\right\}$$
$$< \infty \quad \text{if} \quad a < k+2.$$

Hence (3.14) is satisfied.
Finally, the left-hand side of (3.15) becomes, after some calculations,

$$\int_\Omega [\nabla_{k+1}\{lnV_{k+1}^*(\boldsymbol{\eta})\pi(\boldsymbol{\eta})\}]^2 V_{k+1}^*(\boldsymbol{\eta})\pi(\boldsymbol{\eta})d\boldsymbol{\eta}$$

$$\leq 4\int_\Omega V_{k+1}^*(\boldsymbol{\eta})\pi(\boldsymbol{\eta})d\boldsymbol{\eta} + 8\left[\int_\Omega \frac{\{k(n-1)-a+4\}^2}{(-2\eta_{k+1})^2} V_{k+1}^*(\boldsymbol{\eta})\pi(\boldsymbol{\eta})d\boldsymbol{\eta}\right.$$

$$+\int_\Omega \frac{n^2|\boldsymbol{\eta}_1|^4}{(-2\eta_{k+1})^4} V_{k+1}^*(\boldsymbol{\eta})\pi(\boldsymbol{\eta})d\boldsymbol{\eta}\Bigg]$$

$$+2\int_\Omega \frac{|\boldsymbol{\eta}_1|^4}{(-2\eta_{k+1})^4}\left[\frac{\int_0^\infty \lambda^{\frac{k-a+2}{2}}(n+\lambda)^{-\frac{4-a}{2}}e^{\frac{n+\lambda}{4\eta_{k+1}}|\boldsymbol{\eta}_1|^2}d\lambda}{\int_0^\infty \lambda^{\frac{k-a}{2}}(n+\lambda)^{-\frac{4-a}{2}}e^{\frac{n+\lambda}{4\eta_{k+1}}|\boldsymbol{\eta}_1|^2}d\lambda}\right]^2 V_{k+1}^*(\boldsymbol{\eta})\pi(\boldsymbol{\eta})d\boldsymbol{\eta}$$

$$= 4B_1 + 8(B_2+B_3) + 2B_4.$$

Now,

$$B_1 = c\left\{\int_{-\infty}^0 (-2\eta_{k+1})^{\frac{kn-a+4}{2}} e^{\eta_{k+1}} d\eta_{k+1}\right\}\left\{\int_0^1 u^{\frac{k-a}{2}}du\right\}$$
$$< \infty \quad \text{if} \quad a < k+2.$$

Similarly, if $a < k+2$ $B_2 < \infty$ and $B_3 < \infty$ and using Lemma 3.2 $B_4 < \infty$. Hence (3.15) is satisfied if $a < k+2$, and therefore (3.7) is satisfied. By Lemma 3.1, the proof of Theorem 3.1 is complete.

Therefore, $\boldsymbol{\delta}^\pi(\mathbf{X})$ in (3.4) is also admissible for $\nabla\gamma(\boldsymbol{\eta})$ under the loss (3.1). Note that $\nabla_i\gamma(\boldsymbol{\eta}) = 0$, $i = 1,2,\ldots,k$, $\nabla_{k+1}\gamma(\boldsymbol{\eta}) = -(1/2\eta_{k+1})(=\sigma^2)$, and

$$\boldsymbol{\delta}^\pi(\mathbf{X}) = (\delta_1^\pi(\mathbf{X}),\ldots,\delta_k^\pi(\mathbf{X}),\delta_{k+1}^\pi(\mathbf{X}))$$

with $\delta_i^\pi(\mathbf{X}) = 0, \quad i = 1,2,\ldots,k$ and $\delta_{k+1}^\pi(\mathbf{X}) = \delta_a^{HB}(\mathbf{Y},S), \ 0 \le a < k+2$. This implies that there dose not exist any other estimator $\boldsymbol{\delta}(\mathbf{X})$ such that

$$\begin{aligned} &R_1(\boldsymbol{\eta},\boldsymbol{\delta}) \le R_1(\boldsymbol{\eta},\boldsymbol{\delta}^\pi) \quad \text{for all } \boldsymbol{\eta}\in\Omega \\ \text{and}\ &R_1(\boldsymbol{\eta},\boldsymbol{\delta}) < R_1(\boldsymbol{\eta},\boldsymbol{\delta}^\pi) \quad \text{for some } \boldsymbol{\eta}, \end{aligned} \tag{3.19}$$

where $R_1(\boldsymbol{\eta},\cdot)$ denote the risk of an estimator under the loss (3.1).

Now, $R_1(\boldsymbol{\eta},\boldsymbol{\delta}) = \sum_{i=1}^k V_i(\boldsymbol{\eta})E_{\boldsymbol{\eta}}[(\delta_i(\mathbf{X}))^2] + V_{k+1}(\boldsymbol{\eta})E_{\boldsymbol{\eta}}[(\delta_{k+1}(\mathbf{X})-\nabla_{k+1}\gamma(\boldsymbol{\eta}))^2]$ and $R_1(\boldsymbol{\eta},\boldsymbol{\delta}^\pi) = V_{k+1}(\boldsymbol{\eta})E_{\boldsymbol{\eta}}[(\delta_{k+1}^\pi(\mathbf{X}) - \nabla_{k+1}\gamma(\boldsymbol{\eta}))^2]$. Then with the choice of $\delta_i(\mathbf{X}) = 0, \quad i = 1,2,\ldots,k$,

$$R_1(\boldsymbol{\eta},\boldsymbol{\delta}) = V_{k+1}(\boldsymbol{\eta})E_{\boldsymbol{\eta}}[(\delta_{k+1}^\pi(\mathbf{X}) - \nabla_{k+1}\gamma(\boldsymbol{\eta}))^2].$$

Hence there does not exist any estimator of the form

$$\boldsymbol{\delta}(\mathbf{X}) = (\delta_1(\mathbf{X}),\ldots,\delta_k(\mathbf{X}),\delta_{k+1}(\mathbf{X}))$$

with $\delta_i(\mathbf{X}) = 0, \quad i = 1, 2, \ldots, k$ such that

$$V_{k+1}(\boldsymbol{\eta})E_{\boldsymbol{\eta}}[(\delta_{k+1}(\mathbf{X}) - \nabla_{k+1}\gamma(\boldsymbol{\eta}))^2]$$
$$\leq V_{k+1}(\boldsymbol{\eta})E_{\boldsymbol{\eta}}[(\delta_{k+1}^{\pi}(\mathbf{X}) - \nabla_{k+1}\gamma(\boldsymbol{\eta}))^2] \quad \text{for all} \quad \boldsymbol{\eta} \in \Omega$$

with strict inequality for at least one $\boldsymbol{\eta}$. This implies that $\delta_{k+1}^{\pi}(\mathbf{X}) = \delta_a^{HB}(\mathbf{Y}, S)$ with $0 \leq a < k+2$ is admissible for estimating $\nabla_{k+1}\gamma(\boldsymbol{\eta}) = -(1/2\eta_{k+1}) = \sigma^2$ under the loss (2.4), *i.e.*, under the loss (1.2) since $V_{k+1}(\boldsymbol{\eta}) = 4\eta_{k+1}^2 (= \sigma^{-4}) > 0$ for $-\infty < \eta_{k+1} < 0$. □

ACKNOWLEDGEMENTS

The authors are very grateful to an associate editor and the referees for their valuable suggestions. This work was supported by research funds from Chosun University, 2006.

REFERENCES

BERGER, J. O. AND BERNARDO, J. M. (1992). "On the development of reference priors", *Bayesian Statistics 4* (J. M. Bernardo, *et al.* eds.), 35–60, Oxford University Press, New York.

BOX, G. E. P. AND TIAO, G. C. (1973). *Bayesian Inference in Statistical Analysis*, Addison-Wesley, Massachusetts.

BREWSTER, J. F. AND ZIDEK, J. V. (1974). "Improving on equivariant estimators", *The Annals of Statistics*, **2**, 21–38.

BROWN, L. (1968). "Inadmissiblity of the usual estimators of scale parameters in problems with unknown location and scale parameters", *Annals of Mathematical Statistics*, **39**, 29–48.

DATTA, G. S. AND GHOSH, M. (1995). "Hierarchical Bayes estimators of the error variance in one-way ANOVA models", *Journal of Statistical Planning and Inference*, **45**, 399–411.

DONG, K. H. AND KIM, B. H. (1993). "Sufficient conditions for the admissibility of estimators in the multiparameter exponential family", *Journal of the Korean Statistical Society*, **22**, 55–69.

GHOSH, M. (1994). "On some Bayesian solutions of the Neyman-Scott problem", In *Statistical Decision Theory and Related Topics V* (S. S. Gupta and J. O. Berger. eds.), 267–276, Springer, New York.

MAATTA, J. M AND CASELLA, G. (1990). "Developments in decision theoretic variance estimation (with discussion)", *Statistical Science*, **5**, 90–120.

PORTNOY, S. (1971). "Formal Bayes estimation with application to a random effects model", *The Annals of Mathematical Statistics*, **42**, 1379–1402.

PROSKIN, H. M. (1985). "An admissibility theorem with applications to the estimation of the variance of the normal distribution", Ph. D. Dissertation, Department of Statistics, Rutgers University, New Jersey.

STEIN, C. (1964). "Inadmissibility of the usual estimator of the variance of a normal distribution with unknown mean", *Annals of the Institute of Statistical Mathematics*, **16**, 155–160.

STRAWDERMAN, W. E. (1974). "Minimax estimation of powers of the variance of a normal population under squared error loss", *The Annal of Statistics*, **2**, 190–198.