Journal of the Korean Statistical Society (2006), **35***: 3, pp 225–238*

OBJECTIVE BAYESIAN APPROACH TO STEP STRESS ACCELERATED LIFE TESTS[†]

Dal Ho Kim[1], Woo Dong Lee[2] and Sang Gil Kang[3]

Abstract

This paper considers noninformative priors for the scale parameter of exponential distribution when the data are collected in step stress accelerated life tests. We find the Jeffreys' and reference priors for this model and show that the reference prior satisfies first order matching criterion. Also, we show that there exists no second order matching prior in this problem. Some simulation results are given and we perform Bayesian analysis for proposed priors using some data.

AMS 2000 subject classifications. Primary 62F15; Secondary 62F25.
Keywords. Accelerated life tests, Jeffreys' prior, matching prior, reference prior, step stress.

1. Introduction

In many reliability studies, the life tests were made under various environmental conditions. But for extremely reliable units it is in general impossible to make life tests under the usual conditions because the life times of units under the usual conditions may tend to be large and then the testing time may be very long. As a common approach to overcome this problem, the accelerated life tests (ALTs) are widely used, in which samples of units are subjected to conditions of greater stress than the usual conditions. For example, accelerated test conditions involve higher than usual temperature, voltage, pressure, vibration, cycling rate, load, *etc.*, or some combination of them.

Received June 2006; accepted July 2006.
[†]This research was supported by Kyungpook National University Research Team Fund, 2003.
[1]Corresponding author. Department of Statistics, Kyungpook National University, Daegu 702-701, Korea (e-mail: dalkim@knu.ac.kr)
[2]Department of Asset Management, Daegu Haany University, Gyeongsan 712-715, Korea
[3]Department of Applied Statistics, Sangji University, Wonju 220-702, Korea

The step stress ALT is commonly used in engineering practice. We are interested in the step stress ALT wherein the stress on unfailed units is allowed to change at preassigned times until they fail. Though there are several models that have been commonly used on the step stress ALTs, DeGroot and Goel (1979) proposed tampered random variables (TRV) model which the effect of changing the stress from s_1 to s_2 ($s_1 < s_2$) is to multiply the remaining life of the unit at changing time τ by some unknown factor α ($0 < \alpha < 1$). The proposed model is

$$Y = \begin{cases} X, & X \leq \tau \\ \tau + \alpha(X - \tau), & X > \tau. \end{cases} \tag{1.1}$$

DeGroot and Goel (1979) studied the Bayesian estimation of parameters and optimal design of the model (1.1) when X is exponential distribution. They considered two independent gamma priors for parameter estimation. Although several Bayesian studies have been done for step stress ALTs, most of their works were performed based on subjective conjugate prior (for example, see Pathak *et al.*, 1987).

The present paper focuses on developing noninformative priors for step stress ALTs. We consider Jeffreys' (1961) prior, which is proportional to the positive square root of the determinant of the Fisher information matrix. In spite of its success in one parameter problems, Jeffreys' prior frequently runs into serious difficulties in the presence of nuisance parameters. As an alternative, we derive Bernardo's (1979) reference prior. As well known, Berger and Bernardo (1989, 1992) extended Bernardo's (1979) approach, giving a general algorithm to derive a reference prior by splitting the parameters into several groups according to their order of inferential importance.

On the other hand, we consider Bayesian priors such that the resulting credible intervals for for the scale parameter of exponential distribution when the data are collected in step stress ALTs have coverage probabilities equivalent to their frequentist counterparts. This matching idea goes back to Welch and Peers (1963) and Peers (1965). Interest in such priors revived with the work of Stein (1985) and Tibshirani (1989). Among others, we may cite the work of Mukerjee and Dey (1993), Datta and Ghosh (1995a, b, 1996), Mukerjee and Ghosh (1997).

In this paper, we derive Jeffreys' and reference priors as well as probability matching priors for the scale parameter when the lifetime distribution under normal stress is exponential. Based on the orthogonal transformation in the sense of Cox and Reid (1987), we first find the orthogonal reparametrization for scale parameter and then find reference prior and matching prior. We show that the proposed matching prior is the first order matching and that there exists no second order matching for model (1.1). We also show that the joint posterior is proper. Finally some simulation results and data analysis are given.

2. NONINFORMATIVE PRIORS

2.1. Jeffreys' and reference priors

When the lifetime distribution under the usual condition is exponential distribution with parameter θ, the probability density function (*pdf*) is given by

$$f(x|\theta) = \theta \exp(-\theta x),\ 0 < x < \infty,\ 0 < \theta < \infty. \tag{2.1}$$

Under the step stress ALTs model (1.1), the distribution function is

$$\begin{aligned} G(y|\theta,\alpha) &= \begin{cases} F(y|\theta), & y \le \tau \\ F(\tau + \frac{y-\tau}{\alpha}|\theta), & y > \tau, \end{cases} \\ &= \begin{cases} 1-\exp(-\theta y), & y \le \tau \\ 1-\exp\{-\theta(\tau + \frac{y-\tau}{\alpha})\}, & y > \tau, \end{cases} \end{aligned} \tag{2.2}$$

where $F(\cdot|\theta)$ is a distribution function of *pdf* (2.1), and α $(0 < \alpha < 1)$ is called the tampering coefficient. Then the corresponding *pdf* is

$$g(y|\theta,\alpha) = \begin{cases} \theta\exp(-\theta y), & y \le \tau \\ \frac{\theta}{\alpha}\exp\{-\theta(\tau + \frac{y-\tau}{\alpha})\}, & y > \tau. \end{cases} \tag{2.3}$$

Let $y_1, y_2, \ldots, y_n$ are ALTs data from *pdf* (2.3). Then the likelihood function is

$$L(\theta,\alpha) \propto \theta^n \alpha^{-m_2} \exp\{-\theta(\sum_{i=1}^{m_1} y_i + \sum_{i=1}^{m_2} \frac{y_i - \tau}{\alpha} + m_2\tau)\}, \tag{2.4}$$

where m_1 and m_2 are the number of the untampered and tampered observations, respectively and $n = m_1 + m_2$.

Usually in ALTs one wants to know the information about parameter under normal stress level. In our ALTs model, θ is more important than α. So we consider the orthogonal transformation for θ.

To do this, let $\omega_1 = \theta$ and $\omega_2 = \theta/\alpha,\ \omega_1 < \omega_2$. Then with this parametrization, the likelihood function of (ω_1, ω_2) is given by

$$L(\omega_1,\omega_2) \propto \omega_1^n (\frac{\omega_1}{\omega_2})^{-m_2} \exp\{-\omega_1(\sum_{i=1}^{m_1} y_i + \frac{\omega_2}{\omega_1}\sum_{i=1}^{m_2}(y_i - \tau) + m_2\tau)\}. \tag{2.5}$$

The logarithm of likelihood function (2.5) is

$$l(\omega_1, \omega_2) \propto m_1 \log \omega_1 + m_2 \log \omega_2 - \omega_1 \sum_{i=1}^{m_1} y_i - \omega_2 \sum_{i=1}^{m_2} (y_i - \tau) - \omega_1 m_2 \tau. \quad (2.6)$$

We can find that the random variables m_1 and m_2 are distributed as binomial distribution with parameters $(n, P\{X \leq \tau\})$ and $(n, P\{X > \tau\})$, respectively, where $P\{X > \tau\} = \exp(-\theta\tau)$. So, $E[m_2] = ne^{-\theta t}$. Then the Fisher information matrix of (ω_1, ω_2) is given by

$$I(\omega_1, \omega_2) = \begin{pmatrix} I_{11} & 0 \\ 0 & I_{22} \end{pmatrix},$$

where

$$\begin{aligned} I_{11} &= \frac{n}{\omega_1^2}\{1 - \exp(-\omega_1\tau)\}, \\ I_{22} &= \frac{n}{\omega_2^2} \exp(-\omega_1\tau). \end{aligned}$$

The Jeffreys' prior for ω_1 and ω_2 is

$$\pi^J(\omega_1, \omega_2) \propto (\omega_1\omega_2)^{-1} \{(1 - \exp(-\omega_1\tau)) \exp(-\omega_1\tau)\}^{1/2}, \ 0 < \omega_1 < \omega_2 < \infty. \quad (2.7)$$

Due to the orthogonality of the parameters, following Datta and Ghosh (1995b) choosing rectangular compacts for each ω_1 and ω_2, when ω_1 is of more inferential importance than ω_2, the reference prior for ω_1 and ω_2 is given by

$$\pi^R(\omega_1, \omega_2) \propto (\omega_1\omega_2)^{-1} \{(1 - \exp(-\omega_1\tau))\}^{1/2}, \ 0 < \omega_1 < \omega_2 < \infty. \quad (2.8)$$

Datta and Ghosh (1996) showed that the reference and matching prior have invariance properties under the one to one transformed reparametrization. Hence, the Jeffreys' and reference prior for θ and α is

$$\pi^J(\theta, \alpha) \propto (\theta\alpha)^{-1} \{(1 - \exp(-\theta\tau)) \exp(-\theta\tau)\}^{1/2}, \ 0 < \theta < \infty, \ 0 < \alpha < 1 \quad (2.9)$$

and

$$\pi^R(\theta, \alpha) \propto (\theta\alpha)^{-1} \{(1 - \exp(-\theta\tau))\}^{1/2}, \ 0 < \theta < \infty, \ 0 < \alpha < 1, \quad (2.10)$$

respectively.

2.2. Matching priors

For a prior π, let $\theta^{1-\eta}(\pi;\mathbf{Y})$ be a percentile of the posterior distribution of θ, that is,

$$P^{\pi}\{\theta \le \theta^{1-\eta}(\pi;\mathbf{Y})|\mathbf{Y}\} = 1-\eta. \tag{2.11}$$

We want to find priors which satisfy

$$P^{\pi}\{\theta \le \theta^{1-\eta}(\pi;\mathbf{Y})|\theta,\alpha\} = 1-\eta+o(n^{-u/2}) \tag{2.12}$$

for some $u > 0$, as n goes to ∞. Priors π satisfying (2.12) are called matching priors. If $u = 1$, then π is called a first order matching prior, if $u = 2$, π is called a second order matching prior.

Now, to find the matching priors π, consider the likelihood function of one observation for convenience. The likelihood function of an observation y is

$$L(\theta,\alpha) \propto \theta^{\delta}\exp(-\theta\delta y)\left(\frac{\theta}{\alpha}\right)^{1-\delta}\exp\{-(1-\delta)\theta(\frac{y-\tau}{\alpha}-\tau)\},$$

where

$$\delta = \begin{cases} 1, \; y \le \tau \\ 0, \; y > \tau. \end{cases}$$

First, we apply the orthogonal reparametrization, $\omega_1 = \theta$ and $\omega_2 = \theta/\alpha$ and take the logarithm of the above likelihood function,

$$l(\omega_1,\omega_2) \propto \log\omega_1 - \omega_1\delta y - \delta' \log\omega_1 + \delta' \log\omega_2 - \omega_2\delta'(y-\tau) - \omega_1\delta'\tau,$$

where $\delta' = 1-\delta$.

The Fisher information matrix of (ω_1,ω_2) per observation is given by

$$i(\omega_1,\omega_2) = \begin{pmatrix} i_{11} & 0 \\ 0 & i_{22} \end{pmatrix},$$

where

$$i_{11} = \frac{1}{\omega_1^2}\{1-\exp(-\omega_1\tau)\},$$
$$i_{22} = \frac{1}{\omega_2^2}\exp(-\omega_1\tau).$$

Notice that the transformation from (θ,α) to (ω_1,ω_2) is orthogonal in the sense of Cox and Reid (1987). Due to the work of Tibshirani (1989), the first order probability matching priors when the parameter of interest is ω_1 is given by

$$\pi_1^M(\omega_1,\omega_2) \propto \omega_1^{-1}\{1-\exp(-\omega_1\tau)\}^{1/2}d(\omega_2), \tag{2.13}$$

where $d(\omega_2)$ is an arbitrary function differentiable in its argument. So, the first order matching prior for (θ, α) is

$$\pi_1^M(\theta, \alpha) \propto \alpha^{-2}\{1 - \exp(-\theta\tau)\}^{1/2} g(\theta/\alpha), \tag{2.14}$$

where $g(\cdot)$ is an arbitrary differentiable function in its arguments.

The class of prior given in (2.13) is large, and it may be necessary to narrow down this class of priors. An alternative is to find the class of second order probability matching priors which are given by Mukerjee and Ghosh (1997). They showed that a second order probability matching prior is of the form (2.13), and $d(\omega_2)$ must satisfy the following differential equation,

$$\frac{1}{6} d(\omega_2) \frac{\partial}{\partial \omega_1}\{(i_{11}^{-3/2}) L_{1,1,1}\} + \frac{\partial}{\partial \omega_2}\{i_{11}^{-1/2} L_{112}(i^{22}) d(\omega_2)\} = 0, \tag{2.15}$$

where

$$L_{1,1,1} = E\left[\left(\frac{\partial l(\omega_1, \omega_2)}{\partial \omega_1}\right)^3\right],$$
$$L_{112} = E\left[\frac{\partial^3 l(\omega_1, \omega_2)}{\partial \omega_1^2 \partial \omega_2}\right]$$

and

$$\begin{pmatrix} i^{11} & i^{12} \\ i^{21} & i^{22} \end{pmatrix} = \begin{pmatrix} i_{11} & i_{12} \\ i_{21} & i_{22} \end{pmatrix}^{-1}.$$

Since $\partial^3 l(\omega_1, \omega_2)/\partial \omega_1^2 \partial \omega_2 = 0$, $L_{112} = 0$. So the second term in equation (2.15) is 0. The only way which the prior (2.13) satisfies the second order matching condition (2.15) is $(i_{11}^{-3/2}) L_{1,1,1}$ is function of ω_2 alone or constant.

Now, from the log-likelihood function, one can obtain

$$\frac{\partial l(\omega_1, \omega_2)}{\partial \omega_1} = \delta \omega_1^{-1} - \delta y - \delta' \tau.$$

Then

$$\begin{aligned} L_{1,1,1} &= E\left[\delta(\omega_1^{-1} - Y) - \delta' \tau\right]^3 \\ &= E\left[\delta(\omega_1^{-1} - Y)^3\right] - \tau^3 E[\delta']. \end{aligned}$$

Now, $E[\delta(\omega_1^{-1} - Y)^3]$ can be reexpressed by conditioning on δ as follows:

$$\begin{aligned} E\left[\delta(\omega_1^{-1} - Y)^3\right] &= E\left[\delta(\omega_1^{-1} - Y)^3 | \delta = 1\right] P\{\delta = 1\} \\ &\quad + E\left[\delta(\omega_1^{-1} - Y)^3 | \delta = 0\right] P\{\delta = 0\} \\ &= E\left[(\omega_1^{-1} - Y)^3 | \delta = 1\right] P\{\delta = 1\}. \end{aligned}$$

Note that $P\{\delta = 1\} = 1 - \exp(-\omega_1\tau)$. Moreover, given that an observation is untampered ($Y \le \tau$), the distribution function of Y is, for $0 < y \le \tau$,

$$\begin{aligned} F_{Y|Y\le\tau}(y) &= P\{Y \le y | Y \le \tau\} \\ &= \frac{1-\exp\{-\omega_1 y\}}{1-\exp\{-\omega_1\tau\}}. \end{aligned}$$

From this result, we can calculate the conditional expectations which is included in $E[(\omega_1^{-1} - Y)^3|\delta = 1]$ as follows:

$$\begin{aligned} E\left[Y|\delta=1\right] &= \omega_1^{-1} - \tau A(\omega_1,\tau), \\ E\left[Y^2|\delta=1\right] &= 2\omega_1^{-2} - \tau^2 A(\omega_1,\tau) - 2\tau\omega_1^{-1}A(\omega_1,\tau), \\ E\left[Y^3|\delta=1\right] &= 6\omega_1^{-3} - \tau^3 A(\omega_1,\tau) - 3\tau^2\omega_1^{-1}A(\omega_1,\tau) - 6\tau\omega_1^2 A(\omega_1,\tau), \end{aligned}$$

where $A(\omega_1, \tau) = \exp\{-\omega_1\tau\}(1 - \exp\{-\omega_1\tau\})^{-1}$. So,

$$\begin{aligned} E\left[(\omega_1^{-1} - Y)^3|\delta = 1\right] &= \omega_1^{-3} - 3\omega_1^{-2}E\left[Y|\delta=1\right] + 3\omega_1^{-1}E\left[Y^2|\delta=1\right] \\ &\quad -E\left[Y^3|\delta=1\right] \\ &= \tau^3 A(\omega_1,\tau) + 3\tau\omega_1^{-2}A(\omega_1,\tau) - 2\omega_1^{-3}. \end{aligned}$$

Hence

$$\begin{aligned} L_{1,1,1} &= E\left[(\omega_1^{-1} - Y)^3|\delta = 1\right] P\{\delta = 1\} \\ &= \tau^3\exp(-\omega_1\tau) + 3\tau\omega_1^{-2}\exp(-\omega_1\tau) - 2\omega_1^{-3}\left\{1 - \exp(-\omega_1\tau)\right\} \end{aligned}$$

and

$$\begin{aligned} i_{11}^{-3/2}L_{1,1,1} &= \tau^3\omega_1^3\exp(-\omega_1\tau)\{1-\exp(-\omega_1\tau)\}^{-3/2} \\ &\quad +3\tau\omega_1\exp(-\omega_1\tau)\{1-\exp(-\omega_1\tau)\}^{-3/2} \\ &\quad -2\{1-\exp(-\omega_1\tau)\}^{-1/2}. \end{aligned}$$

The above quantity is not a function of ω_2 or constant. Therefore, there exists no second order matching prior in our model.

Moreover, one can may interest in finding a matching prior for tampering coefficient α, but the general solution of partial differential equation of Tibshirani (1989) does not exist.

The Jeffreys' prior (2.9) does not satisfy first order matching criterion. But if we take the arbitrary function $g(\theta/\alpha)$ in (2.14) as θ/α, then the reference prior (2.10) is in a class of first order probability matching prior.

3. POSTERIOR ANALYSIS

Suppose that y $= (y_1, y_2, \dots, y_n)$ is random sample from *pdf* (2.3). The likelihood function for (θ, α) is (2.4). For the parameters (θ, α), we consider the general form of prior such as

$$\pi_{(i,j,k,l)}(\theta,\alpha) \propto \theta^{-i}\alpha^{-j}\left\{1-\exp(-\theta\tau)\right\}^{k/2}\left\{\exp(-\theta\tau)\right\}^{l/2}, \tag{3.1}$$

where $0 < \theta < \infty$ and $0 < \alpha < 1$. Note that $\pi_{(0,2,1,0)}(\theta,\alpha)$ equals to the matching prior (2.14), and $\pi_{(1,1,1,1)}(\theta,\alpha)$ equals to Jeffreys' prior (2.9) and $\pi_{(1,1,1,0)}(\theta,\alpha)$ equals to reference prior (2.10).

Using the prior (3.1) and the likelihood function (2.4), one can find the joint posterior distribution of θ and α as follows:

$$\begin{aligned}\pi(\theta,\alpha|\mathrm{y}) &\propto L(\theta,\alpha)\pi_{(i,j,k,l)}(\theta,\alpha) \\ &\propto \theta^{n-i}\alpha^{-m_2-j}\exp\{-\theta(m_2\tau+v)-\frac{\theta}{\alpha}w\} \\ &\quad \times(1-\exp\{-\theta\tau\})^{k/2}(\exp\{-\theta\tau\})^{l/2},\end{aligned} \tag{3.2}$$

where $v = \sum_{i=1}^{m_1} y_i$ and $w = \sum_{j=1}^{m_2}(y_j - \tau)$.

We first show the propriety of joint posterior for the prior (3.1).

THEOREM 3.1. *The joint posterior distribution of θ and α is proper if $n - m_2 - i - j + 2 > 0$ and $m_2 + j - 1 > 0$.*

PROOF. One can integrate the joint posterior given in (3.2) with respect to θ and α as follows:

$$\begin{aligned}&\int_0^\infty\int_0^1 \pi(\theta,\alpha|\mathrm{y})d\alpha d\theta \\ &= \int_0^\infty \theta^{n-i}\exp\{-\theta(m_2\tau+v+\frac{l}{2\tau})\}\{1-\exp(-\theta\tau)\}^{k/2} \\ &\quad\times\int_0^1 \alpha^{-m_2-j}\exp\{-\frac{\theta}{\alpha}w\}d\alpha d\theta.\end{aligned}$$

Since, if $m_2 + j - 1 > 0$, then

$$\int_0^1 \alpha^{-m_2-j}\exp\{-\frac{\theta}{\alpha}w\}d\alpha = \frac{\Gamma(m_2+j-1)}{(\theta w)^{m_2+j-1}},$$

the integration of joint posterior is reduced to

$$\frac{\Gamma(m_2+j-1)}{w^{m_2+j-1}}\int_0^\infty \theta^{n-i-m_2-j+1}\exp\{-\theta(m_2\tau+v+\frac{l}{2\tau})\}\{1-\exp(-\theta\tau)\}^{k/2}d\theta.$$

By the fact that $0 < \{1-\exp(-\theta\tau)\}^{k/2} < 1$, the integration is bounded by

$$\frac{\Gamma(m_2+j-1)}{w^{m_2+j-1}}\int_0^\infty \theta^{n-i-m_2-j+1}\exp\{-\theta(m_2\tau+v+\frac{l}{2\tau})\}d\theta.$$

The above quantity is always finite if $n-i-m_2-j+2>0$. This completes the proof. □

Note that the conditions $n-m_2>i+j-2$ and $m_2+j-1>0$ are not a rigid condition. Since in our model, $i+j=2$ and $j=1,2$ the conditions reduces to $n-m_2=m_1>0$ and $m_2>0$. This means that the posterior is finite if one observes at least one observation in normal condition and in accelerated condition.

Next we deals with the marginal posterior, distribution function and Bayes estimate of θ and α. Under the prior (3.1), if $n-m_2>i+j-1$ and $k,l\geq 0$, then we have the following results.

The marginal posteriors of θ and α are given by

$$\begin{aligned}\pi(\theta|\mathrm{y}) &= \theta^{n-i-m_2-j+1}\exp\{-\theta(m_2\tau+v+\tau(\frac{l}{2}))\}\\ &\quad\times\frac{(1-\exp\{-\theta\tau\})^{k/2}}{C_1},\end{aligned}\tag{3.3}$$

where

$$C_1=\int_0^\infty \theta^{n-i-m_2-j+1}\exp\{-\theta(m_2\tau+v+\tau(\frac{l}{2}))\}\times\{1-\exp(-\theta\tau)\}^{k/2}d\theta$$

and

$$\begin{aligned}\pi(\alpha|\mathrm{y}) &= w^{m_2+j-1}\alpha^{-m_2-j}\\ &\quad\times\int_0^\infty \theta^{n-i}\exp\{-\theta(m_2\tau+v+\tau(\frac{l}{2}))-\frac{\theta}{\alpha}w\}\\ &\quad\times\frac{\{1-\exp(-\theta\tau)\}^{k/2}d\theta}{[\Gamma(m_2+j-1)C_1]},\end{aligned}\tag{3.4}$$

respectively.

The marginal posterior distribution functions of θ and α are given by

$$\begin{aligned}F_\theta(\theta|\mathrm{y}) &= \theta^{n-i-m_2-j+2}\\ &\quad\times\int_0^\infty y^{n-i-m_2-j+1}(1+y)^{-(n-i-m_2-j+3)}\\ &\quad\times\exp\{-\theta\frac{y}{1+y}(m_2\tau+v+\tau(\frac{l}{2}))\}\\ &\quad\times\frac{\{1-\exp(-\theta\frac{y}{1+y}\tau)\}^{k/2}dy}{C_1}\end{aligned}\tag{3.5}$$

and

$$F_\alpha(\alpha|\mathrm{y}) = \int_0^\infty \theta^{n-i-m_2-j+1} \exp\{-\theta(m_2\tau + v + \tau(\frac{l}{2}))\} \times\{1-\exp(-\theta\tau)\}^{k/2} \times \left[1 - IG\left(\frac{\theta w}{\alpha}, m_2+j-1\right)\right]\frac{d\theta}{C_1}, \quad (3.6)$$

where $IG(a,b) = \int_a^\infty \{1/\Gamma(b)\} s^{b-1}e^{-s}ds.$ respectively.

The Bayes estimates of θ and α under quadratic loss function are given by

$$\widetilde{\theta} = \int_0^\infty \theta^{n-i-m_2-j+2} \exp\{-\theta(m_2\tau + v + \tau(\frac{l}{2}))\} \times\{1-\exp(-\theta\tau)\}^{k/2}\frac{d\theta}{C_1} \quad (3.7)$$

and

$$\widetilde{\alpha} = \frac{w}{m_2+j-2} \times \int_0^\infty \theta^{n-i-m_2-j+2} \exp\{-\theta(m_2\tau + v + \tau(\frac{l}{2}))\} \times\{1-\exp(-\theta\tau)\}^{k/2}\frac{d\theta}{C_1}, \quad (3.8)$$

respectively.

4. NUMERICAL RESULTS

In this Section, we will show some simulation results and numerical example based on artificial data set.

4.1. Simulation studies

Although probability matching can be justified only asymptotically, our simulation results might indicate that this is indeed achieved for small or moderate sample sizes as well. So we compute the frequentist coverage probabilities for the priors, (2.14), (2.9) and (2.10) when n is small and moderate.

Let $\theta^\gamma(\pi;\mathbf{Y})$ be the posterior γ-quantile of θ given $\mathbf{Y}$ under the prior π. So, $(0, \theta^\gamma(\pi;\mathbf{Y}))$ is the one-sided γ posterior confidence interval. Let $Q_{(\theta,\alpha)}(\gamma;\theta)$ be a frequentist coverage probability of this posterior confidence interval

$$Q_{(\theta,\alpha)}(\gamma;\theta) = P\{0 < \theta \le \theta^\gamma(\pi;\mathbf{Y})\} = \gamma.$$

Similarly, we can define $\alpha^\gamma(\pi; \mathbf{Y})$ and $Q_{(\theta,\alpha)}(\gamma; \alpha)$ to be the posterior γ quantile of α and the corresponding frequentist coverage probability, respectively. In Table 4.1, the estimated $Q_{(\theta,\alpha)}(\gamma; \theta)$ and $Q_{(\theta,\alpha)}(\gamma; \alpha)$ are shown, when $\gamma = 0.05(0.95)$. To obtain the table, we generate 10,000 independent random samples for fixed θ, τ and α from *pdf* (2.3). Note that under the prior π and given $\mathbf{Y}$, the event $\theta \le \theta^\gamma(\pi; \mathbf{Y})$ is equivalent to the event $F_\theta(\theta^\gamma(\pi; \mathbf{Y})|\mathbf{Y}) \le \gamma$. So, we calculate the relative frequency of $F_\theta(\theta^\gamma(\pi; \mathbf{Y})|\mathbf{Y}) \le \gamma$.

From the Table 4.1, we can find the fact that

1. the matching prior $\pi_{(0,2,1,0)}$ and reference prior $\pi_{(1,1,1,0)}$ achieve the target coverage probability for θ relatively well,

2. as we expected for parameter α, the three proposed priors can not achieve the coverage probability. The reference prior $\pi_{(1,1,1,0)}$ meets the coverage probability better than any other priors.

Conclusively, we recommend the reference prior $\pi_{(1,1,1,0)}$ for this step stress ALTs model.

TABLE 4.1 *Frequentist coverage probabilities for θ and α*

	sample size	$\pi_{(0,2,1,0)}$		$\pi_{(1,1,1,1)}$		$\pi_{(1,1,1,0)}$	
	n	0.05	0.95	0.05	0.95	0.05	0.95
θ	5	0.0250	1.0000	0.0035	0.9904	0.0250	1.0000
	10	0.0549	0.9420	0.0355	0.9404	0.0549	0.9420
	15	0.0521	0.9442	0.0392	0.9390	0.0521	0.9442
	20	0.0507	0.9451	0.0402	0.9375	0.0507	0.9451
	25	0.0538	0.9515	0.0411	0.9436	0.0538	0.9515
	30	0.0533	0.9517	0.0424	0.9442	0.0533	0.9517
α	5	0.0093	0.8529	0.0226	0.9306	0.0347	0.9453
	10	0.0178	0.8796	0.0309	0.9204	0.0416	0.9326
	15	0.0209	0.8941	0.0333	0.9219	0.0409	0.9332
	20	0.0249	0.9092	0.0362	0.9303	0.0425	0.9379
	25	0.0252	0.9180	0.0357	0.9371	0.0420	0.9452
	30	0.0282	0.9143	0.0379	0.9317	0.0448	0.9385

4.2. Example

The following artificial data of size 15 are random sample from exponential distribution with $\theta = 2$, $\alpha = 0.5$ and tampering coefficient $\tau = -\log(0.5)/\theta$. τ is the point which a half of 15 sample is tampered and a half is not. The values are

0.553* 0.022 0.093 0.288 0.392* 0.852* 0.512* 0.725*
0.955* 0.227 0.869* 0.373* 0.401* 0.140 0.113

and the starred observations are tampered observation.

From the above sample, various point estimates of θ and α are calculated in Table 4.2. The marginal posterior probability density functions of θ and α are described in Figure 4.1 and Figure 4.2, respectively.

TABLE 4.2 *Point estimates of parameters from artificial data*

	MLE	*Bayes estimator*	
		$\pi_{(1,1,1,1)}$	$\pi_{(1,1,1,0)}$
θ	1.498	1.527	1.592
α	0.419	0.480	0.501

In this example, the Bayes estimate under reference prior $\pi_{(1,1,1,0)}$ gives better estimate than ML and Bayes estimate under Jeffreys' prior $\pi_{(1,1,1,1)}$.

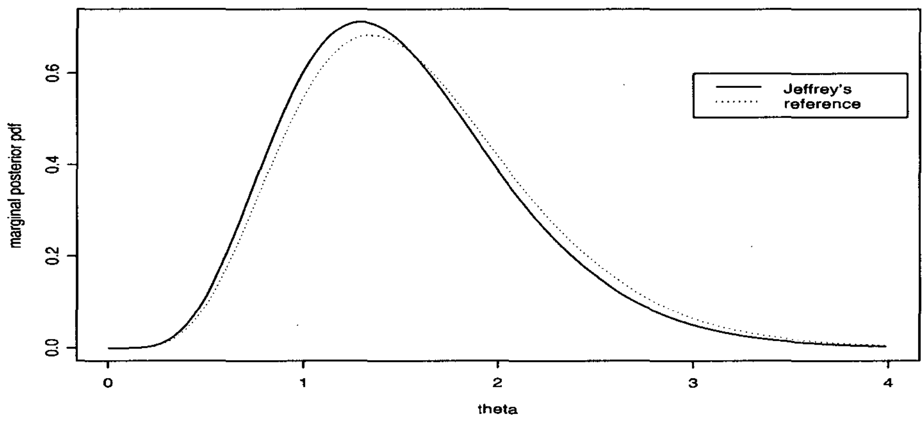

FIGURE 4.1 *Marginal posterior distribution of θ.*

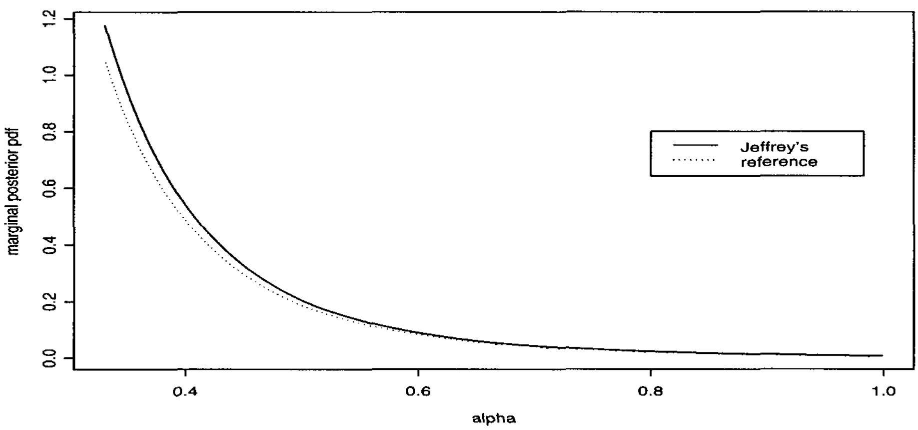

FIGURE 4.2 *Marginal posterior distribution of α.*

REFERENCES

BERGER, J. O. AND BERNARDO, J. M. (1989). "Estimating a product of means: Bayesian analysis with reference priors", *Journal of the American Statistical Association*, **84**, 200–207.

BERGER, J. O. AND BERNARDO, J. M. (1992). "On the development of reference priors", *Bayesian Statistics 4* (J. M. Bernardo, *et al.* eds.), 35–60, Oxford University Press, New York.

BERNARDO, J. M. (1979). "Reference posterior distributions for Bayesian inference (with discussion)", *Journal of the Royal Statistical Society*, Ser. B, **41**, 113–147.

COX, D. R. AND REID, N. (1987). "Parameter orthogonality and approximate conditional inference (with discussion)", *Journal of the Royal Statistical Society*, Ser. B, **49**, 1–39.

DATTA, G. S. AND GHOSH, J. K. (1995a). "On priors providing frequentist validity for Bayesian Inference", *Biometrika*, **82**, 37–45.

DATTA, G. S. AND GHOSH, M. (1995b). "Some remarks on noninformative priors", *Journal of the American Statistical Association*, **90**, 1357–1363.

DATTA, G. S. AND GHOSH, M.(1996). "On the invariance of noninformative priors", *The Annal of Statistics*, **24**, 141–159.

DEGROOT, M. H. AND GOEL, P. K. (1979). "Bayesian estimation and optimal designs in partially accelerated life-testing", *Naval Research Logistics Quarterly*, **26**, 223–235.

JEFFREYS, H. (1961). *Theory of probability*, Oxford University Press, Oxford.

MUKERJEE, R. AND DEY, D. K. (1993). "Frequentist validity of posterior quantiles in the presence of a nuisance parameter: higher order asymptotics", *Biometrika*, **80**, 499–505.

MUKERJEE, R. AND GHOSH, M. (1997). "Second-order probability matching priors", *Biometrika*, **84**, 970–975.

PATHAK, P. K., SINGH, A. K. AND ZIMMER, W. J. (1987). "Empirical Bayesian estimation of mean life from an accelerated life test", *Journal of Statistical Planning and Inference*, **16**, 353–363.

PEERS, H. W. (1965). "On confidence points and Bayesian probability points in the case of several parameters", *Journal of the Royal Statistical Society*, Ser. B, **27**, 9–16.

STEIN, C. M. (1985). "On the coverage probability of confidence sets based on a prior distribution", *Sequential methods in statistics*, 485–514, Banach Center, Warsaw.

TIBSHIRANI, R. (1989). "Noninformative priors for one parameter of many", *Biometrika*, **76**, 604–608.

WELCH, B. L. AND PEERS, H. W. (1963). "On formulae for confidence points based on integrals of weighted likelihoods", *Journal of the Royal Statistical Society*, Ser. B, **25**, 318–329.