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ON AN IMPROVED UNIFIED CONVERGENCE ANALYSIS FOR A CERTAIN CLASS OF EULER–HALLEY TYPE METHODS

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ABSTRACT. Using more precise majorizing sequences than before [6], [10], [11], [14] we provide a finer semilocal convergence analysis for a certain class of Euler–Halley type methods for approximating a solution of an equation in a Banach space setting.

1. INTRODUCTION

In this study we are concerned with the problem of approximating a locally unique solution x^* of a nonlinear equation

$$F(x) = 0 \tag{1.1}$$

where F is a twice-Fréchet differentiable operator defined on a convex subset D of a Banach space X with values in a Banach space Y.

Let $\lambda \in [0, 2]$ be a given parameter. We use the class of Euler–Halley type approximations given by

$$x_{n+1} = T_{F,\lambda}(x_n) = x_n + G_F(x_n) + H_{F,\lambda}(x_n) \quad (x_0 \in D), \ (n \geq 0), \tag{1.2}$$

where

$$G_F(x) = -F'(x)^{-1}F(x)$$
$$Q_{F,\lambda}(x) = \left[I + \frac{\lambda}{2}F'(x)^{-1}F''(x)G_F(x)\right]^{-1},$$

and

$$H_{F,\lambda}(x) = -\frac{1}{2}F'(x)^{-1}F''(x)G_F(x)Q_{F,\lambda}(x)G_F(x)$$

for all $x \in D$.

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This class of approximations includes as special cases many cubically convergent methods: If $\lambda = 0$ we obtain the Euler method [1], [2], [9], [10]; if $\lambda = 1$ we get the Halley method [2], [8], [10] whereas for $\lambda = 2$ method (1.2) reduces to the super-Haley method [7], [10]. A convergence analysis of method (1.2) has been given by us in [6], Gutierrez and Hernandez in [10], Han in [11] and more recently by Wang and Li [14] under various hypotheses that have certain advantages over each other.

In particular in the elegant paper by Wang and Li a unified convergence analysis was provided for method (1.2) using the concept of the average function. This idea has already been used in [12], [13] and in an improved way in [3]–[5] on Newton's method.

Here we show that using more precise majorizing sequences than in [14] and under the same computational cost and the same or weaker hypotheses we can provide finer error bounds on the distances $\|x_{n+1} - x_n\|$, $\|x_n - x^*\|$ $(n \geq 0)$ and a more precise information on the location of the solution x^*.

2. SEMILOCAL CONVERGENCE ANALYSIS OF METHOD (1.2)

Let $x_0 \in D$ such that $F'(x_0)^{-1} \in L(Y, X)$. Let $R > 0$. We denote by $U(x_0, R)$ the ball $U(x_0, R) = \{x \in X \mid \|x - x_0\| < R\}$, whereas $\overline{U}(x_0, R)$ the corresponding closed ball.

We need the definition [5], [12]:

Definition 2.1. An operator $F'(x_0)^{-1}F'(x)$ satisfies the center Lipschitz condition in $U(x_0, r)$ with L_0 average if

$$\|F'(x_0)^{-1}[F'(x) - F'(x_0)]\| \leq \int_0^{\|x-x_0\|} L_0(u)du \tag{2.1}$$

for some positive integrable function L_0 on $[0, R]$. Usually R is taken such that

$$\int_0^R (R - u)L_0(u)du = R. \tag{2.2}$$

Let $r_0^* > 0$ be such that

$$\int_0^{r_0^*} L_0(u)du = 1 \tag{2.3}$$

and define

$$b_0 = \int_0^{r_0^*} uL_0(u)du. \tag{2.4}$$

For $\beta \in [0, b_0]$, define

$$h_0(t) = \beta - t + \int_0^t (t-u)L_0(u)du, \quad t \in [0, R]. \tag{2.5}$$

We need the Lemma [5], [12]:

Lemma 2.2. *The function h_0 nonincreasing in $[0, r_0^*]$ and nondecreasing in $[r_0, R]$. Moreover, if*

$$\beta \leq b_0 \tag{2.6}$$

$h_0(\beta) > 0$, $h_0(r_0) = \beta - b_0 \leq b$, and $h_0(R) = \beta > 0$. That is, h_0 has a unique zero in $[0, r_0]$ denoted by r_1^* and a unique zero r_2^* in $[r_0, R]$, satisfying

$$\beta < r_1^* < \frac{r_0^*}{b_0}\beta < r_0^* < r_2^* < R \tag{2.7}$$

when $\beta < b_0$ and $r_1^* = r_2^*$ when $\beta = b_0$.

We also need the following Lemmas:

Lemma 2.3 ([5]). *If operator $F'(x_0)^{-1}F$ satisfies (2.1) in $U(x_0, r)$ with L_0 average and $r > r_0^*$. Then for each $x \in U(x_0, r_0^*)$, $F'(x)^{-1} \in L(Y, X)$ and*

$$\|F'(x)^{-1}F'(x_0)\| \leq \left[1 - \int_0^{\|x-x_0\|} L_0(u)du\right]^{-1} \tag{2.8}$$

Proof. In view of (2.1) we get

$$\|F'(x_0)^{-1}[F'(x) - F'(x_0)]\| \leq \int_0^{\|x-x_0\|} L_0(u)du < 1. \tag{2.9}$$

It follows by (2.9) and the Banach Lemma on invertible operators [2] that $F'(x)^{-1} \in L(Y, X)$ and (2.8) holds true.

That completes the proof of Lemma 2.3. □

Lemma 2.4 ([5], [12]). *Let $\beta = \|F'(x_0)^{-1}F(x_0)\| \leq b_0$. Assume that $r_1^* < r < r_2^*$ if $\beta < b$, or $r = r_1^*$ if $\beta = b_0$. If operator $F'(x_0)^{-1}F$ satisfies (2.1) in $U(x_0, r)$ with L_0 average, the equation $F(x) = 0$ has a unique solution*

$$x^* \in \overline{U}(x_0 - F'(x_0)^{-1}F(x_0), r_1^* - \beta) \subset \overline{U}(x_0, r_1^*). \tag{2.10}$$

Lemma 2.5. *Assume that*

$$\|F'(x_0)^{-1}F''(x_0)\| = L_0(0) \tag{2.11}$$

$$\|F'(x_0)^{-1}[F''(x) - F''(y)]\| \leq L(\|y - x_0\| + \|x - y\|) - L(\|y - x_0\|) \tag{2.12}$$

for all $y \in \overline{U}(x_0, r)$, $x \in \overline{U}(y, r - \|y - x_0\|)$, *and some positive integrable function* L *on* $[0, R]$. *Then, for each* $x \in U(x_0, r_0)$

$$\|F'(x_0)^{-1}F''(x)\| \le h''(\|x - x_0\|), \tag{2.13}$$

where,

$$h(t) = \beta - t + \int_0^t (t-u)L(u)du, \quad t \in [0, R] \tag{2.14}$$

and

$$\|F'(x)^{-1}F'(x_0)\| \le -\frac{1}{h_0'(\|x - x_0\|)}. \tag{2.15}$$

Proof. In view of (2.11) and (2.12) it follows

$$L_0(u) \le L(u), \quad u \in [0, R] \tag{2.16}$$

and $\frac{L(u)}{L_0}(u)$ can be arbitrarily large [1], [2]. Using (2.11), (2.12), (2.14) and (2.16) we obtain in turn

$$\begin{aligned} \|F'(x_0)^{-1}F''(x)\| &= \|F'(x_0)^{-1}[F''(x) - F''(x_0) + F''(x_0)]\| \\ &\le \|F'(x_0)^{-1}[F''(x) - F''(x_0)]\| + \|F'(x_0)^{-1}F''(x_0)\| \\ &\le L_0(0) + L(\|x - x_0\|) - L(0) \\ &\le L(0) + L(\|x - x_0\|) - L(0) \le h''(\|x - x_0\|), \end{aligned} \tag{2.17}$$

which shows (2.13).

Estimate (2.15) follows immediately from (2.5) and (2.8).

That completes the proof of Lemma 2.5. □

Remark 2.6. If equality holds in (2.16) then our results reduce to the corresponding ones in [14]. Otherwise they constitute an improvement. Indeed, let

$$b = \int_0^{r_0} uL(u)du \tag{2.18}$$

where r_0 is such that

$$\int_0^{r_0} L(u)du = 1. \tag{2.19}$$

If

$$\beta \le b \tag{2.20}$$

denote the corresponding zeros of h by r_1 and r_2. It then follows by (2.3)–(2.5), (2.14), (2.18) and (2.19) that under (2.6) and (2.20) the following hold true:

$$r_0 < r_0^* \tag{2.21}$$

$$r_1^* < r_1 \tag{2.22}$$

$$r_2 < r_2^* \tag{2.23}$$

and

$$b < b_0. \tag{2.24}$$

That is we obtain a better information on the location of the solution and wider upper bound on $\|F'(x_0)^{-1}F(x_0)\|$.

Let $\{t_n\}$ be the majorizing sequence for $\{x_n\}$ given by

$$t_{n+1} = T_{h,\lambda}(t_n) \quad (n \geq 0). \tag{2.25}$$

Define scalar sequence $\{s_n\}$ $(n \geq 0)$ as $\{t_n\}$ but with $h(t_n)^{-1}$ replaced by $h_0^{-1}(t_n)$ $(n \geq 0)$. If equality holds in (2.16) then $s_n = t_n$ $(n \geq 0)$. Otherwise it can easily be seen using induction on $n \geq 0$

$$s_n < t_n \tag{2.26}$$

$$s_{n+1} - s_n < t_{n+1} - t_n \tag{2.27}$$

$$s^* = \lim_{n\to\infty} s_n \leq r_1 = \lim_{n\to\infty} t_n \tag{2.28}$$

and

$$s^* - s_n \leq r_1 - t_n. \tag{2.29}$$

Note that under (2.20) scalar sequence $\{t_n\}$ is nondecreasing and converges to r_1 (see Lemma 2.2 in [14]). Therefore under the same condition (2.20) scalar sequence $\{s_n\}$ is also nondecreasing and converges to s^*.

Let us also define scalar sequence

$$v_{n+1} = T_{h_0,\lambda}(v_n) \quad (n \geq 0). \tag{2.30}$$

It follows that under (2.6) $\{v_n\}$ is nondecreasing and converges to r_1^*. By comparing sequences $\{v_n\}$ and $\{s_n\}$ we deduce

$$v_n \leq s_n \tag{2.31}$$

$$v_{n+1} - v_n \leq s_{n+1} - s_n \tag{2.32}$$

$$r_1^* \leq s^* \tag{2.33}$$

and

$$r_1^* - v_n \leq s^* - s_n, \tag{2.34}$$

although $\{v_n\}$ is not a majorizing sequence for $\{x_n\}$.

We can now state the main unifying result for method (1.2):

Theorem 2.7. *Assume for $r > r_1^*$ conditions (2.11), (2.12) hold true on $\overline{U}(x_0, r)$. If*

$$\beta \leq b_0 \tag{2.35}$$

then sequence $\{x_n\}$ generated by method (1.2) is well defined, for all $\lambda \in [0, 2]$, remains in $U = \overline{U}(x_0 - F'(x_0)^{-1}F(x_0), r_1^* - \beta)$ and converges to a solution $x^* \in U$.

Moreover, for $r \in [r_1^*, r_2^*]$ if $\beta < b_0$ and $r = r_1^*$ if $\beta = b_0$, the equation $F(x) = 0$ has a unique solution in $\overline{U}(x_0, r)$. Furthermore the following estimates hold true for all $n \geq 0$:

$$\|x_{n+1} - x_n\| \leq s_{n+1} - s_n \tag{2.36}$$

and

$$\|x_n - x^*\| \leq s^* - s_n. \tag{2.37}$$

Proof. It follows exactly as Theorem 3.1 in [14] with the only crucial difference that we are using sharper (2.15) instead of

$$\|F'(x)^{-1}F'(x_0)\| \leq -\frac{1}{h'(\|x - x_0\|)} \tag{2.38}$$

used in [14] and majorizing sequence $\{s_n\}$ instead of $\{t_n\}$.

That completes the proof of Theorem 2.7. □

Remark 2.8. Another way of improving the results in [14] is to consider the approximation (see Lemma 3.1 in [14] or [6] or [10]):

$$\begin{aligned} F(x_{n+1}) &= \frac{1}{2}F''(x_n)\{(2 - \lambda)G_F(x_n) + H_{F,\lambda}(x_n)\}H_{F,\lambda}(x_n) \\ &+ \int_0^1 \{F''(x_n + \theta(x_{n+1} - x_n)) - F''(x_n)\}(1 - \theta)d\theta(x_{n+1} - x_n)^2 \end{aligned} \tag{2.39}$$

and the corresponding iteration $\{w_n\}$ given by

$$w_0 = 0,\ w_1 - w_0 = \|x_1 - x_0\|,\ w_{n+2} = w_{n+1} + \overline{T}_{h_0,h,\lambda}(w_{n+1}), \tag{2.40}$$

where,

$$\overline{T}_{h_0,h,\lambda}(w_{n+1}) = \frac{\varepsilon_{n+1}}{-h_0'(w_{n+1})}, \tag{2.41}$$

and

$$\varepsilon_{n+1} = \frac{1}{2}h''(w_{n+1})\{(2-\lambda)G_h(w_{n+1}) + H_{h,\lambda}(w_{n+1})\}H_{h,\lambda}(w_{n+1})$$
$$+ \int_0^1 [L(w_n + \theta(w_{n+1} - w_n)) - L(w_n)](1-\theta)d\theta(w_{n+1} - w_n)^2. \tag{2.42}$$

It follows from the proof of the theorem that $\{w_n\}$ is the finer majorizing sequence for $\{x_n\}$ such that

$$w_n \le s_n \tag{2.43}$$
$$w_{n+1} - w_n \le s_{n+1} - s_n \tag{2.44}$$
$$w^* = \lim_{n\to\infty} w_n \le s^* \tag{2.45}$$

and

$$w^* - w_n \le s^* - s_n. \tag{2.46}$$

Therefore if one finds sufficient convergence conditions for sequence $\{w_n\}$ weaker than the ones given in our Theorem 2.7 or Theorem 3.1 in [14] then the results obtained here or in [14] can be improved even further. We note that sufficient convergence conditions for scalar sequences more general than (2.40) have already been given by us in [1], [2]. Therefore we do not write down explicitly those conditions. Instead we refer the motivated reader to [1], [2].

Remark 2.9. In view of (2.11) there exists a positive integrable function L_1 on $[0, R]$ such that

$$\|F'(x_0)^{-1}(F''(x) - F''(x_0))\| \le L_1(\|x - x_0\|) - L_1(0) \tag{2.47}$$

for all $x \in \overline{U}(x_0, r)$.

Clearly

$$L_1(u) \le L(u) \tag{2.48}$$

holds and $\frac{L(u)}{L_1(u)}$ can be arbitrarily large [1], [2]. If L_1 replaces L_0 in condition (2.5), (2.11) then if we denote the new h function by h_1 we again get

$$\|F'(x)^{-1}F'(x_0)\| \le -\frac{1}{h_1'(\|x - x_0\|)}. \tag{2.49}$$

Indeed by Taylor's formula,

$$F'(x) = F'(x_0) + F''(x_0)(x - x_0) + \int_0^1 \left[F''(x_0 + t(x - x_0)) - F''(x_0)\right] dt(x - x_0) \tag{2.50}$$

which leads to the estimate

$$\begin{aligned}\|F'(x_0)^{-1}(F'(x) - F'(x_0))\| &\leq L_1(0)\|x - x_0\| \\ &\quad + \int_0^1 \|F''(x_0 + t(x - x_0)) - F''(x_0)\| \, \|x - x_0\| dt \\ &\leq \int_0^{\|x - x_0\|} L_1(u) du < 1. \end{aligned} \tag{2.51}$$

It follows from (2.51) and the Banach Lemma on invertible operators that $F'(x)^{-1}$ exists so that (2.49) holds true. That is all the results obtained here hold with L_1 replacing L_0. Note however that although L_1 connects better than L_0 to L it follows from (2.1) and (2.51) that L_1 may be chosen at least as small as L_0.

We now complete this study with an example to show that (2.16) and (2.48) can hold as strict inequalities:

Example 2.10. Let $X = Y = \mathbf{R}$, $x_0 = 0$, $D = [-1, 1]$ and define function F on D by

$$F(x) = e^x - 2x + c \tag{2.52}$$

for some constant c. It can easily be seen that (2.1), (2.12), and (2.47) hold for

$$L_0(u) = L_1(u) = (e - 1)u, \quad \text{and} \quad L(u) = eu. \tag{2.53}$$

Moreover in view of (2.53) we get

$$L_0(u) = L_1(u) < L(u) \quad \text{for all } u \in [0, R]. \tag{2.54}$$

Other choices for the "L" functions can be found in [3]–[7], [10], [14] and the references there.

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