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INTUITIONISTIC FUZZY WEAK CONGRUENCE ON A NEAR-RING MODULE

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ABSTRACT. We introduce the concepts of intuitionistic fuzzy submodules and intuitionistic fuzzy weak congruences on an R-module (Near-ring module). And we obtain the correspondence between intuitionistic fuzzy weak congruences and intuitionistic fuzzy submodules of an R-module. Also, we define intuitionistic fuzzy quotient R-module of an R-module over an intuitionistic fuzzy submodule and obtain the correspondence between intuitionistic fuzzy weak congruences on an R-module and intuitionistic fuzzy weak congruences on intuitionistic fuzzy quotient R-module over an intuitionistic fuzzy submodule of an R-module.

0. INTRODUCTION

The concept of fuzzy set was formulated by Zadeh [26]. Since then, there has been a remarkable growth of fuzzy set theory. The notion of fuzzy relation on a set was defined by Zadeh [27]. Some researchers [9, 20, 21, 23-25] applied the concept of fuzzy sets to congruence theory. In particular, Dutta and Biswas [9] investigated fuzzy congruences on a near-ring module.

In 1986, Atanassov [1] introduced the notion of intuitionistic fuzzy sets as the generalization of fuzzy sets. After that time, Çoker and his colleagues [6, 7, 10], Lee and Lee [22], and Hur and his colleagues [15] applied the notion of intuitionistic fuzzy sets to topology. Also, several researchers [2, 4, 12-14, 16] applied the to algebra. In particular, Bustince and Burillo [5], and Deschrijver and Kerre [8] applied the concept of intuitionistic fuzzy sets to relation. Also, Hur and his colleagues [17]

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investigated several properties of intuitionistic fuzzy equivalence relations. Moreover, Hur and his colleagues [18, 19] introduced the notion of intuitionistic fuzzy congruences on a lattice and on a semigroup, and studied some of their properties.

In this paper, we introduce the concepts of intuitionistic fuzzy submodules and intuitionistic fuzzy weak congruences on an R-module (Near-ring module). And we obtain the correspondence between intuitionistic fuzzy weak congruences and intuitionistic fuzzy submodules of an R-module. Also, we define intuitionistic fuzzy quotient R-module of on R-module over an intuitionistic fuzzy submodule and obtain the correspondence between intuitionistic fuzzy weak congruences on an R-module and intuitionistic fuzzy weak congruences on intuitionistic fuzzy quotient R-module over an intuitionistic fuzzy submodule of an R-module.

1. PRELIMINARIES

We recall some definitions and results that are used in this paper.

For sets X, Y and $Z, f = (f_1, f_2) : X \to Y \times Z$ is called a *complex mapping* if $f_1 : X \to Y$ and $f_2 : X \to Z$ are mappings.

Throughout this paper, we will denote the unit interval $[0, 1]$ as I and for any ordinary relation R on a set X, we will denote the characteristic mapping of R as χ_R.

Definition 1.1 ([1, 6]). Let X be a nonempty set. A complex mapping $A = (\mu_A,\ \nu_A) : X \to I \times I$ is called an *intuitionistic fuzzy set* (in short, *IFS*) in X if for each $x \in X$ $\mu_A(x) + \nu_A(x) \leq 1$, where the mappings $\mu_A : X \to I$ and $\nu_A : X \to I$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\nu_A(x)$) of each $x \in X$ to A, respectively. In particular, $0_\sim$ and $1_\sim$ denote the *intuitionistic fuzzy empty set* and the *intuitionistic fuzzy whole set* in X defined by $0_\sim(x) = (0, 1)$ and $1_\sim(x) = (1, 0)$ for each $x \in X$, respectively.

We will denote the set of all IFSs in X as IFS(X).

Definitions 1.2 ([1]). Let X be a nonempty set and let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be IFSs in X. Then

(1) $A \subset B$ iff $\mu_A \leq \mu_B$ and $\nu_A \geq \nu_B$.

(2) $A = B$ iff $A \subset B$ and $B \subset A$.

(3) $A^c = (\nu_A, \mu_A)$.

(4) $A \cap B = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B)$.
(5) $A \cup B = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B)$.
(6) $[\,]A = (\mu_A, 1 - \mu_A)$, $< > A = (1 - \nu_A, \nu_A)$.

Definition 1.3 ([6]). Let $\{A_i\}_{i \in J}$ be an arbitrary family of IFSs in X, where $A_i = (\mu_{A_i}, \nu_{A_i})$ for each $i \in J$. Then
(a) $\bigcap A_i = (\bigwedge \mu_{A_i}, \bigvee \nu_{A_i})$.
(b) $\bigcup A_i = (\bigvee \mu_{A_i}, \bigwedge \nu_{A_i})$.

Definition 1.4 ([6]). Let A be an IFS in a set X and let $\lambda, \mu \in I$ with $\lambda + \mu \leq 1$. Then the set $A^{(\lambda,\mu)} = \{x \in X : \mu_A(x) \geq \lambda$ and $\nu_A(x) \leq \mu\}$ is called a (λ,μ)*-level subset* of A.

Result 1.A ([14, Proposition 2.2]). Let A be an IFS in a set X and let

$$(\lambda_1, \mu_1), (\lambda_2, \mu_2) \in ImA.$$

If $\lambda_1 \leq \lambda_2$ and $\mu_1 \geq \mu_2$, then $A^{(\lambda_2,\mu_2)} \subset A^{(\lambda_1,\mu_1)}$.

Definition 1.5 ([13]). Let G be a group and let $A \in$ IFS(G). Then A is called an *intuitionistic fuzzy subgroup* (in short, IFG) of G if it satisfies the following conditions :

(i) $\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(xy) \leq \nu_A(x) \vee \nu_A(y)$ for each $x, y \in G$.
(ii) $\mu_A(x^{-1}) \geq \mu_A(x)$ and $\nu_A(x^{-1}) \leq \nu_A(x)$ for each $x \in G$.

We will denote the set of all IFGs of G as IFG(G).

Result 1.B ([13, Proposition 2.6]). *Let A be an IFG of a group G. Then $A(x^{-1}) = A(x)$ and $\mu_A(x) \leq \mu_A(e), \nu_A(x) \geq \nu_A(e)$ for each $x \in G$, where e is the identity element of G.*

Result 1.C ([13, Proposition 2.17 and Proposition 2.18]). *Let A be an IFS of a group G. Then $A \in IFG(G)$ if and only if for each $(\lambda, \mu) \in I \times I$ with $(\lambda, \mu) \leq A(e)$, i.e., $\lambda \leq \mu_A(e)$ and $\mu \geq \nu_A(e)$, $A^{(\lambda,\mu)}$ is a subgroup of G.*

Definition 1.6. An intuitionistic fuzzy nonempty set A in an additive group G is called an *intuitionistic fuzzy normal subgroup* (in short, $IFNG$) of G if it is satisfies the following conditions : for any $x, y \in G$,
(i) $\mu_A(x+y) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(x+y) \leq \nu_A(x) \vee \nu_A(y)$,
(ii) $\mu_A(-x) \geq \mu_A(x)$ and $\nu_A(-x) \leq \nu_A(x)$,
(iii) $A(y + x - y) = A(x)$.

We will denote the set of all IFNG_S of G as IFNG(G).

Result 1.D ([14, Proposition 2.13 and Proposition 2.18]). *Let G be a group and let $A \in IFS(G)$. Then $A \in IFNG(G)$ if and only if $A^{(\lambda,\mu)}$ is a normal subgroup of G for each $(\lambda,\mu) \in ImA$.*

Definition 1.7 ([16]). Let A be an IFNG of an additive group G and let $x \in G$. Then the intuitionistic fuzzy set $x + A$ of G defined by $(x + A)(y) = A(y - x)$ for each $y \in G$, is called the *intuitionistic fuzzy coset* of A determined by x.

Definition 1.8 ([5, 8]). Let X be a set. Then a complex mapping $R = (\mu_R,\ \nu_R) : X \times X \rightarrow I \times I$ is called an *intuitionistic fuzzy relation* (in short, *IFR*) on X if $\mu_R(x, y) + \nu_R(x, y) \leq 1$ for each $(x, y) \in X \times X$, i.e., $R \in$ IFS $(X \times X)$.

We will denote the set of all IFRs on a set X as IFR(X).

Definition 1.9 ([8]). Let X be a set and let $R, Q \in$ IFR(X). Then the *composition* of R and Q, $Q \circ R$, is defined as follows : for any $x, y \in X$,

$$\mu_{Q\circ R}(x,y) = \bigvee_{z\in X} [\mu_R(x,z) \wedge \mu_Q(z,y)]$$

and

$$\nu_{Q\circ R}(x,y) = \bigwedge_{z\in X} [\nu_R(x,z) \vee \nu_Q(z,y)].$$

Definition 1.10 ([5, 8]). An Intutionistic fuzzy Relation R on a set X is called an *intutionisitic fuzzy equivalence relation* (in short, IFER) on X if it satisfies the following conditions :

(i) it is *intutionisitic fuzzy reflexive*, i.e.,$R(x, x) = (1, 0)$ for each $x \in X$.
(ii) it is *intutionisitic fuzzy symmetric*, i.e.,$R(x, y) = R(y, x)$ for any $x, y \in X$.
(iii) it is *intutionisitic fuzzy transitive*, i.e., $R \circ R \subset R$.

We will denote the set of all IFERs on X as IFE(X).

Let R be an intuitionistic fuzzy equivalence relation on a set X and let $a \in X$. We define a complex mapping $Ra : X \rightarrow I \times I$ as follows : for each $x \in X$

$$Ra(x) = R(a, x).$$

Then clearly $Ra \in$ IFS(X). The intuitionistic fuzzy set Ra in X is called an *intuitionistic fuzzy equivalence class* of R containing $a \in X$. The set $\{Ra : a \in X\}$ is called the *intuitionistic fuzzy quotient set of X by R* and denoted by X/R.

Definition 1.11 ([7]). An IFR R on a groupoid S is said to be:

(1) *intuitionistic fuzzy left compatible* if $\mu_R(x,y) \leq \mu_R(zx,zy)$ and $\nu_R(x,y) \geq \nu_R(zx,zy)$, for any $x,y,z \in S$.

(2) *intuitionistic fuzzy right compatible* if $\mu_R(x,y) \leq \mu_R(xz,yz)$ and $\nu_R(x,y) \geq \nu_R(xz,yz)$, for any $x,y,z \in S$.

(3) *intuitionistic fuzzy compatible* if $\mu_R(x,y) \wedge \mu_R(z,t) \leq \mu_R(xz,yt)$ and $\nu_R(x,y) \vee \nu_R(z,t) \geq \nu_R(xz,yt)$, for any $x,y,z,t \in S$.

Definition 1.12 ([17]). An IFER R on a groupoid S is called an:

(1) *intuitionistic fuzzy left congruence* (in short, $IFLC$) if it is intuitionistic fuzzy left compatible.

(2) *intuitionistic fuzzy right congruence* (in short, $IFRC$) if it is intuitionistic fuzzy right compatible.

(3) *intuitionistic fuzzy congruence* (in short, IFC) if it is intuitionistic fuzzy compatible.

We will denote the set of all IFCs [resp. IFLCs and IFRCs] on a groupoid S as IFC(S) [resp. IFLC(S) and IFRC(S)].

Let R be an intuitionistic fuzzy congruence on a semigroup S and let $a \in S$. The intuitionistic fuzzy set Ra in S is called an *intuitionistic fuzzy congruence class of R containing $a \in S$* and we will denote the set of all intuitionistic fuzzy congruence classes of R as S/R.

2. Intuitionistic Fuzzy Submodule

Definition 2.1 ([9]). *A near-ring R* is a system with two binary operations, addition and multiplication, such that :

(i) $(R,+)$ is a group.

(ii) $(R,\cdot)$ is a semigroup.

(iii) $x(y+z) = xy + xz$ for any $x,y,z \in R$.

Definition 2.2 ([9]). An *R - module* (i.e, *near-ring module*) M is a system consisting of an additive group M, a near-ring R and a mapping $\cdot : M \times R \to M$ such that :

(i) $m(x+y) = mx + my$, for each $m \in M$ and any $x,y \in R$.

(ii) $m(xy) = (mx)y$, for each $m \in M$ and any $x,y \in R$.

Definition 2.3 ([9]). Let M and M' be any two R- modules. Then a mapping $f : M \to M'$ is called an R- *homomorphism* if it satisfies the following conditions :

(i) $f(m_1 + m_2) = f(m_1) + f(m_2)$, for any $m_1, m_2 \in M$.

(ii) $f(mr) = f(m)r$, for each $m \in M$ and each $r \in R$.

The submodules of an R- module M are defined to be the kernels of R-homomorphisms.

Result 2.A ([3]). *An additive normal subgroup B of an R-module M is a submodule if and only if $(m + b)r - mr \in B$ for each $m \in M, b \in B$ and $r \in R$.*

Definition 2.4 ([9]). A relation P on an R-module M is called a *congruence* on M if it satisfies the following conditions :

(i) It is an equivalence relation on M.

(ii) If $(a, b) \in P$ and $(c, d) \in P$, then $(a + c, b + d) \in P$ and $(ar, br) \in P$ for any $a, b, c, d \in M$ and each $r \in R$.

Definition 2.5. Let A be an intuitionistic fuzzy nonempty set in an R-module M. Then A is said to be an *intuitionistic fuzzy submodule* (in short, $IFSM$) of M if it is satisfies the following conditions :

(i) $A \in$ IFNG(M).

(ii) $\mu_A\{(x + y)r - xr\} \geq \mu_A(y)$ and $\nu_A\{(x + y)r - xr\} \leq \nu_A(y)$ for any $x, y \in M$ and each $r \in R$.

We will denote the set of all IFSMs of M as IFSM(M). The following is the immediate result of Definition 2.5.

Example 2.5. Consider the additive group $\mathbf{Z}_4 = \{0, 1, 2, 3\}$ and the near-ring $(M(\mathbf{R}), +, \cdot)$, where $M(\mathbf{R})$ denotes the set of all 2×2 matrices over the real numbers, and $+$ and $\cdot$ denote the usual matrix addition multiplication, respectively. Let $\cdot$: $\mathbf{Z}_4 \times M(\mathbf{R}) \to \mathbf{Z}_4$ be the mapping defined as follows: For each $(m, A) \in \mathbf{Z}_4 \times M(\mathbf{R})$

$$mA = m + \quad + m \ (|A| \text{ times}),$$

where $|A|$ denotes the determinant of A. Then clealy $\mathbf{Z}_4$ is an $M(\mathbf{R})$-module.

We defined a complex mapping $A = (\mu_A, \upsilon_A) : \mathbf{Z}_4 \to I \times I$ as follows: For each $m \in \mathbf{Z}_4$,

$$\mu_A(m) = \begin{cases} \frac{2}{3} & \text{if } m = 0 \ or \ 2, \\ \frac{1}{2} & \text{if } m = 1 \ or \ 3, \end{cases}$$

and

$$\nu_A(m) = \begin{cases} \frac{1}{5} & \text{if } m = 0 \text{ or } 2, \\ \frac{1}{3} & \text{if } m = 1 \text{ or } 3, \end{cases}$$

Then we can easily see that $A \in \text{IFNG}(\mathbf{Z}_4)$. Moreover, we can check that the condition (ii) of Definition 2.5 holds. Hence $A \in \text{IFSM}(\mathbf{Z}_4)$. □

Proposition 2.6. *Let B be a non-empty subset of an R-module M. Then $(\chi_B, \chi_{B^c}) \in$ IFSM(M) if and only if B is a submodule of M.*

Proposition 2.7. *Let M be an R-module and let A be an intuitionistic fuzzy nonempty set in M. Then $A \in$ IFSM(M) if and only if for each $(\lambda, \mu) \in$ ImA, $A^{(\lambda,\mu)}$ is a submodule of M. In this case, $A^{(\lambda,\mu)}$ is called a level submodule of M.*

Proof. $(\Rightarrow)$: Suppose $A \in \text{IFSM}(M)$ and let $(\lambda, \mu) \in \text{Im}A$. By Result 1.D, $A^{(\lambda,\mu)}$ is a normal subgroup of M. Let $m \in M$, $b \in A^{(\lambda,\mu)}$, $r \in R$. Since $A \in \text{IFSM}(M)$,

$$\mu_A\{(m + b)r - mr\} \geq \mu_A(b) \geq \lambda$$

and

$$\nu_A\{(m + b)r - mr\} \leq \nu_A(b) \leq \mu.$$

Thus $(m + b)r - mr \in A^{(\lambda,\mu)}$. Hence, By Result 2.A, $A^{(\lambda,\mu)}$ is a submodule of M.

$(\Leftarrow)$: Suppose the necessary condition holds. By Result 1.D, $A \in \text{IFNG}(M)$. Let $x, y \in M$ and let $r \in R$. Let $A(x) = (t_1, s_1)$ and let $A(y) = (t_2, s_2)$ such that $t_2 \leq t_1$ and $s_2 \geq s_1$. Then $x, y \in A^{(t_1 \wedge t_2, s_1 \vee s_2)}$. By the hypothesis and Result 2.A, $(x + y)r - xr \in A^{(t_1 \wedge t_2, s_1 \vee s_2)}$. Thus

$$\mu_A\{(x + y)r - xr\} \geq t_1 \wedge t_2 = t_2$$

and

$$\nu_A\{(x + y)r - xr\} \leq s_1 \vee s_2 = s_2.$$

So $\mu_A\{(x+y)r-xr\} \geq \mu_A(y)$ and $\nu_A\{(x+y)r-xr\} \leq \nu_A(y)$. Hence $A \in \text{IFSM}(M)$. This completes the proof. □

Proposition 2.8. *Let G be an additive group and let $A \in$ IFNG(G). Then $x + A = y + A$ if and only if $A(x - y) = A(0)$ for any $x, y \in G$.*

Proof. $(\Rightarrow)$: Suppose $x + A = y + A$ for any $x, y \in G$. Then for each $z \in G$

$$(x + A)(z) = (y + A)(z), \text{ i.e., } A(z - x) = A(z - y).$$

In particular, $A(y - x) = A(y - y) = A(0)$.

($\Leftarrow$) : Suppose $A(x-y) = A(0)$ for any $x, y \in G$ and let $z \in G$. Then

$$\begin{aligned} \mu_{x+A}(z) &= \mu_A(z-x) \\ &= \mu_A((z-y)+(y-x)) \\ &\geq \mu_A(z-y) \wedge \mu_A(y-x) \text{ (By Definition 1.6(i))} \\ &= \mu_A(z-y) \wedge \mu_A(0) \\ &= \mu_A(z-y) \text{ (By Result 1.B)} \\ &= \mu_{y+A}(z) \end{aligned}$$

and

$$\begin{aligned} \nu_{x+A}(z) &= \nu_A(z-x) = \nu_A((z-y)+(y-x)) \\ &\leq \nu_A(z-y) \vee \nu_A(y-x) = \nu_A(z-y) \vee \nu_A(0) \\ &= \nu_A(z-y) = \nu_{y+A}(z). \end{aligned}$$

By the similar arguments, we have

$$\mu_{y+A}(z) \geq \mu_{x+A}(z) \text{ and } \nu_{y+A}(z) \leq \nu_{x+A}(z).$$

So $(x+A)(z) = (y+A)(z)$ for each $z \in G$. Hence $x + A = y + A$. This completes the proof. □

Theorem 2.9. *Let M be an R-module and let $A \in IFSM(M)$. Then the set M/A of all intuitionistic fuzzy cosets of A is an R-module with respect to the operations defined by $(x+A)+(y+A) = (x+y)+A$ and $(x+A)r = xr+A$ for any $x, y \in M$ and each $r \in R$. If $f : M \to M/A$ is a mapping defined by $f(x) = x + A$ for each $x \in M$, then f is an R- epimorphism with $Kerf = \{x \in M : A(x) = A(0)\}$.*

Proof. Let $x, y, x', y' \in M$ such that $x + A = x' + A$ and $y + A = y' + A$. Then, by Proposition 2.8, $A(x - x') = A(0)$ and $A(y - y') = A(0)$. Thus

$$\begin{aligned} \mu_A(x+y-y'-x') &= \mu_A\{(-x'+x)+(y-y')\} \text{ (Since } A \in \text{ IFNG}(M)) \\ &\geq \mu_A(-x'+x) \wedge \mu_A(y-y') \\ &= \mu_A(x-x') \wedge \mu_A(y-y') = \mu_A(0) \end{aligned}$$

and

$$\begin{aligned} \nu_A(x+y-y'-x') &= \nu_A\{(-x'+x)+(y-y')\} \leq \nu_A(-x'+x) \vee \nu_A(y-y') \\ &= \nu_A(x-x') \vee \nu_A(y-y') = \nu_A(0). \end{aligned}$$

By Result 1.B, $\mu_A(x+y-y'-x') = \mu_A(0)$ and $\nu_A(x+y-y'-x') = \nu_A(0)$, i.e., $A(x+y-y'-x') = A(0)$. So $(x+y)+A = (x'+y')+A$, i.e, $(x+A)+(y+A) = (x'+A)+(y'+A)$. Hence the operation $(x+A)+(y+A) = (x+y)+A$ is well-defined. Again, let $x, y \in M$ such that $x+A = y+A$. Let $r \in R$. Then, by Proposition 2.8, $A(x-y) = A(0)$. Thus

$$\begin{aligned}\mu_A(xr-yr) &= \mu_A((y-y+x)r-yr)\\ &\geq \mu_A(-y+y) \text{ (By Definition 1.6(ii))}\\ &= \mu_A(0)\end{aligned}$$

and

$$\nu_A(xr-yr) = \nu_A((y-y+x)r-yr) \leq \nu_A(-y+y) = \nu_A(0).$$

Thus, by Result 1.B, $\mu_A(xr-yr) = \mu_A(0)$ and $\nu_A(xr-yr) = \nu_A(0)$, i.e, $A(xr-yr) = A(0)$. So $xr+A = yr+A$, i.e, $(x+A)r = (y+A)r$. Hence the operation $(x+A)r = xr+A$ is well-defined. It is easy to show that M/A is an R-module.

Let $x, y \in M$ and let $r \in R$. Then

$$f(x+y) = (x+y)+A = (x+A)+(y+A) = f(x)+f(y)$$

and

$$f(xr) = xr+A = (x+A)r = f(x)r.$$

Thus f is an R-homomorphism. Moreover, it is clear that f is surjective. So f is an R-epimorphism. Now let $x \in M$. Then

$$\begin{aligned}x \in Kerf &\Leftrightarrow f(x) = 0+A\\ &\Leftrightarrow x+A = 0+A\\ &\Leftrightarrow A(x) = A(0).\end{aligned}$$

Hence $Kerf = \{x \in M : A(x) = A(0)\}$. This completes the proof. □

Definition 2.10. The R-module M/A is called the *intuitionistic fuzzy quotient R-module of M over A.*

3. Intuitionistic Fuzzy Weak Congruence

Definition 3.1. Let M be an R-module. An intuitionistic fuzzy nonempty relation P on M is called an *intuitionistic fuzzy weak equivalence relation* (in short, $IFWER$) if it satisfies the following conditions :

(i) P is *intuitionistic fuzzy weak reflexive*, i.e, for each $x \in M$,

$$P(x,x) = (\bigvee_{y,z\in M} \mu_P(y,z), \bigwedge_{y,z\in M} \nu_P(y,z)).$$

(ii) P is *intuitionistic fuzzy symmetric*, i.e, $P(x,y) = P(y,x)$ for any $x, y \in M$.

(iii) P is *intuitionistic fuzzy transitive*, i.e, $P \circ P \subset P$.

We will denote the set of all IFWERs on M as $\text{IFE}_W(M)$.

Definition 3.2. Let P be an IFWER on an R-module M. Then P is called an *intuitionistic fuzzy weak congruence* (in short, *IFWC*) on M if for any $a, b, c, d \in M$ and each $r \in R$,

$$\mu_P(a+c, b+d) \geq \mu_P(a,b) \wedge \mu_P(c,d), \mu_P(ar, br) \geq \mu_P(a,b)$$

and

$$\nu_P(a+c, b+d) \leq \nu_P(a,b) \vee \nu_P(c,d), \nu_P(ar, br) \leq \nu_P(a,b).$$

We will denote the set of all IFWCs on M as $\text{IFC}_W(M)$.

Example 3.2. Consider the $M(\mathbf{R})$-module $\mathbf{Z}_4$ in Example 2.5. We define a complex mapping $P = (\mu_P, \nu_P) : \mathbf{Z}_4 \times \mathbf{Z}_4 \to I \times I$ as follows :

P	0	1	2	3
0	(0.9, 0.1)	(0.6, 0.3)	(0.9, 0.1)	(0.6, 0.3)
1	(0.6, 0.3)	(0.9, 0.4)	(0.6, 0.3)	(0.9, 0.1)
2	(0.9, 0.1)	(0.6, 0.3)	(0.9, 0.1)	(0.6, 0.3)
3	(0.6, 0.3)	(0.9, 0.1)	(0.6, 0.3)	(0.9, 0.1)

Then we can easily see that $P \in \text{IFC}_W(\mathbf{Z}_4)$.

The following is the immediate result of Definition 3.2.

Proposition 3.3. *Let P be a relation on an R-module M. Then P is a congruence on M if and only if $(\chi_P, \chi_{P^c}) \in IFC_W(M)$. In fact, $(\chi_P, \chi_{P^c}) \in IFC(W)$.*

Proposition 3.4. *Let P be an IFWC on an R-module M. Then $P(x,y) = P(x-y, 0)$ for any $x, y \in M$.*

Proof. Let $x, y \in M$. Then

$$\mu_P(x-y, 0) = \mu_P(x-y, y-y) \geq \mu_P(x,y) \wedge \mu_P(-y,-y) = \mu_P(x,y)$$

and

$$\nu_P(x-y, 0) = \nu_P(x-y, y-y) \leq \nu_P(x,y) \vee \nu_P(-y,-y) = \nu_P(x,y).$$

Also we can easily see that $\mu_P(x-y,0) \leq \mu_P(x,y)$ and $\nu_P(x-y,0) \geq \nu_P(a,y)$. Hence $P(x,y) = P(x-y,0)$ for any $x, y \in M$. □

Remark 3.5. In Proposition 2.13 of [19], Hur and his colleagues proved that if P is an intuitionistic fuzzy congruence on a groupoid S, then for each $(\lambda, \mu) \in I \times I$ with $\lambda + \mu \leq 1$, $P^{(\lambda,\mu)}$ is a congruence on S. But our definition of intuitionistic fuzzy reflexivity enables us to establish both necessary and sufficient condition of the theorem which is as follows.

Theorem 3.6. *Let P be an IFR on an R-module M. Then P is an IFWC on M if and only if $P^{(\lambda,\mu)}$ is a congruence on M for each $(\lambda, \mu) \in ImP$.*

Proposition 3.7. *Let P be an IFWC on an R-module M. We define a complex mapping $A_P = (\mu_{A_P}, \nu_{A_P}) : M \to I \times I$ as follows : for each $a \in M$,*

$$A_P(a) = P(a, 0).$$

Then $A_P \in$ IFSM(M). In this case, A_P is called the intuitionistic fuzzy submodule of M induced by P.

Proof. It is clear that $A_P \in \text{IFS}(M)$ from the definition of A_P. Since $P \neq 0_\sim$, there exists an $(x_0, y_0) \in M \times M$ such that $P(x_0, y_0) \neq (0,1)$. Then $A_P(0) = P(0,0) = (\bigvee_{x,y \in M} \mu_P(x,y), \bigwedge_{x,y \in M} \nu_P(x,y)) \neq (0,1)$. Thus $A_P \neq 0_\sim$. Let $a, b \in M$. Then

$$\mu_{A_P}(a+b) = \mu_P(a+b,0) \geq \mu_P(a,0) \wedge \mu_P(b,0) = \mu_{A_P}(a) \wedge \mu_{A_P}(b)$$

and

$$\nu_{A_P}(a+b) = \nu_P(a+b,0) \leq \nu_P(a,0) \vee \nu_P(b,0) = \nu_{A_P}(a) \vee \nu_{A_P}(b).$$

Also,

$$\begin{aligned} \mu_{A_P}(-a) &= \mu_P(-a,0) = \mu_P(-a+0, -a+a) \\ &\geq \mu_P(-a,-a) \wedge \mu_P(0,a) \\ &= \mu_P(0,a) = \mu_P(a,0) = \mu_{A_P}(a) \end{aligned}$$

and

$$\begin{aligned} \nu_{A_P}(-a) &= \nu_P(-a,0) = \nu_P(-a+0, -a+a) \\ &\leq \nu_P(-a,-a) \vee \nu_P(0,a) \\ &= \nu_P(0,a) = \nu_P(a,0) = \nu_{A_P}(a). \end{aligned}$$

Again,

$$\begin{aligned}\mu_{A_P}(a+b-a) &= \mu_P(a+b-a,0) = \mu_P(a+b-a, a+0-a)\\ &\geq \mu_P(a+b, a+0) \wedge \mu_P(-a,-a)\\ &= \mu_P(a+b, a+0) \geq \mu_P(a,a) \wedge \mu_P(b,0)\\ &= \mu_P(b,0) = \mu_{A_P}(b)\end{aligned}$$

and

$$\begin{aligned}\nu_{A_P}(a+b-a) &= \nu_P(a+b-a,0) = \nu_P(a+b-a, a+0-a)\\ &\leq \nu_P(a+b, a+0) \vee \nu_P(-a,-a)\\ &= \nu_P(a+b, a+0) \leq \nu_P(a,a) \vee \nu_P(b,0)\\ &= \nu_P(b,0) = \nu_{A_P}(b).\end{aligned}$$

So $A_P \in \mathrm{IFNG}(M)$.

Now let $a, b \in M$ and let $r \in R$. Then

$$\begin{aligned}\mu_{A_P}\{(a+b)r - ar\} &= \mu_P((a+b)r - ar, 0)\\ &= \mu_P((a+b)r - ar, ar - ar)\\ &\geq \mu_P((a+b)r, ar) \wedge \mu_P(-ar, -ar)\\ &= \mu_P((a+b)r, ar) \geq \mu_P(a+b, a)\\ &\geq \mu_P(a,a) \wedge \mu_P(b,0) \geq \mu_P(b,0) = \mu_{A_P}(b)\end{aligned}$$

and

$$\begin{aligned}\nu_{A_P}\{(a+b)r - ar\} &= \nu_P((a+b)r - ar, 0)\\ &= \nu_P((a+b)r - ar, ar - ar)\\ &\leq \nu_P((a+b)r, ar) \vee \nu_P(-ar, -ar)\\ &= \nu_P((a+b)r, ar) \leq \nu_P(a+b, a)\\ &\leq \nu_P(b,0) = \nu_{A_P}(b).\end{aligned}$$

Hence $A_P \in \mathrm{IFSM}(M)$. This completes the proof. □

Proposition 3.8. *Let A be an IFSM of an R- module M. We define a complex mapping $P_A = (\mu_{P_A},\ \nu_{P_A}) : M \times M \to I \times I$ as follows : for any $x, y \in M$,*

$$P_A(x,\ y) = A(x-y).$$

Then $P_A \in IFC_W(M)$. In this case, P_A is called the intuitionistic fuzzy weak congruence on M induced by A.

Proof. It is clear that $P_A \in$ IFR(M) from the definition of P_A. Since $A \neq 0_\sim$, $P_A \neq 0_\sim$. Let $x \in M$. Then for any $y, z \in M$,

$$\mu_{P_A}(x, x) = \mu_A(0) \geq \mu_A(y - z) = \mu_{P_A}(y, z)$$

and

$$\nu_{P_A}(x, x) = \nu_A(0) \leq \nu_A(y - z) = \nu_{P_A}(y, z).$$

Thus $\mu_{P_A}(x, x) = \bigvee_{y,z\in M} \mu_{P_A}(y, z)$ and $\nu_{P_A}(x, x) = \bigwedge_{y,z\in M} \nu_{P_A}(y, z)$. So $P_A(x, x) = (\bigvee_{y,z\in M} \mu_{P_A}(y, z), \bigwedge_{y,z\in M} \nu_{P_A}(y, z))$, i.e, P_A is intuitionistic fuzzy weakly reflexive. It is clear that P_A is intuitionistic fuzzy symmetric. Let $x, y \in M$. Then for each $z \in M$,

$$\begin{aligned} \mu_{P_A}(x, y) &= \mu_A(x - y) = \mu_A(x - z + z - y) \\ &\geq \mu_A(x - z) \wedge \mu_A(z - y) = \mu_{P_A}(x, z) \wedge \mu_{P_A}(z, y) \end{aligned}$$

and

$$\begin{aligned} \nu_{P_A}(x, y) &= \nu_A(x - y) = \nu_A(x - z + z - y) \\ &\leq \nu_A(x - z) \vee \nu_A(z - y) = \nu_{P_A}(x, z) \vee \nu_{P_A}(z, y). \end{aligned}$$

Thus

$$\mu_{P_A}(x, y) \geq \bigvee_{z\in M} [\mu_{P_A}(x, z) \wedge \mu_{P_A}(z, y)]$$

and

$$\nu_{P_A}(x, y) \leq \bigwedge_{z\in M} [\nu_{P_A}(x, z) \vee \nu_{P_A}(z, y)].$$

So $P_A \circ P_A \subset P_A$, *i.e.*, P_A is intuitionistic fuzzy transitive. Hence $P_A \in \text{IFE}_W(M)$.

Let $x, y, x', y' \in M$ and let $r \in R$. Then

$$\begin{aligned} \mu_{P_A}(x + x', y + y') &= \mu_A(x + x' - y' - y) = \mu_A(-y + x + x' - y') \\ &\geq \mu_A(-y + x) \wedge \mu_A(x' - y') \\ &= \mu_{P_A}(x, y) \wedge \mu_{P_A}(x', y') \end{aligned}$$

and

$$\begin{aligned} \nu_{P_A}(x + x', y + y') &= \nu_A(x + x' - y' - y) = \nu_A(-y + x + x' - y') \\ &\leq \nu_A(-y + x) \vee \nu_A(x' - y') \\ &= \nu_{P_A}(x, y) \vee \nu_{P_A}(x', y'). \end{aligned}$$

Also,

$$\begin{aligned} \mu_{P_A}(xr, yr) &= \mu_A(xr - yr) = \mu_A((y - y + x)r - yr) \\ &\geq \mu_A(-y + x) = \mu_{P_A}(x, y) \end{aligned}$$

and

$$\begin{aligned} \nu_{P_A}(xr, yr) &= \nu_A(xr - yr) = \nu_A((y - y + x)r - yr) \\ &\leq \nu_A(-y + x) = \nu_{P_A}(x, y). \end{aligned}$$

Hence $P_A \in \mathrm{IFC}_W(M)$. This completes the proof. □

Example 3.8. Let A be the IFSM of the $M(\mathbf{R})$-module $\mathbf{Z}_4$ in Example 2.5. We define a complex mapping $P_A = (\mu_{P_A}, \nu_{P_A}) : \mathbf{Z}_4 \times \mathbf{Z}_4 \to I \times I$ as follows : For any $m, n \in \mathbf{Z}_4$,

$$P_A(m, n) = A(m - n).$$

Then, by proposition 3.8, P_A is the intuitionistic fuzzy weak confruence on $\mathbf{Z}_4$ induced by A. In fact, P_A is defined as follows :

P_A	0	1	2	3
0	$(\frac{2}{3}, \frac{1}{5})$	$(\frac{1}{2}, \frac{1}{3})$	$(\frac{2}{3}, \frac{1}{5})$	$(\frac{1}{2}, \frac{1}{3})$
1	$(\frac{1}{2}, \frac{1}{3})$	$(\frac{2}{3}, \frac{1}{5})$	$(\frac{1}{2}, \frac{1}{3})$	$(\frac{2}{3}, \frac{1}{5})$
2	$(\frac{2}{3}, \frac{1}{5})$	$(\frac{1}{2}, \frac{1}{3})$	$(\frac{2}{3}, \frac{1}{5})$	$(\frac{1}{2}, \frac{1}{3})$
3	$(\frac{1}{2}, \frac{1}{3})$	$(\frac{2}{3}, \frac{1}{5})$	$(\frac{1}{2}, \frac{1}{3})$	$(\frac{2}{3}, \frac{1}{5})$.

Theorem 3.9. *Let M be an R-module. Then there exists an inclusion-preserving bijection from IFSM(M) to IFC$_W$(M).*

Proof. We define two mappings $\Psi : \mathrm{IFSM}(M) \to \mathrm{IFC}_W(M)$ and $\Phi : \mathrm{IFC}_W(M) \to \mathrm{IFM}(M)$ as follows, respectively :

$$\Psi(A) = P_A \text{ for each } P \in \mathrm{IFSM}(M)$$

and

$$\Phi(P) = A_P \text{ for each } P \in \mathrm{IFC}_W(M).$$

¿From Proposition 3.8 and Proposition 3.7, it is clear that Ψ and Φ are well-defined. Let $A \in \mathrm{IFSM}(M)$ and let $a \in M$. Then

$$\begin{aligned} [(\Phi \circ \Psi)(A)](a) &= [\Phi(\Psi(A))](a) = [\Phi(P_A)](a) = A_{P_A}(a) \\ &= P_A(a, 0) = A(a - 0) = A(a) \\ &= [id_{\mathrm{IFM}(M)}(A)](a). \end{aligned}$$

Thus $\Phi \circ \Psi = id_{\mathrm{IFSM}(M)}$. So Ψ is injective. Let $A, B \in \mathrm{IFSM}(M)$ such that $A \subset B$. Let $(x, y) \in M \times M$. Then

$$\mu_{P_B}(x,y) = \mu_B(x-y) \geq \mu_A(x-y) = \mu_{P_A}(x,y)$$

and

$$\nu_{P_B}(x,y) = \nu_B(x-y) \leq \nu_A(x-y) = \nu_{P_A}(x,y).$$

Thus $P_A \subset P_B$, i.e., $\Psi(A) \subset \Psi(B)$. So Ψ is inclusion- preserving. Let $P \in \mathrm{IFC}_W(M)$ and let $x, y \in M$. Then

$$\begin{aligned}
[(\Psi \circ \Phi)(P)](x,y) &= [\Psi(\Phi(P))](x,y) = [\Psi(A_P)](x,y) \\
&= P_{A_P}(x,y) = A_P(x-y) = P(x-y,0) \\
&= P(x,y) \text{ (by Proposition 3.4)} \\
&= [id_{\mathrm{IFC}W}(P)](x,y).
\end{aligned}$$

Thus $\Psi \circ \Phi = id_{\mathrm{IFC}_W(M)}$. So Ψ is surjective. This completes the proof. □

Proposition 3.10. *Let P be an IFWC on an R-module M. Then for each $(\lambda, \mu) \in$ ImP, ${A_P}^{(\lambda,\ \mu)} = \{x \in M : x \equiv 0(P^{(\lambda,\ \mu)})\}$ is the submodule induced by the congruence $P^{(\lambda,\ \mu)}$.*

Proof. Let $a \in M$. Then

$$\begin{aligned}
a \in {A_P}^{(\lambda,\ \mu)} &\Leftrightarrow \mu_{A_P}(a) \geq \lambda \text{ and } \nu_{A_P}(a) \leq \mu \\
&\Leftrightarrow \mu_P(a,0) \geq \lambda \text{ and } \nu_P(a,0) \leq \mu \\
&\Leftrightarrow (a,0) \in P^{(\lambda,\ \mu)} \\
&\Leftrightarrow a \equiv 0(P^{(\lambda,\ \mu)}) \\
&\Leftrightarrow a \in \{x \in M : x \equiv 0(P^{(\lambda,\ \mu)})\}.
\end{aligned}$$

Hence $A_P^{(\lambda,\mu)}$ is the submodule induced by the congruence $P^{(\lambda,\mu)}$. □

Proposition 3.11. *Let A be an IFSM of an R-module M. Then for each $(\lambda,\ \mu) \in$ ImA, ${P_A}^{(\lambda,\ \mu)}$ is the congruence on M induced by $A^{(\lambda,\ \mu)}$.*

Proof. For each $(\lambda, \mu) \in \mathrm{Im}A$,let Q be the congruence on M induced by $A^{(\lambda,\ \mu)}$ and let $x, y \in M$. Then

$$(x,y) \in Q \Leftrightarrow x - y \in A^{(\lambda,\ \mu)}.$$

Let $(x, y) \in {P_A}^{(\lambda,\ \mu)}$. Then

$$\mu_{P_A}(x,y) = \mu_A(x-y) \geq \lambda \text{ and } \nu_{P_A}(x,y) = \nu_A(x-y) \leq \mu.$$

Thus $x-y \in A^{(\lambda,\ \mu)}$. So $(x,y) \in Q$, i.e., ${P_A}^{(\lambda,\ \mu)} \subset Q$. By the similar arguments, we have $Q \subset {P_A}^{(\lambda,\ \mu)}$. Hence ${P_A}^{(\lambda,\ \mu)} = Q$. This completes the proof. □

Definition 3.12. Let M be an R-module and let $P \in \mathrm{IFC}_W(M)$. Then $Q \in \mathrm{IFC}_W(M)$ is said to be *P-invariant* if $P(x,y) = P(x',\ y')$ implies that $Q(x,y) = Q(x',\ y')$ for any $(x,y),\ (x',y') \in M \times M$.

Lemma 3.13. *Let M be an R-module and let $A \in IFSM(M)$. We define a complex mapping $P/P = (\mu_{P_A/P_A}, \nu_{P_A/P_A}) : M/A \times M/A \to I \times I$ as follows : for any $x,y \in M$,*

$$P_A/P_A(x+A, y+A) = P_A(x,y).$$

Then $P_A/P_A \in IFC_W(M/A)$.

Proof. From the definition of P_A/P_A, it is clear that $P_A/P_A \in \mathrm{IFR}(M/A)$. Suppose $x+A = x'+A$ and $y+A = y'+A$. Then $A(x-x') = A(0)$ and $A(y-y') = A(0)$. Thus, by the definition of P_A,

$$P_A(x,x') = (\bigvee_{p,q\in M} \mu_{P_A}(p,q),\ \bigwedge_{p,q\in M} \nu_{P_A}(p,q))\ (*)$$

and

$$P_A(y,y') = (\bigvee_{p,q\in M} \mu_{P_A}(p,q),\ \bigwedge_{p,q\in M} \nu_{P_A}(p,q))\ (**)$$

On the other hand,

$$\begin{aligned}
\mu_{P_A}(x,y) &\geq \mu_{P_A}(x,x') \wedge \mu_{P_A}(x',y) \text{ (Since } P_A \text{ is intuitionistic fuzzy transitive)}\\
&= \mu_{P_A}(x',\ y) \text{ (By } (*))\\
&\geq \mu_{P_A}(x',\ y') \wedge \mu_{P_A}(y',\ y) \text{ (Since } P_A \text{ is intuitionistic fuzzy transitive)}\\
&= \mu_{P_A}(x',y') \wedge \mu_{P_A}(y,y') \text{ (Since } P_A \text{ is intuitionistic fuzzy symmetric)}\\
&= \mu_{P_A}(x',\ y') \text{ (By } (**))
\end{aligned}$$

and

$$\begin{aligned}
\nu_{P_A}(x,y) &\leq \nu_{p_A}(x,x') \vee \nu_{P_A}(x',y) = \nu_{P_A}(x',\ y)\\
&\leq \nu_{P_A}(x',\ y') \vee \mu_{P_A}(y',\ y) = \nu_{P_A}(x',\ y').
\end{aligned}$$

By the similar arguments, we have

$$\mu_P(x',\ y') \geq \mu_P(x,y) \text{ and } \nu_P(x',\ y') \leq \nu_P(x,y).$$

Thus $P(x,y) = P(x',\ y')$, i.e., $P/P(x+A, y+A) = P/P(x'+A, y'+A)$.
So P_A/P_A is well-defined. The rest of the proof is easy. This completes the proof. □

Theorem 3.14. *Let M be an R-module and let $A \in IFSM(M)$. Then there exists a one-to-one correspondence between the set*

$$IFC_{W,P_A}(M)$$

and the set

$$IFC_{W,P_A/P_A}(M/A),$$

where $IFC_{W,P_A}(M)$ and $IFC_{W,P_A/P_A}(M/A)$ denote the set of P_A-invariant IFC_S on M and the set of P_A/P_A-invariant IFC_S on M/A, respectively.

Proof. Let $Q \in \mathrm{IFC}_{W,P_A}(M)$. We define a complex mapping

$$Q/P_A = (\mu_{Q/P_A}, \nu_{Q/P_A}) : M/A \times M/A \to I \times I$$

as follows : for any $x, y \in M$,

$$Q/P_A(x + A, y + a) = Q(x, y).$$

Suppose $x + A = x' + A$ and $y + A = y' + A$. Then $P_A(x, y) = P_A(x', y')$. Since Q is P_A-invariant, $Q(x, y) = Q(x', y')$. Thus Q/P_A is well-defined. Moreover, it is clear that $Q/P_A \in \mathrm{IFR}(M/A)$ from the definition of Q/P_A. It is easy to show that Q/P_A is a P_A/P_A-invariant IFWC on M/A.

We define a mapping $\Phi : \mathrm{IFC}_{W,P_A}(M) \to \mathrm{IFC}_{W,P_A/P_A}(M/A)$ by $\Phi(Q) = Q/P_A$ for each $Q \in \mathrm{IFC}_{W,P_A}(M)$. Let $Q_1, Q_2 \in \mathrm{IFC}_{W,P_A}(M)$ such that $Q_1 \neq Q_2$. Then there exist $x, y \in M$ such that $Q_1(x, y) \neq Q_2(x, y)$. Thus

$$Q_1/P_A(x + A, y + A) = Q_1(x, y) \neq Q_2(x, y) = Q_2/P_A(x + A, y + A).$$

So $\Phi(Q_1) \neq \Phi(Q_2)$, i.e., Φ is injective.

Now let $Q' \in \mathrm{IFC}_{W,P_A/P_A}(M/A)$. We define a complex mapping $Q = (\mu_Q, \nu_Q) : M \times M \to I \times I$ as follows : for any $x, y \in M$,

$$Q(x, y) = Q'(x + A, y + A).$$

Let $x \in M$, Then

$$\begin{aligned} \mu_Q(x, x) &= u_{Q'}(x + A, x + A) \\ &= \bigvee_{u+A, v+A \in M/A} \mu_{Q'}(u + A, v + A) = \bigvee_{u,v \in M} \mu_Q(u, v) \end{aligned}$$

and

$$\begin{aligned} \nu_Q(x, x) &= \nu_{Q'}(x + A, x + A) \\ &= \bigwedge_{u+A, v+A \in M/A} \nu_{Q'}(u + A, v + A) = \bigwedge_{u,v \in M} \nu_Q(u, v). \end{aligned}$$

Thus Q is intuitionistic fuzzy weakly reflexive. It is easy to see that Q is intuitionistic fuzzy symmetric. Let $x, y \in M$. Then

$$\begin{aligned}\mu_Q(x,y) &= \mu_{Q'}(x+A, y+A) \\ &\geq \bigvee_{z+A\in M/A} [\mu_{Q'}(x+A, z+A) \wedge \mu_{Q'}(z+A, y+A)] \\ &= \bigvee_{z\in M} [\mu_Q(x,z) \wedge \mu_Q(z,y)]\end{aligned}$$

and

$$\begin{aligned}\nu_Q(x,y) &= \nu_{Q'}(x+A, y+A) \\ &\leq \bigwedge_{z+A\in M/A} [\nu_{Q'}(x+A, z+A) \vee \nu_{Q'}(z+A, y+A)] \\ &= \bigwedge_{z\in M} [\nu_Q(x,z) \vee \nu_Q(z,y)].\end{aligned}$$

Thus Q is intuitionistic fuzzy transitive. So $Q \in \mathrm{IFE}_W(M)$.

Let $x, y, a, b \in M$ and let $r \in R$. Then

$$\begin{aligned}\mu_Q(x+a, y+b) &= \mu_{Q'}(x+a+A, y+b+A) \\ &= \mu_{Q'}((x+A)+(a+A), (y+A)+(b+A)) \\ &\geq \mu_{Q'}(x+A, y+A) \wedge \mu_{Q'}(a+A, b+A) \\ &\quad (\text{Since } Q' \in \mathrm{IFC}_{W,P_A/P_A}(M/A)) \\ &= \mu_Q(x,y) \wedge \mu_Q(a,b)\end{aligned}$$

and

$$\begin{aligned}\nu_Q(x+a, y+b) &= \nu_{Q'}(x+a+A, y+b+A) \\ &= \nu_{Q'}((x+A)+(a+A), (y+A)+(b+A)) \\ &\leq \nu_{Q'}(x+A, y+A) \vee \nu_{Q'}(a+A, b+A) \\ &= \nu_Q(x,y) \vee \nu_Q(a,b).\end{aligned}$$

Also,

$$\begin{aligned}\mu_Q(xr, yr) &= \mu_{Q'}(xr+A, yr+A) = \mu_{Q'}((x+A)r, (y+A)r) \\ &\geq \mu_{Q'}(x+A, y+A) \ (\text{Since } Q' \in \mathrm{IFC}_{W,P_A/P_A}(M/A)) \\ &= \mu_Q(x,y)\end{aligned}$$

and

$$\begin{aligned}\nu_Q(xr, yr) &= \nu_{Q'}(xr + A, yr + A) = \nu_{Q'}((x + A)r, (y + A)r) \\ &\leq \nu_{Q'}(x + A, y + A) = \nu_Q(x, y).\end{aligned}$$

So $Q \in \text{IFC}_W(M)$.

Let $x, y, u, v \in M$ and suppose $P_A(x, y) = P_A(u, v)$. Then

$$P_A/P_A(x + A, y + A) = P_A/P_A(u + A, v + A).$$

Since $Q' \in \text{IFC}_{W,P_A/P_A}(M/A)$, $Q'(x + A, y + A) = Q'(\mu + A, \nu + A)$.
Thus $Q(x, y) = Q(u, v)$. So Q is P-invariant. Let $(x + A, y + A) \in M/A \times M/A$. Then $Q/P_A(x + A, y + A) = Q(x, y) = Q'(x + A, y + A)$. Thus $Q' = Q/P_A = \Phi(Q)$. So Φ is surjective. Hence Φ is bijective. This completes the proof. □

Theorem 3.15. Let M be an R-module and let $A \in \text{IFSM}(M)$. If $(\lambda, \mu) = (\bigvee \text{Im}\mu_{P_A}, \bigwedge \text{Im}\nu_{P_A})$, then $M/A \cong M/P_A^{(\lambda,\mu)}$.

Proof. We define a mapping $\Phi : M/A \to M/P_A^{(\lambda,\mu)}$ by $\Phi(x + A) = xP_A^{(\lambda,\mu)}$ for each $x \in M$, where $xP_A^{(\lambda,\mu)}$ denotes the congruence class of x by the congruence $P_A^{(\lambda,\mu)}$. For any $x, y \in M$, suppose $x + A = y + A$. Then $A(x - y) = A(0)$. Since $P_A(x, y) = A(x - y)$,

$$P_A(x, y) = (\bigvee \text{Im}\mu_{P_A}, \bigwedge \text{Im}\nu_{P_A}) = (\lambda, \mu).$$

Then $(x, y) \in P_A^{(\lambda,\mu)}$. Thus $xP_A^{(\lambda,\mu)} = yP_A^{(\lambda,\mu)}$. So $\Phi(x + A) = \Phi(y + A)$.
Hence Φ is well-defined.

Let $x, y \in M$ and let $r \in R$. Then

$$\begin{aligned}\Phi((x + A) + (y + A)) &= \Phi(x + y + A) = (x + y)P_A^{(\lambda,\mu)} \\ &= xP_A^{(\lambda,\mu)} + yP_A^{(\lambda,\mu)} \\ &= \Phi(x + A) + \Phi(y + A)\end{aligned}$$

and

$$\begin{aligned}\Phi((x + A)r) &= \Phi(xr + A) = xrP_A^{(\lambda,\mu)} = (xP_A^{(\lambda,\mu)})r \\ &= (\Phi(x + A))r.\end{aligned}$$

So ϕ is an R-homorphism.

For any $x, y \in M$, suppose $\Phi(x + A) = \Phi(y + A)$. Then $xP_A^{(\lambda,\mu)} = yP_A^{(\lambda,\mu)}$. Thus $(x, y) \in P_A^{(\lambda,\mu)}$, i.e., $P_A(x, y) = (\lambda, \mu)$. Since $P_A(x, y) = A(x - y)$, $A(x - y) = (\lambda, \mu)$.

Then $A(x-y)=A(0)$. So $x+A=y+A$, i.e., Φ is injective. It is clear that Φ is surjective. Hence $M/A \cong MP_A^{(\lambda,\mu)}$. This completes the proof. □

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