Ordering Policy for Planned Maintenance with Salvage Value

Young T. Park[1†] and Sun Jing[2]

[1] Systems Management Engineering, Sungkyunkwan University, Suwon 440-746, Korea
[2] School of Economics and Management, Tsinghua University, Beijing 100084, China
E-mails: [1]ytpark@skku.edu, [2]sunj3@em.tsinghua.edu.cn

Abstract

A spare ordering policy is considered for planned maintenance. Introducing the ordering, uptime, downtime, inventory costs and salvage value, we derive the expected cost effectiveness. The problem is to determine jointly the ordering time for a spare and the preventive replacement time for the operating unit which maximize the expected cost effectiveness. Some properties regarding the optimal policy are derived, and a numerical example is included to explain the proposed model.

Key Words: Spare, Ordering, Maintenance, Salvage Value, Cost Effectiveness

1. Introduction

Consider a 1-unit system, where each failed unit is scrapped without repair and each spare is provided only by an order. The original unit begins operating at time 0. If the original unit fails before a specified time t_0, we place an order immediately at the failure time instant and replace the failed unit with the new one as soon as it is delivered. On the other hand, if the operating unit does not fail up to t_0, we place an order for a spare at t_0 and replace the unit as follows: (i) If the unit fails between t_0 and another specified time $t_1(\geq t_0)$, the failed unit is replaced as soon as a spare is available. (ii) When the unit does not fail up to t_1, it is replaced preventively at t_1 if a spare is available, or as soon as the ordered spare is delivered.

The time between successive replacements is a cycle and the behavior in each cycle repeats. The decision variables are the scheduled times t_0 and t_1 for spare ordering and preventive replacement maximizing the expected cost effectiveness. The cost effectiveness is defined as "steady-stated availability/expected cost rate" which reflects the efficiency per dollar outlay. This criterion is useful in the case that the benefits obtained from the system oper-

†Corresponding Author

ation are not reducible to monetary terms as in weapon systems [2].

Symbols

$f(t),\ F(t),\ m$	pdf, cdf, and mean value of the lifetime of a unit
$\overline{F}(t)$	$1-F(t)$
$h(t)$	$f(t)/\overline{F}(t)$, instantaneous failure rate of a unit
$h_x(t)$	$[F(t+x)-F(t)]/\overline{F}(t)$, interval failure rate of a unit
$g(x), G(x),\ m_x$	pdf, cdf, and mean value of lead time
t_0	scheduled time for spare ordering
t_1	scheduled time for preventive replacement(; $t_1 \geq t_0$)
c_0	ordering cost
c_u	uptime cost per unit time operation
c_d	downtime cost per unit time due to spare shortage
c_h	holding cost of a spare per unit time
v_s	salvage value per unit time for residual lifetime
$U(t_0,\ t_1)$	expected uptime per cycle
$C(t_0,\ t_1)$	expected cost per cycle
$E(t_0,\ t_1)$	$U(t_0,\ t_1)/C(t_0,\ t_1)$expected cost effectiveness

Other symbols are defined when needed.

2. Cost Effectiveness Model

From the renewal reward theorem, the expected cost rate for an infinite time span is the expected cost per cycle divided by the expected cycle length. Since the time between successive replacements is a cycle, the following five mutually exclusive and exhaustive possibilities exist as in Park and Park [4]:

(i) the operating unit fails before t_0

(ii) the operating unit fails between t_0 and the arrival of the ordered spare

(iii) the operating unit fails between the arrival of the ordered spare and t_1

(iv) the ordered spare arrives before t_1 and the operating unit does not fail up to t_1.

(v) the ordered spare arrives after t_1 and the operating unit does not fail before the arrival of the ordered spare.

The expected cycle length is

$$\int_0^{t_0}(t+m_x)f(t)dt+\int_0^{\infty}\int_{t_0}^{t_0+x}(t_0+x)f(t)g(x)dtdx+\int_0^{t_1-t_0}\int_{t_0+x}^{t_1}t\,f(t)g(x)dtdx$$

$$+\int_0^{t_1-t_0}\int_{t_1}^{\infty}t_1 f(t)g(x)dtdx+\int_{t_1-t_0}^{\infty}\int_{t_0+x}^{\infty}(t_0+x)f(t)g(x)dtdx$$

$$= m_x + \int_0^{t_0}\overline{F}(t)dt + \int_0^{t_1-t_0}\int_{t_0+x}^{t_1}\overline{F}(t)g(x)dtdx \tag{1}$$

Downtime occurs in the cases (i) and (ii), and thus the expected downtime per cycle is

$$m_x\int_0^{t_0}f(t)dt + \int_0^{\infty}\int_{t_0}^{t_0+x}(t_0+x-t)f(t)g(x)dtdx = \int_0^{\infty}\int_{t_0}^{t_0+x}F(t)g(x)dtdx \tag{2}$$

Since uptime per cycle is cycle length minus downtime, the expected uptime per cycle is

$$U(t_0,t_1) = m_x + \int_0^{t_0}\overline{F}(t)dt + \int_0^{t_1-t_0}\int_{t_0+x}^{t_1}\overline{F}(t)g(x)dtdx - \int_0^{\infty}\int_{t_0}^{t_0+x}F(t)g(x)dtdx$$

$$= \int_0^{\infty}\int_0^{t_0+x}\overline{F}(t)g(x)dtdx + \int_0^{t_1-t_0}\int_{t_0+x}^{t_1}\overline{F}(t)g(x)dtdx \tag{3}$$

The expected cost per cycle is the sum of the ordering, uptime, downtime and spare holding costs and salvage value. Since the number of orders per cycle is one, the ordering cost per cycle is C_0. From (3), the expected uptime cost per cycle is

$$c_u[\int_0^{\infty}\int_0^{t_0+x}\overline{F}(t)g(x)dtdx + \int_0^{t_1-t_0}\int_{t_0+x}^{t_1}\overline{F}(t)g(x)dtdx] \tag{4}$$

From (2), the expected downtime cost per cycle is

$$c_d\int_0^{\infty}\int_{t_0}^{t_0+x}F(t)g(x)dtdx \tag{5}$$

Holding of a spare occurs in the cases (iii) and (iv), and the expected holding cost per cycle is

$$c_h\left[\int_0^{t_1-t_0}\int_{t_0+x}^{t_1}(t-t_0-x)f(t)g(x)dtdx+\int_0^{t_1-t_0}\int_{t_1}^{\infty}(t_1-t_0-x)f(t)g(x)dtdx\right]$$

$$= c_h\int_0^{t_1-t_0}\int_{t_0+x}^{t_1}\overline{F}(t)g(x)dtdx \tag{6}$$

It seems reasonable that salvage value of a used unit, which is still operable, is proportional to the expected residual lifetime [3]. Salvage value occurs in the cases (iv) and (v), and the expected salvage value per cycle is

$$v_s\left[\int_0^{t_1-t_0}\int_{t_1}^{\infty}\overline{F}(t)g(x)dtdx+\int_{t_1-t_0}^{\infty}\int_{t_0+x}^{\infty}\overline{F}(t)g(x)dtdx\right]$$

$$= v_s\int_{t_1-t_0}^{\infty}\overline{F}(t_0+x)G(x)dx \tag{7}$$

Thus the expected cost per cycle is

$$C(t_0,t_1) = c_o+c_u[\int_0^{\infty}\int_0^{t_0+x}\overline{F}(t)g(x)dtdx+\int_0^{t_1-t_0}\int_{t_0+x}^{t_1}\overline{F}(t)g(x)dtdx]+c_d\int_0^{\infty}\int_{t_0}^{t_0+x}F(t)g(x)dtdx$$

$$+\ c_h\int_0^{t_1-t_0}\int_{t_0+x}^{t_1}\overline{F}(t)g(x)dtdx\ -\ v_s\int_{t_1-t_0}^{\infty}\overline{F}(t_0+x)G(x)dx \tag{8}$$

Since each replacement is a regeneration point, the cost effectiveness "steady-stated availability/expected cost rate" can be rewritten as "expected uptime in a cycle/expected cost per cycle." Thus, the expected cost effectiveness is

$$E(t_0,t_1)=U(t_0,t_1)/C(t_0,t_1) \tag{9}$$

where
$U(t_0, t_1)$ and $C(t_0, t_1)$ are given by (3) and (8) respectively.

3. Optimal Policy

Theorem 1. For any fixed ordering time t_0, the expected cost effectiveness, $E(t_0, t_1)$, is maximized at either $t_1=t_0$ or $t_1=\infty$.

Proof. Differentiating $E(t_0, t_1)$in (9) with respect to t_1 yields

$$\partial E(t_0,t_1)/\partial t_1 = A(t_0)[\overline{F}(t_1)G(t_1-t_0)/C(t_0,t_1)^2] \tag{10}$$

where

$$A(t_0)=c_o-v_sm+c_d\int_0^{\infty}\int_{t_0}^{t_0+x}F(t)g(x)dtdx-c_h\int_0^{\infty}\int_{t_0}^{t_0+x}\overline{F}(t)g(x)dtdx \tag{11}$$

If $A(t_0) < 0$, then $\partial E(t_0, t_1)/\partial t_1 < 0$ for all $t_1 \in (t_0, \infty)$ and thus $E(t_0, t_1)$ is maximized at $t_1 = t_0$. Similarly if $A(t_0) > 0$, $E(t_0, t_1)$ is maximized at $t_1 = \infty$. If $A(t_0) = 0$, all values of t_1 give the same cost effectiveness and both t_0 and ∞ are as good as any. Hence an optimal value of t_1 is either t_0 or ∞.

Thus we need only consider the two cases ($t_1 = t_0$ and $t_1 = \infty$) in order to obtain the optimal ordering policy which maximizes the cost effectiveness. Now, let us treat the two cases.

Policy 1: Replacement on spare's arrival

In this case the spare on arrival replaces the original unit irrespective of the condition of the original one. Substituting $t_1 = t_0$ into (9), we obtain

$$E_1(t_0) \equiv E(t_0, t_0) = U_1(t_0)/C_1(t_0) \tag{12}$$

where

$$U_1(t_0) = \int_0^\infty \int_0^{t_0+x} \overline{F}(t) g(x) dt dx \tag{13}$$

$$C_1(t_0) = c_o + c_u \int_0^\infty \int_0^{t_0+x} \overline{F}(t) g(x) dt dx + c_d \int_0^\infty \int_{t_0}^{t_0+x} F(t) g(x) dt dx - v_s \int_0^\infty \overline{F}(t_0 + x) G(x) dx \tag{14}$$

Define the numerator of the derivative of $E_1(t_0)$ in (12) divided by $\overline{F}(t_0)$ as

$$p_1(t_0) = [1 - \int_0^\infty h_x(t_0) g(x) dx] \cdot C_1(t_0) - U_1(t_0) \cdot [(c_d - c_u - v_s) \int_0^\infty h_x(t_0) g(x) dx + c_u + v_s] \tag{15}$$

Then, we have the following theorem regarding the optimum ordering time $t_{01}*$ which maximizes $E_1(t_0)$.

Theorem 2. (1) Suppose that $h(t)$ is strictly increasing.

(i) If $p_1(0) \le 0$, then the optimum ordering time $t_{01}* = 0$, i.e., place an order for a spare at the same instant when a unit is put in service.

(ii) If $p_1(0) > 0$ and $p_1(\infty) < 0$, then there exists a finite and unique optimum ordering time $t_{01}*$ $(0 < t_{01}* < \infty)$ satisfying $p_1(t_{01}*) = 0$.

(iii) If $p_1(\infty) \ge 0$, then the optimum ordering time $t_{01}* = \infty$, i.e., place an order for a spare at the instant of failure of the operating unit.

(2) Suppose that $h(t)$ is non-increasing. Then, $t_{01}*$ is either 0 or ∞.

Proof. Differentiating $E_1(t_0)$ with respect to t_0 and setting it equal to zero implies $p_1(t_0) = 0$. Further,

$$p_1'(t_0) = -[\int_0^\infty h_x'(t_o)g(x)dx]\cdot C_1(t_0) - U_1(t_0)\cdot[(c_d - c_u - v_s)\int_0^\infty h_x'(t_0)g(x)dx] \tag{16}$$

Notice that the difference between downtime cost and uptime cost should be larger than salvage value (i.e., $c_d - c_u \geq v_s$) to justify system operation. Since the interval failure rate $h_x(t_0)$ and the instantaneous failure rate $h(t)$ have the same monotone properties (see Barlow and Proschan [1, p.23]), $p_1(t)$ is strictly decreasing if $h(t)$ is strictly increasing, and $p_1(t)$ is non-decreasing if $h(t)$ is non-increasing. Thus, the existence of $t_{01}*$ in the theorem follows trivially.

Policy 2: No preventive replacement

In this case the delivered spare is put into inventory if the original unit is still operating, and not used until the original one fails. Substituting $t_1 = \infty$ into (9), we obtain

$$E_2(t_0) \equiv E(t_0, \infty) = U_2(t_0)/C_2(t_0) \tag{17}$$

where

$$U_2(t_0) = \int_0^\infty \int_0^{t_0+x} \overline{F}(t)g(x)dtdx + \int_0^\infty \int_{t_0+x}^\infty \overline{F}(t)g(x)dtdx = m \tag{18}$$

$$C_2(t_0) = c_o + c_u m + c_d \int_0^\infty \int_{t_0}^{t_0+x} F(t)g(x)dtdx + c_h \int_0^\infty \int_{t_0+x}^\infty \overline{F}(t)g(x)dtdx \tag{19}$$

Define the numerator of the derivative of $E_2(t_0)$ in (17) divided by $\overline{F}(t_0)$ as

$$p_2(t_0) = -m[(c_d + c_h)\int_0^\infty h_x(t_0)g(x)dx - c_h] \tag{20}$$

Then, we have the following theorem regarding the optimum ordering time $h(t)t_{02}*$ which maximizes $E_2(t_0)$.

Theorem 3. (1) Suppose that $h(t)$ is strictly increasing.

(i) If $p_2(0) \leq 0$, then the optimum ordering time $t_{02}* = 0$.

(ii) If $p_2(0) > 0$ and $p_2(\infty) < 0$, then there exists a finite and unique optimum ordering time $t_{02}*$ $(0 < t_{02}* < \infty)$ satisfying $p_2(t_{02}*) = 0$.

(iii) If $p_2(\infty) \geq 0$, then the optimum ordering time $t_{02}* = \infty$.

(2) Suppose that $h(t)$ is non-increasing. Then, $h(t)$ is either 0 or ∞.

Proof. The proof is omitted since it is similar to the proof Theorem 2.

Theorem 1 shows that either Policy 1 or Policy 2 is optimal, and Theorems 2 and 3 show the existence of the optimum ordering times for the two policies. The following theorem shows that which one is the global optimal ordering policy according to the cost parameters.

Lemma. For any fixed ordering time t_0, one and only one of the following three statements is true:

(a) $E_1(t_0) > E_2(t_0) > 1/(c_u + c_h + v_s)$

(b) $E_1(t_0) = E_2(t_0) = 1/(c_u + c_h + v_s)$

(c) $E_1(t_0) < E_2(t_0) < 1/(c_u + c_h + v_s)$

Proof. (a) Suppose that $E_1(t_0) \equiv E(t_0, \infty) > 1/(c_u + c_h + v_s)$. Since $A(t_0)$ in (11) can be rewritten as

$$A(t_0) = [1 - (c_u + c_h + v_s)E(t_0, t_1)] \cdot C(t_0, t_1), \tag{21}$$

$A(t_0) < 0$ and thus $\partial E(t_0, t_1)/\partial t_1 < 0$ for all $t_1 \in (t_0, \infty)$ in (10), which implies Policy 1 is optimal. Hence, if $E_2(t_0) > 1/(c_u + c_h + v_s)$, then $E_1(t_0) > E_2(t_0) > 1/(c_u + c_h + v_s)$.

Similarly, we can prove that (b) if $E_2(t_0) = 1/(c_u + c_h + v_s)$, then $E_1(t_0) = E_2(t_0) = 1/(c_u + c_h + v_s)$, and (c) if $E_2(t_0) < 1/(c_u + c_h + v_s)$, then $E_1(t_0) < E_2(t_0) < 1/(c_u + c_h + v_s)$.

Since the three conditions in the proofs of (a), (b), and (c) (namely, $E_2(t_0) > 1/(c_u + c_h + v_s)$, $E_2(t_0) = 1/(c_u + c_h + v_s)$, $E_2(t_0) < 1/(c_u + c_h + v_s)$) are mutually exclusive and exhaustive, one and only one of the three statements in the lemma is true.

Theorem 4. Let $t_{01}*$ and $t_{02}*$ be the optimum ordering times which maximize $E_1(t_0)$ and $E_2(t_0)$ respectively. Then, one and only one of the following three statements is true:

(a) $E_1(t_{01}*) > E_2(t_{02}*) > 1/(c_u + c_h + v_s)$

(b) $E_1(t_{01}*) = E_2(t_{02}*) = 1/(c_u + c_h + v_s)$

(c) $E_1(t_{01}*) < E_2(t_{02}*) < 1/(c_u + c_h + v_s)$

Proof. (a) Suppose that $E_2(t_0*) > 1/(c_u + c_h + v_s)$. Then, from the Lemma, we obtain

$$E_1(t_{02}*) > E_2(t_{02}*) > 1/(c_u + c_h + v_s). \tag{22}$$

From the optimality of $E_1(t_0)$,

$$E_1(t_{01}*) \geq E_1(t_{02}*). \tag{23}$$

Applying inequality (23) to (22), it follows

$$E_1(t_{01}*) > E_2(t_{02}*) > 1/(c_u + c_h + v_s).$$

(b) Suppose that $E_2(t_0*) = 1/(c_u + c_h + v_s)$. Then, from the Lemma, we get

$$E_1(t_{02}*) = E_2(t_{02}*) = 1/(c_u + c_h + v_s), \tag{24}$$

and from the optimality of $E_2(t_0)$,

$$E_2(t_0) \le E_2(t_{02}*) = 1/(c_u + c_h + v_s) \text{ for all } t_0$$

which implies from the Lemma,

$$E_1(t_0) \le E_2(t_0) \le 1/(c_u + c_h + v_s) \text{ for all } t_0. \tag{25}$$

From (24) and (25), we get

$$E_1(t_{02}*) \ge E_1(t_0) \text{ for all } t_0.$$

Thus, $t_{02}*$ maximizing $E_2(t_0)$ also maximizes $E_1(t_0)$, i.e.,

$$E_1(t_{02}*) = E_1(t_{01}*). \tag{26}$$

Substituting (26) into (24), it follows

$$E_1(t_{01}*) = E_2(t_{02}*) = 1/(c_u + c_h + v_s).$$

(c) Suppose that $E_2(t_0*) < 1/(c_u + c_h + v_s)$. Then,

$$E_2(t_0) < 1/(c_u + c_h + v_s) \text{ for all } t_0$$

which implies from the Lemma,

$$E_1(t_0) < E_2(t_0) < 1/(c_u + c_h + v_s) \text{ for all } t_0.$$

Thus,

$$E_1(t_{01}*) < E_2(t_{02}*) < 1/(c_u + c_h + v_s).$$

Since the three conditions in the proofs of (a), (b), and (c) are mutually exclusive and exhaustive, this completes the proof.

Corollary. If $\dfrac{m}{c_0 + c_u m + c_d m_x} > \dfrac{1}{c_u + c_h + v_s}$, Policy 1 is optimal.

Proof. Since $E_2(t_{02}*) \ge E_2(\infty) = \dfrac{m}{c_0 + c_u m + c_d m_x}$, $E_2(t_{02}*) > \dfrac{1}{c_u + c_h + v_s}$ if $\dfrac{m}{c_0 + c_u m + c_d m_x} > \dfrac{1}{c_u + c_h + v_s}$. Thus, from (a) of Theorem 4, $\dfrac{m}{c_0 + c_u m + c_d m_x} > \dfrac{1}{c_u + c_h + v_s}$ is a sufficient condition for Policy 1 to be optimal.

4. Numerical Example

For the purpose of illustration let us consider the following case: Both the lifetime and lead time are gamma distributed with integer modulus.

Lifetime cdf $F(t) = 1 - [1 + 0.003t + (0.003t)^2/2]\exp(-0.003t)$, where mean $m = 1{,}000$.

Lead time cdf $G(x) = 1-(1+0.02x)\exp(-0.02x)$, where mean $m_x = 100$.

The cost parameters are $c_0 = \$8{,}000$, $c_u = \$10$, $c_d = \$80$, $c_h = \$20$, $v_s = \$5$.

In this case example, Policy 1 is optimal since $\dfrac{m}{c_0 + c_u m + c_d m_x} = \dfrac{1}{26} > \dfrac{1}{c_u + c_h + v_s} = \dfrac{1}{35}$.

Figure 1 shows how the expected cost rate $E_1(t_0)$ and $E_2(t_0)$ changes with respect to the scheduled ordering time t_0. The optimum ordering time $t_{01}* = 541$ and the corresponding cost rate is $E_1(t_{01}*) = 0.0414$.

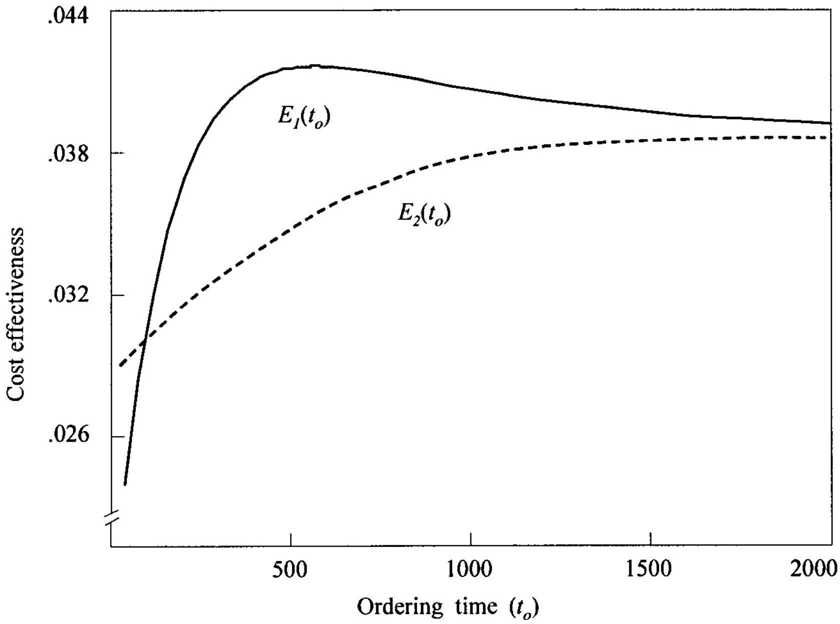

Figure 1. Cost effectiveness $E_1(t_0)$ as function of ordering time t_0.

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