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Geometric Hermite Curves Based on Curvature Variation Minimization

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Abstract – **Based on the smoothness criterion of minimum curvature variation of the curve, tangent angle constraints guaranteeing an optimized geometric Hermite (OGH) curve both mathematically and geometrically smooth is given, and new methods for constructing composite optimized geometric Hermite (COH) curves are presented in this paper. The comparison of the new methods with Yong and Cheng's methods based on strain energy minimization is included.**

***Key Words*: Curvature variation, Geometric smoothness, Hermite**

1. Introduction

Given endpoint conditions (positions and tangent vectors) can determine one and only one Hermite curve. Although the Hermite curve has the minimum strain energy (adopting the integrated squared second derivative of the curve as the approximation of the strain energy) among all C^1 cubic polynomial curves satisfying the same endpoint conditions, its shape may be unplea-sant. It may have loops, cusps or folds, namely, not geometrically smooth. Hence, additional degrees of freedom are needed to meet the geometric smoothness requirements. Obviously, adjusting the magnitudes of the given tangent vectors can make the Hermite curve geometrically smooth, and such Hermite curve is known as geometric Hermite curve.

Research on geometric Hermite curves can be classified into two categories. The first one focuses on building a low degree geometric Hermite curve with high order geometric continuity and approximation accuracy. The second one focuses on producing a G^1 geometric Hermite curve without loops, cusps and folds. Meek and Walton [6,7] use a T-cubic curve to get pleasing shape by implicitly restricting the directions of the input tangent vectors. T-cubic curves can be joined with circular arcs to form nice spirals if the curve segment is short enough. Yong and Cheng [9] present a new class of curves named optimized geometric Hermite (OGH) curves, such a curve is defined by optimizing the magnitudes of the endpoint tangent vectors in the Hermite interpolation process in order to minimize the strain energy of the curve, and they also give the geometric smoothness conditions and techniques for constructing 2-segment and 3-segment composite optimized geometric Hermite (COH) curves. Hence, an explicit way can be used to quantize the smoothness of a curve in the geometric Hermite interpolation process both mathematically and geometrically. However, the criterion to measure the smoothness of a curve is not unique, usually minimum strain energy (MSE) or minimum curvature variation (MCV) is used. Yong and Cheng use MSE as the smoothness criterion, but it cannot make the curvature variation minimum at the same time. Hence, the curves may have unpleasing shape in some cases.

For above disadvantage, MCV is used as the new smoothness criterion of curve in this paper and the integrated squared third derivative of curve is chosen as the approximate form of the curvature variation, i.e., object function. The extended definitions of OGH and COH curves and the tangent angle constraints (tangent direction preserving conditions and geometric smoothness conditions) under which an OGH curve would be mathematically and geometrically smooth are given. New methods for constructing 2-segment and 3-segment COH curves are presented, and by extension, they can cover tangent angles of all possible cases. Finally, the COH curves based on new criterion are compared with the COH curves by Yong and Cheng.

2. Description of the problem

The extended definition of the optimized geometric Hermite (OGH) curves based on MCV is given first.

Definition 1 Given two endpoints P_0 and P_1, and two endpoint tangent vectors V_0 and V_1, a cubic polynomial curve P(t), $t \in [t_0, t_1]$, is called an optimized geometric Hermite (OGH) curve with respect to the endpoint conditions $\{P_0, P_1, V_0, V_1\}$ if it has the smallest curvature variation among all cubic Hermite curves $\overline{P}(t)$, $t \in [t_0, t_1]$ satisfying the following conditions:

$$\overline{P}(t_0) = P_0,\ \overline{P}(t_1) = P_1, \overline{P}'(t_0) = \alpha_0 V_0, \overline{P}'(t_1) = \alpha_1 V_1 \qquad (1)$$

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where α_0 and α_1 are arbitrary real numbers, and the cubic Hermite curve $\overline{P}(t)$, $t \in [t_0, t_1]$, satisfying the constraints (1) can be expressed as

$$\overline{P}(t)=(2s+1)(s-1)^2p_0+(-2s+3)s^2P_1$$
$$+(1-s)^2s(t_1-t_0)\alpha_0V_0+(s-1)s^2(t_1-t_0)\alpha_1V_1 \quad (2)$$

where $s = (t-t_0)/(t_1-t_0)$. The object function, i.e., the approximate curvature variation of the curve $\overline{P}(t)$ on $[t_0, t_1]$ is defined as $E=\int_{t_0}^{t_1}[\overline{P}'''(t)]^2dt$, where $\overline{P}'''(t)$ is the third derivative of $\overline{P}(t)$.

Such an OGH curve absolutely exists and has the smallest curvature variation, i.e., the curve is mathematically smooth under such smoothness criterion. However, there exist two matters: one is that the endpoint tangent vectors of the OGH curve may be opposite to the given endpoint tangent vectors; and the other is that it may have loops, cusps or folds. Either of these is certainly not desired. So we should discuss the tangent angle constraints that ensure tangent direction preserving and geometric smoothness of the OGH curve.

3. Tangent angle constraint

The explicit values of α_0 and α_1 which define the OGH curve $P(t)$ can be got easily from definition 1. The theorem is as follows:

Theorem 1 Given two endpoints P_0 and P_1, two endpoint tangent vectors V_0 and V_1, and a parameter space $[t_0, t_1]$, the value of α_0 and α_1 related to an OGH curve P(t), $t \in [t_0, t_1]$ with respect to the endpoint conditions $\{P_0, P_1, V_0, V_1\}$ is obtained as follows:
if V_0 and V_1 are unparallel, then

$$\begin{cases}\alpha_0=\dfrac{2[(P_1-P_0)\cdot V_0]V_1^2-2[(P_1-P_0)\cdot V_1](V_0\cdot V_1)}{[V_0^2(V_1^2)-(V_0\cdot V_1)^2](t_0-t_1)}\\ \alpha_1=\dfrac{2[(P_1-P_0)\cdot V_1]V_1^2-2[(P_1-P_0)\cdot V_0](V_0\cdot V_1)}{[V_0^2(V_1^2)-(V_0\cdot V_1)^2](t_0-t_1)}\end{cases} \quad (3)$$

if V_0 are V_1 parallel, then α_0 and α_1 satisfy the equation

$$\alpha_0V_0^2+\alpha_1V_0\cdot V_1=\frac{2(P_1-P_0)\cdot V_0}{t_1-t_0} \quad (4).$$

Proof. From Eq. (2), the curvature variationof E of $\overline{P}(t)$ can be represented as a function of α_0 and α_1 as follows:

$$E=\frac{144}{(t_1-t_0)^5}(P_1-P_0)^2-\frac{144}{(t_1-t_0)^4}(P_1-P_0)(\alpha_0V_0+\alpha_1V_1)$$
$$+\frac{36}{(t_1-t_0)^3}(\alpha_0V_0+\alpha_1V_1)^2$$

The optimization problem is equivalent to finding the minimum point of the above equation. Theorem 1 can be obtained by solving the corresponding linear equations.

α_0 and α_1 defined in Eqs. (3) and (4) are called the optimized coefficients of the tangent vectors of $P(t)$ at t_0 and t_1, respectively. Obviously, α_0 and α_1 are not necessarily positive, so the magnitudes of the endpoint tangent vectors of the OGH curve may be zero, or the directions may be opposite to the given tangent vectors. Neither of these is desired. Hence, we discuss the tangent angle conditions ensuring α_0 and α_1 positive, subsequently.

Theorem 2 $P(t)$, $t \in [t_0, t_1]$, is an OGH curve with respect to the endpoint conditions $\{P_0, P_1, V_0, V_1\}$, α_0 and α_1 are the optimized coefficients of the tangent vectors of $P(t)$ at t_0 and t_1, respectively. α_0 and α_1 are positive if and only if the following tangent direction preserving conditions

$$\begin{aligned}&\sin(\theta-\varphi)\neq 0 \text{ and } \cos\theta > \cos(\theta-2\varphi)\\ &\qquad\text{and } \cos\varphi > \cos(\varphi-2\theta)\\ &\text{or } \theta=\varphi \quad \text{and } \cos\theta>0\\ &\text{or } \theta-\varphi=\pm\pi\end{aligned} \quad (5)$$

are satisfied, where or θ is the counterclockwise angle form from vector $\overrightarrow{P_0P_1}$ to V_0, φ is the counterclockwise angle from vector to $\overrightarrow{P_0P_1}$ to V_1, $\theta, \varphi \in [0,2\pi)$, and θ, φ are named tangent angles.

Proof. Without loss of generality, we assume $P_0=[0,0]^T$, $P_1=[1,0]^T$, V_0 and V_1 are both unit vectors. Thus, $V_0=[\cos\theta, \sin\theta]^T$, $V_1=[\cos\varphi, \sin\varphi]^T$. We discuss in two cases:

1) When V_0 and V_1 are unparallel, substitute the coordinates of P_0, P_1, V_0, V_1 into Eq.(3), we obtain

$$\begin{cases}\alpha_0=\dfrac{2[\cos\theta-\cos\varphi\cos(\theta-\varphi)]}{\sin^2(\theta-\varphi)(t_1-t_0)}\\ \alpha_1=\dfrac{2[\cos\varphi-\cos\theta\cos(\theta-\varphi)]}{\sin^2(\theta-\varphi)(t_1-t_0)}\end{cases}$$

Obviously, V_0 are V_1 unparallel if and only if $\sin(\theta-\varphi)\neq 0$, which can ensure denominator of the above two equations not to be zero. Therefore,

$\alpha_0>0$, if and only if $\cos\theta-\cos\varphi\cos(\theta-\varphi)>0$,

$\alpha_1>0$, if and only if $\cos\varphi-\cos\theta\cos(\theta-\varphi)>0$.

Simplifying the above two inequalities, we get

$\cos\theta>\cos(\theta-2\varphi)$ and $\cos\varphi>\cos(\varphi-2\theta)$.

2) When V_0 and V_1 are parallel, i.e., $\sin(\theta-\varphi)=0$, α_0 and α_1 merely satisfy Eq. (4), substitute the coordinates of P_0, P_1, V_0, V_1 into Eq. (4), we obtain

$$\alpha_0+\alpha_1\cos(\theta-\varphi)=2\cos\theta/(t_1-t_0) \quad (6)$$

a) If V_0 and V_1 are in the same direction, i.e., $\theta=\varphi$, then Eq. (6) is equivalent to

$$\alpha_0+\alpha_1=2\cos\theta/(t_1-t_0).$$

Thus, $\alpha_0,\alpha_1>0$ if and only if $\cos\theta>0$.

b) If V_0 and V_1 are in the opposite direction, i.e., $\theta-\varphi=\pm\pi$, then Eq. (6) is equivalent to

$$\alpha_0-\alpha_1=2\cos\theta/(t_1-t_0).$$

Obviously, whatever θ is, we can always find positive α_0 and α_1 that satisfy the equation.

Summarizing the above two cases 1) and 2), we obtain the conclusion of theorem 2.

If the given tangent angles satisfy the condition (5), then the constructed OGH curve can preserve directions of the given tangent vectors and has the smallest curvature variation, i.e., mathematically smooth, but it may not be geometrically smooth, i.e., it may have loops, cusps or folds. So we discuss the geometric smoothness conditions below.

Theorem 3 $P(t)=[x(t),y(t)]^T$, $t\in[t_0,t_1]$ is an OGH curve with respect to the endpoint conditions $\{P_0,P_1,V_0,V_1\}$, then $P(t)$ is geometric smoothness if it satisfies the conditions:

$$\sin(\theta-\varphi)\neq 0 \textit{ and } \tan\theta\tan\varphi<0 \qquad (7)$$

$$\text{or } \sin(\theta-\varphi)=0 \textit{ and } \tan 0<\alpha_0\cos\theta<2\cos^2\theta$$

where α_0 is the optimized coefficient of the tangent vector of $P(t)$ at t_0, θ,φ are defined in theorem 2. conditions (7) is called the geometric smoothness conditions.

Proof. To make the OGH curve geometrically smooth, i.e., loop-, cusp- and fold-free, it's sufficient to guarantee $\forall t\in[t_0,t_1]$, $x'(t)>0$ (or $x'(t)<0$). Then obviously, there aren't any points on $[t_0,t_1]$ making $x'(t)=0$, thus $P(t)$ doesn't have any cusps. Moreover, $x(t)$ is an increasing (or decreasing) function in this case, thus $P(t)$ doesn't have any loops or folds.

Without loss of generality, we assume $P_0=[0,0]^T$, $P_1=[1,0]^T$, $[t_0,t_1]=[0,1]$, and V_0 and V_1 are both unit vectors, then $V_0=[\cos\theta,\sin\theta]^T$, $V_1=[\cos\varphi,\sin\varphi]^T$. Obviously, all these assumptions don't change the sign of $x'(t)$. Hence, we have

$$x(t)=[-2+\alpha_0\cos\theta+\alpha_1\cos\varphi]t^3+[3-2\alpha_0\cos\theta-\alpha_1\cos\varphi]t^2+\alpha_0\cos\theta t$$

then

$$x(t)=3[-2+\alpha_0\cos\theta+\alpha_1\cos\varphi]t^2+2[3-2\alpha_0\cos\theta-\alpha_1\cos\varphi]t+\alpha_0\cos\theta \qquad (8)$$

We discuss in two cases:

1) Whileare V_0 and V_1 unparallel, i.e.,, $\sin(\theta-\varphi)\neq 0$, α_0 and α_1 are obtained from Eqs. (3), i.e.,

$$\begin{cases}\alpha_0=\dfrac{2[\cos\theta-\cos\varphi\cos(\theta-\varphi)]}{\sin^2(\theta-\varphi)}\\ \alpha_1=\dfrac{2[\cos\varphi-\cos\theta\cos(\theta-\varphi)]}{\sin^2(\theta-\varphi)}\end{cases}$$

Substituting to (8), we get

$$x'(t)=\frac{2[\cos^2\varphi-\cos^2\varphi]}{\sin^2(\theta-\varphi)}t-\frac{2\cos\theta\sin\varphi}{\sin(\theta-\varphi)}.$$

Let notate the coefficient of t in $x'(t)$, then

a) When $\cos^2\varphi=\cos^2\theta$, $A=0$. Then, $\forall t\in[0,1]$, $x'(t)>0$ (or 0), as long as

$$\frac{2\cos\theta\sin\varphi}{\sin(\theta-\varphi)}>0(\text{or}<0),\text{ i.e., }\cos\theta\sin\varphi\neq 0.$$

Obviously, the condition is true if $\tan\theta\tan\varphi<0$.

b) When $\cos^2\varphi\neq\cos^2\theta$, $A\neq 0$. Then $\forall t\in[0,1]$, $x'(t)>0$ (or <0), as long as

$$\begin{cases}x'(0)=\dfrac{2\cos\theta\sin\varphi}{\sin(\theta-\varphi)}\\ x'(1)=\dfrac{2\sin\theta\cos\varphi\sin(\theta-\varphi)}{\sin^2(\theta-\varphi)}>0\end{cases}$$

or

$$\begin{cases}x'(0)=\dfrac{2\cos\theta\sin\varphi}{\sin(\theta-\varphi)}<0\\ x'(1)=\dfrac{2\sin\theta\cos\varphi\sin(\theta-\varphi)}{\sin^2(\theta\ \varphi)}<0\end{cases}$$

the above inequalities are equivalent to $\dfrac{\cos\theta\sin\varphi}{\sin\theta\cos\varphi}<0$, i.e., $\tan\theta\tan\varphi<0$.

2) While V_0 and V_1 are parallel, i.e., $\sin(\theta-\varphi)=0$, $\theta_0$and α_1 merely satisfy Eq. (4).

① If V_0 and V_1 are in the same direction, i.e.,$\theta=\varphi$, Eq.(4) is equivalent to $\alpha_0+\alpha_1=2\cos\theta$.

Substituting to (8), we get

$$x'(t)=-6\sin^2\theta t^2+2(3-\alpha_0\cos\theta-2\cos^2\theta)t+\alpha_0\cos\theta$$

Let B notate the coefficient of t^2 in $x'(t)$. Obviously, $B\leq 0$. Thus,

a) When $\theta=0$, $B=0$, then $\forall t\in[0,1]$, $x'(t)>0$ (or <0), as long as

$$\begin{cases}x'(0)=\alpha_0>0\\ x'(1)=2-\alpha_0>0\end{cases}\quad\text{or}\quad\begin{cases}x'(0)=\alpha_0<0\\ x'(1)=2-\alpha_0<0\end{cases},$$

Simplifying the above inequalities, we get $0<\alpha_0<2$.

b) When $\theta=\pi$, $B=0$, then $\forall t\in[0,1]$, $x'(t)>0$ (or <0), as long as

$$\begin{cases}x'(0)=-\alpha_0>0\\ x'(1)=2+\alpha_0>0\end{cases} \quad\text{or}\quad \begin{cases}x'(0)=-\alpha_0<0\\ x'(1)=2+\alpha_0<0\end{cases},$$

Simplifying the above inequalities, we get $-2 < \alpha_0\ 0$.

c) When $\sin\theta \neq 0$, $B < 0$, then $\forall\ t \in [0,1]$, $x'(t) > 0$ (or < 0), as long as

$$\begin{cases}x'(0)=\alpha_0\cos\theta>0\\ x'(1)=2\cos^2\theta-\alpha_0\cos\theta>0\end{cases}$$

or

$$\begin{cases}x'(0)=\alpha_0\cos\theta<0\\ x'(1)=2\cos^2\theta-\alpha_0\cos\theta<0\end{cases}.$$

Simplifying the above inequalities, we get $0 < \alpha_0 \cos\theta < 2\cos^2\theta$.

Obviously, the conclusions of a) and b) can be included in c), i.e., $0 < \alpha_0 \cos\theta < 2\cos^2\theta$.

② If V_0 and V_1 are in the opposite direction, i.e., $\theta - \varphi = \pm\pi$, the proof process is similar to ①.

Summarizing the above two cases 1) and 2), we obtain the conclusion of Theorem 3.

From Theorems 2 and 3, we conclude that conditions (5) and (7) can both be satisfied when $(\theta,\varphi) \in (0, \pi/2) \times (3\pi/2, 2\pi) \cup (3\pi/2, 2\pi) \times (0, \pi/2)$. Obviously, if the given tangent angles are in the region, the corresponding OGH curve is the most ideal for having minimum curvature variation, preserving tangent vector directions and loop-, cusp- and fold-free. While the curve is not pleasing if either (5) or (7) cannot be satisfied, hence, we should consider COH curves, which can achieve the whole smoothness requirements by ensuring automatic satisfaction of conditions (5) and (7) for each OGH segment.

4. Methods for constructing COH curves

The definition of a COH curve is as follows:

Definition 2 A piecewise cubic polynomial curve is called a composite optimized geometric Hermite (COH) curve if the curve is G^1 and each segment of the curve is an OGH curve.

Below, for the given tangent angles not satisfying (5) and (7) simultaneously, the corresponding methods for constructing 2-segment or 3-segment COH curves will be given. Firstly, methods for constructing 2-segment COH curves are given. In these methods, the joint and the tangent vector at the joint of the two OGH segments are denoted Q and V_Q, respectively. The counterclockwise angles at the endpoints of these OGH segments with respect to their base lines are denoted ϕ_1, ϕ_2, ϕ_3 and ϕ_4 respectively (see Fig.1 (1)-(2)).

Method M1. If $(\theta, \varphi) \in [0, \pi/2) \times (0, \pi/2)$, then Q are V_Q determined by setting

$$\phi_1 = \begin{cases}\theta, & \theta\in(0,\pi/2)\\ \varphi/4, & \theta=0\end{cases}$$

Q, on the perpendicular bisector of $\overline{P_0P_1}$, and

$$\phi_2 = \begin{cases}(\theta+\varphi)/3, & \theta\in(0,\pi/2)\\ (\theta+\varphi)/6, & \theta=0\end{cases}.$$

Method M2. If $(\theta, \varphi) \in [0, \pi/2) \times (\pi, 3\pi/2)$, then Q and V_Q are determined by setting

$$\phi_1 = \theta/2,\ \phi_4 = (2\pi - \varphi)/2, \text{ and } \phi_2 = \phi_3.$$

Methods for constructing 3-segment COH curves are given next. In these methods, the joints and the tangent vectors at the joints of the three OGH segments are donatedand Q_0, Q_1 and V_{q0}, V_{q1}, respectively. The counterclockwise angles at the endpoints of these OGH segments with respect to their base lines are donated ϕ_1, ϕ_2, ϕ_3, ϕ_4 ϕ_5 and ϕ_6 respectively (see Fig. 1 (3)-(6)).

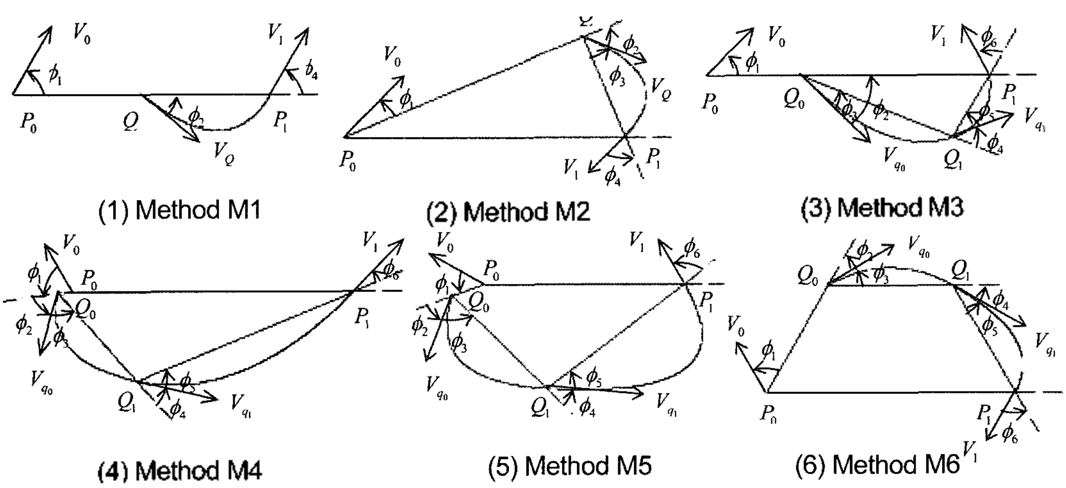

Fig. 1. Methods for constructing 2-segment and 3-segment COH curves.

Method M3. If $(\theta, \varphi) \in [0, \pi/2) \times (\pi/2, \pi)$, then Q_0, Q_1, V_{q0}, V_{q1} are determined by setting

$$\phi_1 = \begin{cases} \theta, & \theta \in (0, \pi/2) \\ \pi/18, & \theta = 0 \end{cases}, \left|\overline{P_0Q_0}\right| = \left|\overline{P_1P_1}\right|/3, \phi_2 = \phi_1,$$

$$\phi_3 = \frac{\phi_2}{2}, \phi_6 = \begin{cases} \phi/2, & \theta \in (\pi/2, \pi) \\ 4\pi/9, \end{cases}, \text{ and } \phi_4 = \phi_5.$$

Method M4. If $(\theta, \varphi) \in [\pi/2, \pi) \times (0, \pi/2]$, then Q_0, Q_1, V_{q0}, V_{q1} are determined by setting

$$\phi_1 = \begin{cases} \pi/3, & \theta = \pi/2 \\ 7\pi/8 - 3\theta/4, & \theta \in (\pi/2, \pi) \end{cases}, \left|\overline{P_0Q_0}\right| = \left|\overline{P_0P_1}\right|/8,$$

$\phi_2 = \phi_1$, $\overrightarrow{Q_0Q_1}$ bisecting the counterclockwise angle from V_{q0} to $\overrightarrow{Q_0P_0}$, $\phi_6 = \phi/2$, and $\phi_4 = \phi_5$.

Method M5. If $(\theta, \varphi) \in (\pi/2, \pi] \times (\pi/2\pi,)$, then Q_0, Q_1, V_{q0}, V_{q1} are determined by setting

$\phi_1 = 5\pi/6 - 2\theta/3, \left|\overline{P_0Q_0}\right| = \left|\overline{P_0P_1}\right|/6$, $\phi_2 = \phi_1$, $\overline{Q_0Q_1}$ bisecting the counterclockwise angle from V_{q0} to $\overrightarrow{Q_0P_1}$,

$$\phi_6 = \begin{cases} 5\pi/8 - \varphi/4, & \varphi \in [17\pi/30, \pi] \\ \varphi - \pi/12, & \varphi \in [\pi/2, 17\pi/30] \end{cases}, \text{ and } \phi_4 = \phi_5.$$

Method M6. If $(\theta, \varphi) \in (\pi/2, \pi] \times [\pi/3\pi/2)$, then Q_0, Q_1, V_{q0}, V_{q1} are determined by setting

$$\phi_1 = \begin{cases} \theta/2, & \theta \in (\pi/2, \pi) \\ 7\pi/16, & \theta = \pi \end{cases}, \left|\overline{P_0Q_0}\right| = \left|\overline{P_0P_1}\right|/2, \overrightarrow{Q_0Q_1} \,//$$

$$\overrightarrow{P_0P_1}, \phi_6 = \begin{cases} \pi - \varphi/2, & \varphi \in (\pi, 3\pi/2) \\ 7\pi/16, & \varphi = \pi \end{cases}, \phi_2 = \phi_3, \text{ and } \phi_4 = \phi_5$$

It can be easily obtained that the tangent angles of each OGH segment of the COH curves generated by above methods are all in the region $(0, \pi/2) \times (3\pi/2, 2\pi) \cup (3\pi/2, 2\pi) \times (0, \pi/2)$, so these methods can guarantee automatic satisfaction of conditions (5) and (7) for each segment and consequently, the satisfaction of the whole smoothness requirement of the COH curve.

5. Extension of the constructing methods

Let M0 denote the method generating OGH curves (i.e., 1-segment COH curves). Obliviously, the above seven methods M0~M6 cannot cover the entire $\theta\varphi$-space, $[0.2\pi) \times 0,2\pi)$. So we extend the methods as follows:

$P(t)$ is an COH curve generated by method M (shown in Fig. 2), $P^T(t)$ is a new curve symmetric to $P(t)$ with respect to the base line of $P(t)$, $P^R(t)$ is a new curve reversing $P(t)$, and $P^{RT}(t)$ is a new curve symmetric to $P^R(t)$ with respect to the base line of $P^R(t)$, then the methods generating $P^T(t)$, $P^R(t)$ and $P^{RT}(t)$ are called M^T, M^R and M^{RT}, respectively.

If the applicable region of method is $(\theta, \varphi) \in [\theta_1, \theta_2]$, $\times [\varphi_1, \varphi_2]$, then the applicable regions of methods M^T, M^R and M^{RT} are $(\theta^T, \varphi^T) \in [2\pi - \theta_2, 2\pi - \theta_1,] \times [2\pi - \varphi_2, 2\pi - \varphi_1]$, $(\theta^R, \varphi^R) \in [2\pi - \varphi_2, 2\pi - \varphi_1,] \times [2\pi - \theta_2, 2\pi - \theta_1]$ and $(\theta^{RT}, \varphi^{RT}) \in [\varphi_1, \varphi_2] \times [\theta_1, \theta_2]$ respectively, all these regions are generally called extension regions of M. Obviously, after above extension, methods M_i, M_i^T, M_i^R, and M_i^{RT}, $i = 0,1...6$, with the addition of $\theta = 0$, $\varphi = 0$ (OGH curve is a line in this case) can cover the entire $\theta\varphi$-space, $[0,2\pi) \times [0,2\pi)$.

6. Comparison of the COH Curves with Different Object Functions

The discussion above gives the tangent angle constraints that ensure the OGH curves with the object function of curvature variation (i.e., based on the smoothness criterion of MCV) mathematically and geometrically smooth and new methods for constructing COH curves. Following these curves are compared with those COH curves based on MSE by Yong and Cheng.

As shown in Fig. 3, (a)-(h) are examples of the COH curves based on MCV being better than those based on MSE; (i)-(n) are examples of the COH curves based on MSE being better than those based on MCV; and (o)-(p) are Examples of the COH curves with two object functions both being unpleasing. Note : Symbols ① and ② in figure denote the COH curve based on MCV and the COH curve based on MSE, respectively.

From the examples, we can draw the conclusion as follows:

when

$$(\theta, \varphi) \in [0, \pi/3] \times [0, \pi] \cup [2\pi/3, \pi] \times [\pi, 3\pi/2] \cup (\pi/2, 2\pi/3)$$

$$(\pi/2, 2\pi/3) \times [\pi, 4\pi/3] \cup [2\pi/3, \pi] \times [\pi/6, \pi/2] \cup [\pi/(3, 2\pi/3)]$$

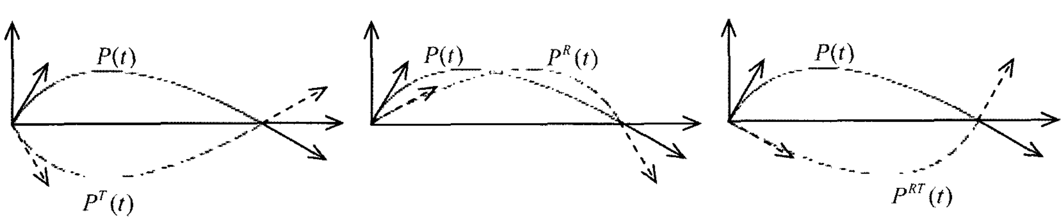

Fig. 2. Extension of methods for constructing COH curves.

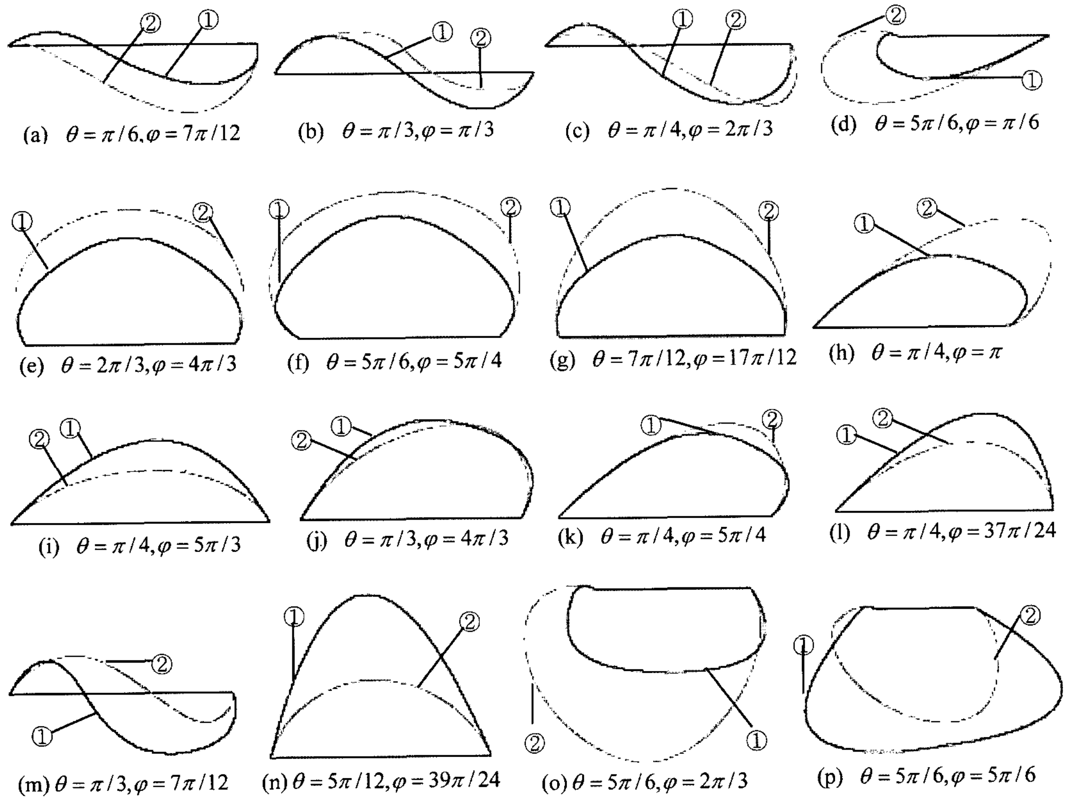

Fig. 3. Comparison of the COH curves with different object functions.

$\times[4\pi/3, 3\pi/2]$, Shapes of the COH curves based on MCV are more pleasant.

When,
$(\theta,\varphi)\in[0,\pi/3]\times[\pi,2\pi]\cup[\pi/3,2\pi/3]\times[3\pi/2,5\pi/3]$

$\cup[\pi/3,\pi/2]\times[\pi,4\pi/3]\cup[\pi/3,2\pi/3]\times[\pi/3,2\pi/3]$, shapes of the COH curves based on MSE are more pleasant.

When $(\theta,\varphi)\in[2\pi/3,\pi]\times(\pi/2\pi)$, shapes of the COH curves based on MCV and MSE are both some unpleasing.

We mark the above three regions R1, R2 and R3. After extension, they can cover tangent angles of all possible cases, and the results of comparisons in the extension regions are the same as their respective original regions. Furthermore, the proportion of the region R1 to the entire -space is approximately equal to that of the region R2.

7. Conclusion

The comparison above shows that the smoothness criterion of curves is not unique, so different ones should be adopted in different cases to achieve more pleasing shapes. This paper gives the conclusion that which criterion generates better curves when the tangent angles in different regions. Our discussion shows that the combination of the new methods with the Yong and Cheng's methods can achieve a much better result. To the region in which curves based on two criterions are both unpleasant, new criterion should be considered, such as minimum curve length, that's the future work.

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