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Characterization Results of the NBUCA Class of Life Distributions

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Abstract. In this paper, some characterization results of the $NBUCA$ class of life distributions are obtained. Behavior of the life distributions of the present class is developed in terms of the monotonicity of the residual life of k-out of-n systems given that the $(n-k)$th failure has occurred at time $t \geq 0$. Similar conclusions based on the residual life of parallel system are also presented. Next, we focus upon the aging process of a system with independent but not necessarily identical $NBUCA$ components. Finally, it is proved that if the lifetimes of a series systems with a random number of identical components have the $NBUCA$ property then its units also have the same property.

Key Words : *stochastic order, increasing convex order, k-out of-n systems, NBU, $NBUC$, $NBUCA$.*

1. INTRODUCTION

During the past decades, various classes of life distributions have been proposed in order to model different aspects of aging. Some of such classes are defined by stochastic comparisons of the residual life of a used unit with the lifetime of a new one. In this section, we present some of these comparisons and such classes (see Shaked and Shanthikumar (1994) and Ahmad et al.(2006)). Throughout this paper,

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X and Y are two non-negative random variables representing equipment lives, F and G are their corresponding distributions, and $\overline{F}$ and $\overline{G}$ are corresponding survival functions. Also, $=_{ST}$ denotes equality in law or distribution. We say that a random variable X is smaller than Y in the:

(i) stochastic order (denoted by $X \leq_{ST} Y$), if

$$\overline{F}(x) \leq \overline{G}(x), \text{ for all } x \ ;$$

(ii) increasing convex order (denoted by $X \leq_{ICX} Y$), if

$$\int_x^{\infty} \overline{F}(u)du \leq \int_x^{\infty} \overline{G}(u)du, \quad \text{for all } x;$$

(iii) increasing convex average order (denoted by $X \leq_{ICXA} Y$), if

$$\int_0^{\infty} \int_x^{\infty} \overline{F}(u)\ du\ dx \leq \int_0^{\infty} \int_x^{\infty} \overline{G}(u)\ du\ dx, \quad \text{for all } x.$$

In the economics theory, the above orders are respectively known as *first-order stochastic dominance* denoted by $X\ FSD\ Y$, *second-order stochastic dominance* denoted by $X\ SSD\ Y$, and *weak third-order stochastic dominance* denoted by $X\ WTSD\ Y$ (for more details, see Deshpande et al. (1986), Kaur et al. (1994) and Ahmad et al. (2006)).

For any random variable X, let

$$X_t = [X - t | X > t], \quad t \in \{x : F(x) < 1\},$$

denote a random variable whose distribution is the same as the conditional distribution of $X - t$ given that $X > t$. When X is the lifetime of a device, X_t can be regarded as the residual lifetime of the device at time t, given that the device has survived up to time t. The comparison of the residual life at different times has been used to give definitions of aging classes. We say that X is

(i) new better than used (denoted by $X \in NBU$) if

$$X_t \leq_{ST} X, \quad \text{for all } t \geq 0;$$

(ii) new better than used in the increasing convex order (denoted by $X \in NBUC$) if

$$X_t \leq_{ICX} X, \quad \text{for all } t \geq 0.$$

The classes NBU and $NBUC$ have proved to be very useful in performing analyses of life lengths as well as usable in replacement policies. Hence a lot of results related to these two classes have been obtained in the literature (see for instance, Bryson and Siddiqui (1969), Barlow and Proschan (1981), Cao and Wang (1991), Li et al. (2000), Franco et al. (2001) and Hu and Xie. (2002)).

Along a similar line, Ahmad et al. (2006) proposed the new better than used in convex average order ($NBUCA$) class of life distributions, expanding the $NBUC$ class to a much bigger and a more practical one, whose definition is also recalled here.

Definition 1.1. A non-negative random variable X is said to be new better than used in the increasing convex average order denoted by $NBUCA$ if

$$X_t \leq_{ICXA} X,$$

equivalently, $X \in NBUCA$ iff

$$\int_0^\infty \int_x^\infty \overline{F}(u+t)\, du\, dx \leq \overline{F}(t) \int_0^\infty \int_x^\infty \overline{F}(u)\, du\, dx \quad \text{for all } t \geq 0. \tag{1.1}$$

It is obvious that, $X \in NBUCA$ if, and only if

$$X_t \;\; WTSD \;\; X \quad \text{for all } t \geq 0.$$

Its dual class is new worse than used in increasing convex average ordering, denoted by $NWUCA$, which is defined by reversing the inequality in (1.1). The following chain of implications can be easily established,

$$NBU \;\Rightarrow\; NBUC \;\Rightarrow\; NBUCA.$$

Recently, many authors have centered their attention to study the behavior of aging properties in coherent structure such as parallel (series) systems, k-out-of-n systems, convolution, mixtures, and renewal process. For more details, see Barlow and Proschan (1981), Chen (1994), Belzunce et al.(1999), Pellerey and Petakos (2002), Hu and Xie (2002), Li and Chen (2004) and Ahmad et al. (2006).

In this investigations, some characterization results of the $NBUCA$ class of life distributions are obtained. In the next section, we present behavior of the life distributions, of the present class, in terms of the monotonicity of the residual life of a k-out of-n system given that the $(n-k)$th failure has occurred at time $t \geq 0$. Similar conclusions based on the residual life of parallel systems are presented as well. We also focus upon the aging process of a system with independent but not necessarily identical $NBUCA$ components. Further, it is proved that if the lifetimes of a series systems with the random number of identical components have the $NBUCA$ property then its units also have the same property. Finally, in Section 3 we provide a brief conclusion and remarks of the current and/or future research.

2. CHARACTERIZATIONS RESULTS

The k-out of-n system consists of n independent and identically distributed components and works as long as at least k components are working; that is, it works if at most $n-k$ components have failed. Thus, the life of a k-out of-n system can be characterized by the $(n-k+1)$th order statistic $X_{n-k+1,n}$. In fact, a series system is an n-out-of-n system and a parallel system is a 1-out-of-n system. Given that the $(n-k)$th failure has occurred, then the system will fail when the $(n-k+1)$th failure occurs. Thus, in order to understand how the aging property of the elements affect the aging procedure of the total life of the whole system, it is of special interest to study the aging procedure of the residual life after the $(n-k)$th failure.

In fact, given that the $(n-k)$th failure has occurred at time $t \geq 0$, the residual life of a k-out-of-n system is given as the following conditional random variable,

$$\begin{aligned} RLS_{k,n,t} &= (X_{n-k+1,n} - X_{n-k,n} | X_{n-k,n} = t) \\ &= \min_{1 \leq i \leq k} (X_i)_t, \end{aligned}$$

where $(X_1)_t, ..., (X_k)_t$ are k independent identically distributed $(i.i.d)$ copies of X_t. The life of a series system of k $i.i.d$ components is

$$LS_k = X_{1,k} = \min \{X_1, ..., X_k\}.$$

Their respective survival functions are

$$\begin{aligned} P(RLS_{k,n,t} > x) &= \left[\frac{\overline{F}(x+t)}{\overline{F}(t)} \right]^k \\ &= \overline{F}_t^k(x), \qquad x \geq 0, \end{aligned}$$

and

$$P(LS_k > x) = \overline{F}^k(x).$$

In this way, using stochastic comparisons, some authors characterized several aging distributions by the stochastic ordering of the residual life of the k-out-of-n system, given that the $(n-k)$th failure has occurred at different times. In particular, Langberg et al. (1980) presented the following characterizations,

$$X \in NBU \Leftrightarrow RLS_{k,n,t} \leq_{ST} LS_k, \text{ for all } t \geq 0,$$

and for any integer k such that $1 \leq k < n$.

Afterward, Belzunce et al. (1999), Li and Zuo (2002) and Li and Chen (2004) provided additional results on some other stochastic orders and aging notions. To state and prove our results, we need to recall the following lemma (see Ahmad et al. (2006)).

Lemma 2.1. Assume that $W(x)$ is a *Lebesgue- Stieltjes* measure, not necessarily positive. If $h(x)$ is non-negative and increasing, and

$$\int_0^\infty \int_t^\infty dW(x)dt \geq 0, \quad \text{for all } t \geq 0,$$

then $\int_0^\infty \int_t^\infty h(x)dW(x)dt \geq 0$.

Theorem 2.1. For any integer $1 \leq k < n$ and for all $t \geq 0$, if $RLS_{k,n,t} \leq_{ICXA} LS_k$ then X is $NBUCA$.

Proof. First, note that

$$h(x) = \left[\overline{F}^{k-1}(x) + \overline{F}^{k-2}(x)\overline{F}_t(x) + ... + \overline{F}(x)\overline{F}_t^{k-2}(x) + \overline{F}_t^{k-1}(x)\right]^{-1} \tag{2.1}$$

is non-negative and increasing in $x \geq 0$. By the assumption $RLS_{k,n,t} \leq_{ICXA} LS_k$ for all $t \geq 0$, we have

$$\int_0^\infty \int_y^\infty \left[\overline{F}^k(x) - \overline{F}_t^k(x)\right] dx\, dy \geq 0,$$

and by (2.1) and Lemma 2.1, it follows that,

$$\int_0^\infty \int_y^\infty \left[\overline{F}(x) - \overline{F}_t(x)\right] dx\, dy \geq 0, \quad \text{for all } t \geq 0$$

and hence X is $NBUCA$.

On the other hand, the residual life of a parallel system given the $(n-r)$th failure has occurred at time $t \geq 0$,

$$\begin{aligned} RLP_{r,n,t} &= (X_{n,n} - X_{n-r,n} | X_{n-r,n} = t) \\ &= \max_{1 \leq i \leq r} (X_i)_t\ , \end{aligned}$$

and the life length of the parallel system of r components

$$LP_r = X_{r,r} = \max\{X_1, ..., X_r\}$$

have their survival functions as, for all $x \geq 0$,

$$\begin{aligned} P(RLP_{r,n,t} &> x) = P\left(X_{n,n} - X_{n-r,n} > x | X_{n-r,n} = t\right) \\ &= 1 - F_t^r(x), \end{aligned}$$

and

$$P(LP_r > x) = 1 - F^r(x),$$

where $F_t(x) = 1 - \overline{F}_t(x)$.

The next theorem gives behavior of $NBUCA$ in terms of the conditioned residual life of a parallel system.

Theorem 2.2. For any integer $1 \leq r < n$ and for all $t \geq 0$, if X is $NBUCA$ then

$$RLP_{r,n,t} \leq_{ICXA} LP_r.$$

Proof. The $NBUCA$ property states that, for all $t \geq 0$

$$\int_0^\infty \int_y^\infty \left[\overline{F}(x) - \overline{F}_t(x)\right] \, dx \, dy = \int_0^\infty \int_y^\infty \left[F_t(x) - F(x)\right] \, dx \, dy \geq 0.$$

Since, for any $t \geq 0$, the function

$$g(x) = F^{r-1}(x) + F^{r-2}(x)F_t(x) + ... + F(x)F_t^{r-2}(x) + F_t^{r-1}(x)$$

is non-negative and increasing, it follows from Lemma 2.1, that for all $t \geq 0$

$$\int_0^\infty \int_y^\infty [P(LP_r > x) - P\left(RLP_{r,n,t} > x\right)] \, dx \, dy = \int_0^\infty \int_y^\infty [F_t^r(x) - F^r(x)] \, dx \, dy \geq 0$$

Thus, it holds that

$$RLP_{r,n,t} \leq_{ICXA} LP_r \quad \text{for all } t \geq 0,$$

and this completes the proof.

Suppose now that the system is composed of independent but not identical components. We will investigate the residual life of the 1-out of-n system given the first $n - r$ failed components and the $(n - r)$th failure at time t. For convenience, they are still denoted by $RLP_{r,n,t}$ and LP_r.

Given the first $n - r$ failed components and $(n - r)$th failure time t, there should exist some $(i_1, ..., i_r) \subset (1, ..., n)$ such that

$$P(LP_r \leq x) = \prod_{j=1}^{r} F_{i_j}(x),$$

and

$$P(RLP_{r,n,t} \leq x) = \prod_{j=1}^{r} F_{i_j,t}(x).$$

Next we give the following result.

Theorem 2.3. For any integer $1 \leq r < n$, if $X_i's$, $i = 1, 2, ..., n$, are all $NBUCA$, then

$$RLP_{r,n,t} \leq_{ICXA} LP_r \quad \text{for all } \; t \geq 0. \tag{2.2}$$

Proof. From the definition of the $NBUCA$ class we have, for all $t \geq 0$,

$$X_{i_j,t} \leq X_{i_j}, \quad j = 1, ..., r.$$

That is to say, for any $y \geq 0$,

$$\int_0^\infty \int_y^\infty \left[\overline{F}_{i_j}(x) - \overline{F}_{i_j,t}(x)\right] dxdy = \int_0^\infty \int_y^\infty \left[F_{i_j,t}(x) - F_{i_j}(x)\right] dxdy \geq 0.$$

Since $F_{i_2,t}(x)$ and $F_{i_1}(x)$ are non-negative and increasing, it follows from Lemma 2.1 that, for all $t \geq 0$,

$$\begin{aligned}\int_0^\infty \int_y^\infty & [F_{i_1,t}(x)F_{i_2,t}(x) - F_{i_1}(x)F_{i_2}(x)]\, dxdy \\ &= \int_0^\infty \int_y^\infty F_{i_2,t}(x)\,[F_{i_1,t}(x) - F_{i_1}(x)]\, dxdy \\ &\qquad + \int_0^\infty \int_y^\infty F_{i_1}(x)\,[F_{i_2,t}(x) - F_{i_2}(x)]\, dxdy \geq 0\,.\end{aligned}$$

By induction, we have for all $t \geq 0$,

$$\int_0^\infty \int_y^\infty \left[\prod_{j=1}^r F_{i_j,t}(x) - \prod_{j=1}^r F_{i_j}(x)\right] dxdy \geq 0,$$

and this is just (2.2).

Let now $X_1, X_2, \ldots$ be a sequence of independent and identical distributed (*i.i.d*) random variables and K be a positive integer-valued *random variable* which is independent of the X_i and have a discrete density function p_K. Put

$$X_{(1:K)} \equiv \min\{X_1, X_2, \ldots, X_K\}.$$

The random variables $X_{(1:K)}$ arise naturally in reliability theory as the lifetimes of a series systems, with the *random number* K of identical components with lifetimes $X_1, X_2, \ldots, X_K$. In life-testing, if a random censoring is adopted, then the completely observed data constitute a sample of random size, say $X_1, X_2, \ldots, X_K$, where $K > 0$ is a random variable of integer value. In survival analysis, $X_{(1:K)}$ arises naturally as the minimal survival time of a transplant operation, where K of them are defective and hence may cause death.

Our next result investigates the reversed preservation property of the increasing convex average order under series systems which are composed of a random number of *i.i.d.* components. The proof is similar to that of Theorem 2.2 in Ahmad and Kayid (2006) and hence is omitted.

Proposition 2.1. Let $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ each be a sequence of *i.i.d.* random variables copies of X and Y, respectively, and K is independent of X_i*'s* and Y_i*'s*. If $X_{(1:K)} \leq_{ICXA} Y_{(1:K)}$, then $X \leq_{ICXA} Y$.

Corollary 2.1. Let $X_1, X_2, \ldots,$ be a sequence of *i.i.d.* random variables copies of X, and K is independent of X_i*'s*. If $X_{(1:K)}$ is $NBUCA$, then X is also $NBUCA$.

Proof. First, note that $X_{(1:K)}$ is $NBUCA$, then it holds that,

$$\left(X_{(1:K)}\right)_t \leq_{ICXA} X_{(1:K)}, \quad t \geq 0.$$

Since (see, Li and Zuo (2004))

$$\min\{(X_1)_t, (X_2)_t, ..., (X_K)_t\} =_{ST} \left(X_{(1:K)}\right)_t, \quad t \geq 0,$$

it holds that

$$\min\{(X_1)_t, (X_2)_t, ..., (X_K)_t\} \leq_{ICXA} X_{(1:K)}, \quad t \geq 0.$$

By Theorem 2.4, we have, for all $t \geq 0$, $X_t \leq_{ICXA} X$, and hence X is also $NBUCA$.

3. CONCLUSIONS

This paper has investigated the residual life of a k-out-of-n system given that the $(n-k)$th failure has occurred at time $t \geq 0$ for the life distributions belong to the $NBUCA$ class. Results in Section 2 study the behavior of the residual life of a system with *i.i.d.* $NBUCA$ components. They can be regarded as supplement to those known results in this line of research. In the rest of that section, we mainly focus upon the aging process of a system with independent but not necessarily identical $NBUCA$ components, the results there are extensions of those in the literature. It would be also interesting to investigate the aging procedure of the residual life of a k-out-of-n system with other well-known aging notions such as new better than used in expectation ($NBUE$) and harmonic new better than used in expectation ($HNBUE$). Finally, we investigate the reversed preservation property of the increasing convex average order and the $NBUCA$ class under series systems which are composed of a *random number* of *i.i.d.* components.

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