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LIMIT CYCLES IN A CUBIC PREDATOR-PREY DIFFERENTIAL SYSTEM

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ABSTRACT. We propose a cubic differential system, which can be considered a generalization of the predator-prey models, studied by many authors recently (see [18, 20], for instance). The properties of the equilibrium points, the existences, nonexistence, the uniqueness conditions and the relative positions of the limit cycles are investigated. An example is used to show our theorems are easy to be used in applications.

1. Introduction

Since the very famous papers of Poincaré (1881, 1882, 1885, 1886), the concept of limit cycle has been attracted attentions from many mathematicians. Even in the famous speech entitled: "Mathematical Problems", given by David Hilbert at the Second International Congress of Mathematicians, Paris 1900, the limit cycle was one of the important topics. In Hilbert 23 Problems, the 16th, is on limit cycles - finding the maximum number of limit cycles of the differential equations:

$$\frac{dx}{dt} = X_n(x,y), \ \frac{dy}{dt} = Y_n(x,y),$$

where $X_n(x,y)$ and $Y_n(x,y)$ are polynomials whose degrees are not greater than n.

Then in the 1930s', van der Pol and Andronov showed that the closed orbit in the phase plane of a self-sustained oscillation occurring in a vacuum tube circuit was a limit cycle as considered by Poincaré. After that, the existence, nonexistence, uniqueness and other properties of limit cycles have been studied extensively by scientists in all fields in addition to mathematicians, (see, for instance, Ye, et al. [19], Qin [16].)

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The study of limit cycles normally includes two aspects: one is the existence, stability and instability, number and relative positions of limit cycles, and the other is the creating and disappearing of limit cycles along with the varying of the parameters in system (e.g. bifurcation). For the exact number of limit cycles and their relative positions, the known results are not many because determining the number and positions of limit cycles is not easy. That is the reason why the 16th Hilbert problem still remains open even for the case when $n = 2$ after one hundred years, although some important progress has been made recently [4], [10], [12], [16], [17].

The development of the qualitative analysis of ordinary differential equations is derived not only by the Hilbert Problems but also by the study of the nonlinear oscillations in many other fields, such as discontinuous automatic control systems [16], bio-chemical reactions [5], [7], [15], immune response and predator-prey systems, and other problems in mathematical bi-sciences [1], [6], [13], [14]. Qualitative analysis is now a powerful tool in the study of nonlinear phenomena in all areas in science and technology, and it is developing very rapidly. In this paper, we study a cubic differential system, which can be considered a generalization of the predator-prey model studied recently by many authors [11], [18], [20]. We will analyze the properties of the equilibrium points, the stability and instability, the existence and non-existence of limit cycles, the uniqueness conditions and the relative positions of the limit cycles. Since the paper of May [14], finding the conditions that there is one and only one limit cycle in a predator-prey system has been considered a primary problem in mathematical ecology. In many articles, the conditions presented are just sufficient [20], but we are going to show a condition that is both sufficient and necessary for the uniqueness of the limit cycle in the system. Our work is definitely useful for a further study in nonlinear oscillations.

2. The cubic system and its equilibrium points

We consider the system

$$
\begin{aligned}
\frac{dx}{dt} &= b_1x + b_2x^2 - b_3x^3 - b_4xy \\
\frac{dy}{dt} &= -cy + (\alpha x - \beta y)y,
\end{aligned}
\tag{2.1}
$$

where b_1 is nonnegative, $b_3, b_4, c, \alpha, \beta$ are positive, and the sign of b_2 is undetermined.

System (2.1) can be considered a special case of the following model for predator-prey interaction:

$$\begin{aligned} \frac{dx}{dt} &= f(x) - g(x,y) \\ \frac{dy}{dt} &= u(g(x,y),y) - v(y) \end{aligned} \tag{2.2}$$

where, x and y represent densities of prey and predator, respectively. The functions f, g, u, and v represent the rates of prey reproduction, prey death due to predation, predator reproduction, and predator death, respectively. Gilpin (see Kuno [11], for instance) used a function of the form $f(x) = ax - bx^2 + cx^3$ in his predator-prey model, which can be describe both over- and under-crowding effects in the prey population. And many Chinese authors ([18, 20]) have used some other forms for $f(x)$ and other functions of (2.2).

By the variable transform: $x = \frac{c}{\alpha}\bar{x}$, $y = \frac{c}{\beta}\bar{y}$, $dt = \frac{1}{c}d\tau$, and then replace $\bar{x}, \bar{y}, \tau$ with x, y, t, system (2.1) is transferred to

$$\begin{aligned} \frac{dx}{dt} &= x\left(a_1 + a_2 x - a_3 x^2\right) - kxy \\ \frac{dy}{dt} &= y\left(-1 + x - y\right), \end{aligned} \tag{2.3}$$

where $a_1 = \frac{b_1}{c}$ is nonnegative, $a_3 = \frac{b_3 c}{\alpha^2}$ and $k = \frac{b_2}{\beta}$ are positive, and $a_2 = \frac{b_2}{\alpha}$.

It is easy to see that the system has six equilibrium points: $O(0,0)$, $A(x_1, 0)$, $B(x_2, 0)$, $C(0, -1)$, $D(x_3, x_3 - 1)$, and $E(x_4, x_4 - 1)$, where

$$\begin{aligned} x_{1,2} &= \frac{a_2 \mp \sqrt{a_2^2 + 4a_1 a_3}}{2a_3}, \\ x_{3,4} &= \frac{(a_2 - k) \mp \sqrt{(a_2 - k)^2 + 4(a_1 + k)a_3}}{2a_3}. \end{aligned}$$

Let

$$\Delta = (a_2 - k)^2 + 4(a_1 + k)a_3. \tag{2.4}$$

Then we have

$$x_4 = \frac{a_2 - k + \sqrt{\Delta}}{2a_3}.$$

A simple calculation tells that $O(0,0)$ is a saddle, $C(0,-1)$ an unstable node, and $A(x_1, 0)$ stable node; $B(x_2, 0)$ is a saddle if $a_1 + a_2 > a_3$, and stable node if $a_1 + a_2 < a_3$. $D(x_3, x_3 - 1)$ is always a saddle because the corresponding characteristic polynomial has two eigenvalues with different signs.

For $E(x_4, x_4 - 1)$, if $a_1 + a_2 = a_3$, then $x_4 - 1 = 0$, and at that time $E(x_4, x_4 - 1) = (1, 0)$ is a stable node. If $a_1 + a_2 < a_3$, $x_4 - 1 < 0$ and $E(x_4, x_4 - 1)$ is a saddle.

Now suppose that $a_1 + a_2 > a_3$. Then $E(x_4, x_4 - 1)$ is in the interior of the first quadrant. In this case, let

$$p = (-1 - a_2 + 2k)x_4 + (1 - 2a_1 - 2k). \tag{2.5}$$

If $a_2 < (2k-1) + \frac{1-2a_1-2k}{x_4}$, (or $p > 0$), E is an unstable node or focus; if $p < 0$, E is a stable node or focus. When $p = 0$, E is a center of the corresponding linear system, but still a focus of the nonlinear system (2.3). (Note that the condition $a_1 + a_2 > a_3$ implies $x_4 < x_2$).

Let $\Omega = \{(x, y) | x \geq 0, y \geq 0\}$, and $\Omega^+ = \{(x, y) | x > 0, y > 0\}$. Regarding the boundedness of the solutions, we have

THEOREM 2.1. *All the solutions of system* (2.3) *are bounded for* $t > 0$.

Proof. Since both x and y axes are boundaries of system (2.3), any trajectory of (2.3) starting at $(x(0), y(0)) \in \Omega^+$ will be remained in Ω^+. Now suppose there exists $t_1 > 0$ such that $x(t_1) = x_2$, by (2.3)

$$\left.\frac{dx}{dt}\right|_{t=t_1} = \left.\frac{dx}{dt}\right|_{x=x_2} = -kx_2y < 0. \tag{2.6}$$

That is any trajectory attached $x = x_2$ will cross the line $x = x_2$ from the right to the left. Therefore, for any $x(0) < x_2$, $x(t) \leq x_2$ when $t > 0$. Suppose $x(0) \geq x_2$. Since x_2 is the only positive root of the equation $a_1 + a_2x - a_3x^2 = 0$ for any $T \geq x_2$, $a_1 + a_2T - a_3T^2 < 0$, hence

$$\left.\frac{dx}{dt}\right|_{x=T} < 0. \tag{2.7}$$

Therefore, for all $t > 0$, $x(t) \leq \max\{x(0), x_2\}$. The proof that $x(t)$ is bounded is completed.

The boundedness of $y(t)$ can be shown by the phase portrait analysis on the region D bounded by x and y axes and the lines l_1 and l_2 defined as follows:

$$l_1 \;:\; x + y - \epsilon = 0,$$
$$l_2 \;:\; x = \eta, \text{ where } \eta \text{ is a constant, } \eta > \max\{x(0), x_2\}.$$

It is easy to see that, because x is bounded,

$$(2.8)\qquad \begin{aligned} \frac{dl_1}{dt} &= \frac{dx}{dt}+\frac{dy}{dt} \\ &= a_1 + a_2x - a_3x^2 - kxy - y + xy - y^2 \\ &= (a_1+1)x + (a_1 - 2 + k)x^2 - a_3x^3 \\ &\quad +(3x - kx - 1)\epsilon - \epsilon^2 \\ &< 0 \quad \text{for sufficiently large } \epsilon. \end{aligned}$$

Therefore, the region D is invariant under system (2.3), and $y(t)$ is also bounded for $t > 0$. The proof of Theorem 2.1 is completed. □

Regarding the stability, we have

THEOREM 2.2. *When $a_1 + a_2 < a_3$, the equilibrium $B(x_2, 0)$ is global asymptotically stable.*

Proof. We first point it out that there is no equilibrium point in Ω^+ when $a_1 + a_2 < a_3$. By the proof of Theorem 2.1, all the trajectories for $t > 0$ are bounded, and their ω limit sets may only be singular points, closes orbits or singular closed orbits. Since both the x and y axes are trajectories of (2.3), and there is no other equilibrium in Ω, all the trajectories in Ω must approach to $B(x_2, 0)$. This completes the proof of Theorem 2.2. □

3. Existence and uniqueness of limit cycles

Our discussion is in Ω^+ because this is the only place where the limit cycles may exist in system (2.3). We first take care of the case when there is no limit cycles in (2.3).

THEOREM 3.1. *If $a_1 + a_2 > a_3$ and $k \le 1$, system (2.3) has no limit cycles.*

Proof. Construct the Dulac function $\delta(x, y) = x^{-2}y^{-1}$ and let

$$P(x,y) = x\left(a_1 + a_2x - a_3x^2\right) - kxy,\ Q(x,y) = y(-1 + x - y).$$

Then, we have

$$(3.9)\quad \operatorname{div}(\delta P, \delta Q) = \frac{\partial \delta P}{\partial x} + \frac{\partial \delta Q}{\partial y} = -a_1x^{-2}y^{-1} - a_3y^{-1} + (k-1)x^{-2}.$$

Since our discussion is in Ω^+, if $k \le 1$, $\operatorname{div}(\delta P, \delta Q) < 0$, and in any sub-region of Ω^+, $\operatorname{div}(\delta P, \delta Q) \ne 0$. By the Dulac theorem [3], system (2.3) does not exist limit cycles in Ω^+. □

Before we prove the next theorem, we need to introduce the following lemma.

LEMMA 3.2. (Cherkas and Zhilevich [2]) *If all the functions in the generalized Linéard system*

$$\begin{aligned} \frac{dx}{dt} &= -h(y) - A(x) \\ \frac{dy}{dt} &= g(x) \end{aligned} \tag{3.10}$$

are continuously differentiable and the following assumptions are satisfied:

(A1) *$xg(x) > 0$, for $x \neq 0$; $yh(y) > 0$ for $y \neq 0$; $A(0) = 0, A'(0) < 0$.*

(A2) *$h(y)$ is non-decreasing and the curve $h(y) + A(x) = 0$ is defined for all $x \in (-\infty, \infty)$.*

(A3) *there exists an interval (x_1', x_2') with $x_1' < 0 < x_2'$, such that there is no limit cycle for $x \leq x_1'$, and $x \geq x_2'$, and $\frac{A'(x)}{g(x)}$ is non-decreasing for x in $(x_1', 0)$ and $(0, x_2')$.*

Then system (3.10) *has at most one limit cycle, and if it exists it is stable.*

REMARK. The assumption (A1) in Lemma 3.2 can be reduced to

(A1') $xg(x) > 0$, for $x \neq 0$, $x \in (x_1', x_2')$; $yh(y) > 0$ for $y \neq 0$; $A(0) = 0, A'(0) < 0$, where (x_1', x_2') is as defined in the assumption (A3).

We may also use some other theorems to prove our results (see [8, 9], for example).

Now we are in a position to prove the following uniqueness theorem of limit cycle in system (2.3).

THEOREM 3.3. *When $a_1 + a_2 > a_3$ and $k > 1$, the necessary and sufficient condition for there exists one and only one limit cycle in system* (2.3) *is $p > 0$.*

Proof. The existence of limit cycle follows from the proof of Theorem 2.1. As to the uniqueness, we make a change of variables: $x = u + x_4$, $y = v + y_4$, where $y_4 = x_4 - 1$. Then we have

$$\begin{aligned} \frac{du}{dt} &= (u + x_4)(a_1 + a_2(u + x_4) - a_3(u + x_4)^2) \\ &\quad -k(u + x_4)(v + y_4) \\ \frac{dv}{dt} &= (v + y_4)(u - v). \end{aligned} \tag{3.11}$$

We use another change of variables:

$$\begin{aligned} v &= x_4^{-1/k} y_4 e^z (u + x_4)^{1/k} - y_4 \\ \tau &= k(u + x_4)^{\frac{k+1}{k}} t. \end{aligned}$$

Then system (3.11) is transferred to the generalized Liénard system

$$\begin{aligned} \frac{du}{dt} &= -x_4^{-1/k} y_4 (e^z - 1) - \frac{a_3 u^2 - (a_2 - 2a_3 x_4)u - k y_4}{k(u + x_4)^{1/k}} \\ &\quad - x_4^{-1/k} y_4, \\ \frac{dz}{dt} &= \frac{a_3 u^2 + (k - a_2 + 2a_3 x_4)u}{k^2 (u + x_4)^{\frac{k+1}{k}}}. \end{aligned} \tag{3.12}$$

Let

$$\begin{aligned} h(z) &= x_4^{-1/k} y_4 (e^z - 1), \\ A(u) &= \frac{a_3 u^2 - (a_2 - 2a_3 x_4)u - k y_4}{k(u + x_4)^{1/k}} + x_4^{-1/k} y_4, \\ g(u) &= \frac{a_3 u^2 + (k - a_2 + 2a_3 x_4)u}{k^2 (u + x_4)^{\frac{k+1}{k}}}. \end{aligned} \tag{3.13}$$

We want to show that the assumptions of Lemma 3.2 are satisfied. Actually, for the assumptions (A1) to (A3), we just need to show they are valid in the interval $u \in (-x_4, x_2 - x_4)$ since there is no limit cycle when $x \leq 0$ (or $u \leq x_4$) and $x \geq x_2$ (or $u \geq x_2 - x_4$). By (2.4), for $u \neq 0$,

$$ug(u) = \frac{(a_3 u + \sqrt{\Delta})u^2}{k^2 (u + x_4)^{\frac{k+1}{k}}} > 0$$

since

$$\begin{aligned} a_3 u + \sqrt{\Delta} &= a_3(x - x_4) + \sqrt{\Delta} \\ &> \sqrt{\Delta} - a_3 x_4 \text{ (since } x > 0) \\ &= \sqrt{\Delta} - \frac{a_2 - k + \sqrt{\Delta}}{2} \\ &= \frac{\sqrt{\Delta} - a_2 + k}{2} \\ &> 0, \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} zh(z) &= x_4^{-1/k} y_4 z(e^z - 1) \text{ for } z \neq 0, \\ h'(z) &> 0, \\ A(0) &= 0. \end{aligned}$$

Notice that

$$A'(u) = \frac{a_3(2k-1)(u+x_4)^2 + a_2(1-k)(u+x_4) + a_1}{k^2(u+x_4)^{\frac{k+1}{k}}}. \tag{3.15}$$

Then by (2.5),

$$\begin{aligned} & a_3(2k-1)x_4^2 + a_2(1-k)x_4 + a_1 \\ = \;& (2k-1)((k-1)x_4 - a_1 - k + 1 - p) + a_2(1-k)x_4 + a_1 \\ = \;& (1-k)((1+a_2-2k)x_4 + 2a_1) + (1-2k)(k+p-1) \\ = \;& (1-k)(1-2k-p) + (1-2k)(k-1) + (1-2k)p \\ = \;& -(1-k)p + (1-2k)p \\ = \;& -kp \\ < \;& 0, \quad \text{since } p > 0. \end{aligned} \tag{3.16}$$

Thus, $A'(0) < 0$.

It is also easy to know that $g(u)$ can be written as

$$g(u) = \frac{a_3(u+x_4)^2 + (k-a_2)(u+x_4) - (a_1+k)}{k^2(u+x_4)^{\frac{k+1}{k}}}. \tag{3.17}$$

Let $w = u + x_4$. Then by (3.15) and (3.17), we have

$$\frac{A'(u)}{g(u)} = \frac{a_3(2k-1)w^2 + a_2(1-k)w + a_1}{a_3w^2 + (k-a_2)w - (a_1+k)}. \tag{3.18}$$

Thus

$$\left(\frac{A'(u)}{g(u)}\right)' = \frac{\phi(w)}{(a_3w^2 + (k-a_2)w - (a_1+k))^2}, \tag{3.19}$$

where

(3.20)

$$\phi(w) = a_3k(2k-a_2-1)w^2 + 2a_3k(1-2a_1-2k)w - (a_1 - a_1a_2 + a_2)k + a_2k^2.$$

When $k > 1$, $p > 0$, by (2.5),

$$2k - a_2 - 1 = \frac{p + (2k + 2a_1 - 1)}{x_4} > 0.$$

Therefore, $\phi(w)$ reaches its minimum at $w = \bar{w}$, where

$$\phi(\bar{w}) = \frac{a_3 k(2k + 2a_1 - 1)^2}{1 + a_2 - 2k} - (a_1 - a_1 a_2 + a_2)k + a_2 k^2.$$

Notice that, by (2.5),

$$p = -a_3 x_4^2 + (k-1)x_4 - a_1 - k + 1.$$

Thus it can be reached that

$$\begin{aligned} (3.21) \qquad \phi(\bar{w}) &= kp\Big(\frac{k - a_2 + \sqrt{(a_2 - k)^2 + 4a_3(a_1 + k)}}{2} \\ &\quad + \frac{a_3(2k + 2a_1 - 1)}{2k - a_2 - 1}\Big) > 0. \end{aligned}$$

Thus, we have

$$\left(\frac{A'(u)}{g(u)}\right)' > 0.$$

Therefore, the assumptions of Lemma 3.2 are satisfied, and system (2.3) only has one limit cycle, which is stable.

On the other hand, if $p \leq 0$ system (2.3) has no limit cycle or more than one limit cycles. The condition is also necessary. □

4. Relative position of the limit cycles

In order to estimate the relative position of the limit cycles, we first prove the following lemma.

LEMMA 4.1. *All the solutions of the auxiliary system*

$$\begin{aligned} (4.22) \qquad \frac{dx}{dt} &= x(a_1 + a_2 x_4 - a_3 x_4^2 - ky) \\ \frac{dy}{dt} &= y(-1 + x - y_4) \end{aligned}$$

are periodic.

Proof. Let $(x_0, y_0) \neq (x_4, y_4)$, $(0 < x_0 < x_2, y_0 > 0)$. Then the trajectory Γ of (4.22) starting at (x_0, y_0) satisfies

$$(4.23) \qquad \int_{y_0}^{y} \frac{a_1 + a_2 x_4 - a_3 x_4^2 - ky}{y} dy = \int_{x_0}^{x} \frac{-1 + x - y_4}{x} dx.$$

Let $\Gamma = (x(t), y(t))$. Suppose Γ is not a closed orbit. We can find two points: $(x(t_1), y(t_1))$, $(x(t_2), y(t_2))$, $t_1 < t_2$ such that $x(t_1) = x(t_2) = x_4$, $y(t_1) < y(t_2) < y_4$. Then

$$\begin{aligned} &\int_{y_0}^{y(t_2)} \frac{a_1 + a_2x_4 - a_3x_4^2 - ky}{y} dy \\ (4.24) \qquad = &\int_{y_0}^{y(t_1)} \frac{a_1 + a_2x_4 - a_3x_4^2 - ky}{y} dy \\ &+ \int_{y(t_1)}^{y(t_1)} \frac{a_1 + a_2x_4 - a_3x_4^2 - ky}{y} dy. \end{aligned}$$

Since when $y < y_4$,

$$(4.25) \qquad a_1 + a_2x - a_3x^2 - ky > 0,$$

we have

$$\begin{aligned} (4.26) \qquad &\int_{y_0}^{y(t_2)} \frac{a_1 + a_2x_4 - a_3x_4^2 - ky}{y} dy \\ > &\int_{y_0}^{y(t_1)} \frac{a_1 + a_2x_4 - a_3x_4^2 - ky}{y} dy. \end{aligned}$$

However,

$$\begin{aligned} (4.27) \qquad &\int_{x_0}^{x_4} \frac{-1 + x - y_4}{x} dx = \int_{y_0}^{y(t_1)} \frac{a_1 + a_2x_4 - a_3x_4^2 - ky}{y} dy \\ < &\int_{y_0}^{y(t_2)} \frac{a_1 + a_2x_4 - a_3x_4^2 - ky}{y} dy = \int_{x_0}^{x_4} \frac{-1 + x - y_4}{x} dx. \end{aligned}$$

The contradiction completes the proof of Lemma 4.1. □

THEOREM 4.2. *Let* $\bar{x} = \min\{x | a_1 + a_2x - a_3x^2 = a_1 + a_2x_4 - a_3x_4^2, x > x_4\}$ *and define*

$$(4.28) \qquad A = \left\{(x, y) | x_4 \le x \le \bar{x}, y_4 \le y \le a_1 + a_2x_4 - a_3x_4^2\right\}.$$

If for $x \le x_4$,

$$\begin{aligned} (4.29) \qquad &\left(a_1 + a_2x - a_3x^2 - ky\right)(-1 + x - y_4) \\ \ge &\left(a_1 + a_2x_4 - a_3x_4^2 - ky\right)(-1 + x - y) \end{aligned}$$

is satisfied, then the region A *is inside of all the limit cycles of system (2.3).*

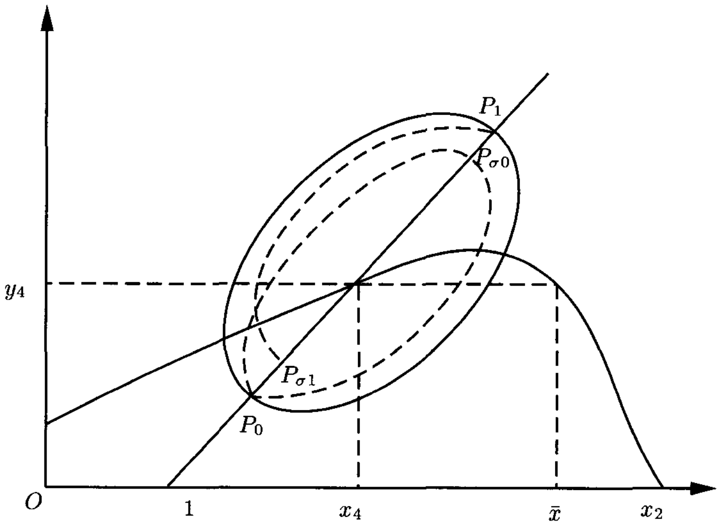

FIGURE 1. The bigger the $r(P)$, the smaller the $r(\sigma(P))$

Proof. Suppose L is a limit cycle of (2.3) surrounding (x_4, y_4). By the phase portrait analysis, L intersects the prey isocline $a_1 + a_2x - a_3x^2 - ky = 0$ exactly at two points, denoted as $P_0(x_0, y_0)$ and $P_1(x_1, y_1)$,where $y_0 < y_1$. Consider the solution of system (4.22) with the initial condition $x(0) = x_0, y(0) = y_0$. Lemma 4.1 implies that the solutions of the system are periodic. Furthermore, each orbit has two intersection points with the predator isocline $-1 + x - y = 0$, denoted as $P_0(x_0, y_0)$ and $\sigma(P_0) = P_{\sigma 0}(x_{\sigma 0}, y_{\sigma 0})$, where the σ satisfies

$$\sigma(P_{\sigma 0}) = \sigma(\sigma(P_0)) = P_0. \tag{4.30}$$

Let $r(P)$ be the distance from point $P(x, y)$ to $(1, 0)$ along with the predator isocline. It is easy to know that the bigger the $r(P)$, the smaller the $r(\sigma(P))$.

By the definition of $\bar{x}$, if $(\bar{x}, y_e)$ is inside L, so is A. Suppose $(\bar{x}, y_e)$ is not inside L. Consider two vectors in the space:

$$\begin{aligned} \bar{V_1} &= \left(x\left(a_1 + a_2x - a_3x^2 - ky\right), y\left(-1 + x - y\right), 0\right), \\ \bar{V_2} &= \left(x\left(a_1 + a_2x_4 - a_3x_4^2 - ky\right), y\left(-1 + x - y_4\right), 0\right) \end{aligned} \tag{4.31}$$

and their vector product:

$$\begin{aligned} \bar{V_1} \times \bar{V_2} = \; & \left(0, 0, x\left(a_1 + a_2x - a_3x^2 - ky\right) y\left(-1 + x - y_4\right)\right. \\ & \left. -x\left(a_1 + a_2x_4 - a_3x_4^2 - ky\right) y\left(-1 + x - y\right)\right). \end{aligned} \tag{4.32}$$

Since for $x_4 < x < \bar{x}$, by the definition of $\bar{x}$,

$$a_1 + a_2 x - a_3 x^2 > a_1 + a_2 x_4 - a_3 x_4^2 \tag{4.33}$$

and when $y \geq y_4$, for $(x, y) \in A$

$$-1 + x - y_4 \geq -1 + x - y_4 > 0$$

Thus, for $x_4 < x < \bar{x}$

$$\begin{aligned} & x\left(a_1 + a_2 x - a_3 x^2 - ky\right) y \left(-1 + x - y_4\right) \\ & -x\left(a_1 + a_2 x_4 - a_3 x_4^2 - ky\right) y \left(-1 + x - y\right) > 0. \end{aligned} \tag{4.34}$$

Then by (4.29), for $0 < x \leq \bar{x}$, (4.34) always holds.

Therefore, in the region $\{(x, y) | 0 < x < \bar{x}, 0 < y\}$ the flow of system (2.3) is always directed outwards with respect to the flow of (4.22). In other words, when $x_4 < x < \bar{x}$, the trajectory L must be outside the trajectory Γ', Γ' is shown as the curve $P_0 P_{\sigma 0}$ in Figure 1. That is,

$$r(P_1) > r(P_{\sigma 0}). \tag{4.35}$$

Similarly, for $0 < x \leq x_4$, consider two trajectories starting at P_1: L and Γ'', Γ'' is shown as the curve $P_1 P_{\sigma_1}$ in Figure 1, we have,

$$r(P_0) < r(\sigma(P_1)) = r(P_{\sigma_1}). \tag{4.36}$$

Since $r(P_1) > r(P_{\sigma 0})$, we have

$$r(\sigma(P_1)) < r(\sigma(P_{\sigma 0})) = r(P_0). \tag{4.37}$$

This is a contradiction that completes the proof of Theorem 4.2. □

THEOREM 4.3. *Let ϵ be defined as in (2.8). Then, all the limit cycles of system (2.3) are inside the region B,where $B = B_1 \cup B_2$,*

$$\begin{aligned} B_1 &= \{(x, y) | 0 \leq x \leq x_4, 0 \leq y \leq -x_4 + \epsilon\}, \\ B_1 &= \{(x, y) | x_4 \leq x \leq x_2, 0 \leq y \leq -x + \epsilon\}. \end{aligned} \tag{4.38}$$

Proof. Define vectors $\bar{V}$ and $\bar{T}$ as the following:

$$\begin{aligned} \bar{V} &= \left(\frac{dx}{dt}, \frac{dy}{dt}, 0\right) \\ \bar{T} &= (t_1, t_2, t_3) \\ &= \begin{cases} (-1, 0, 0,) \text{ if } 0 \leq x \leq x_4, y = -x_4 + \epsilon \\ (-1, 1, 0,) \text{ if } x_4 \leq x \leq x_2, y = -x + \epsilon. \end{cases} \end{aligned} \tag{4.39}$$

Since

$$\bar{T} \times \bar{V} = \left(0, 0, -\left(\frac{dy}{dt} + t_2 \frac{dx}{dt}\right)\right), \tag{4.40}$$

if we can prove, for $0 \le x \le x_2$ and $\frac{dy}{dt} + t_2 \frac{dx}{dt} \le 0$, B is invariant under (2.3). By (4.39), for $0 \le x \le x_4$ and $t_2 = 0$,

$$\frac{dy}{dt} + t_2 \frac{dx}{dt} = (-x_4 + \epsilon)(-1 + x + x_4 - \epsilon) < 0. \tag{4.41}$$

For $x_4 \le x \le x_2$, $t_2 = 1$, and $y = -x + \epsilon$,

$$\begin{aligned} \frac{dy}{dt} + \frac{dx}{dt} &= y(-1 + x - y) + x(a_1 + a_2 x - a_3 x^2 - ky)|_{y=-x+\epsilon} \\ &< 0 \text{ (for same } \epsilon \text{ as in 2.8).} \end{aligned}$$

Thus we have proved that B contains all the limit cycles of system (2.3). The proof of Theorem 4.3 is completed. □

Combine the above two theorems, we have the relative position of the limit cycles of system (2.3):

THEOREM 4.4. *If system* (2.3) *has any limit cycle, and if* (4.29) *holds, then the limit cycle must be inside the region* $B \setminus A$*, with* A *and* B *as defined in Theorems 4.2 and 4.3.*

5. Applications to the predator-prey systems

We use an example to illustrate our theorems. Letting $a_1 = 0$ in system (2.3), we have

$$\begin{aligned} \frac{dx}{dt} &= x\left(a_2 x - a_3 x^2\right) - kxy \\ \frac{dy}{dt} &= y\left(-1 + x - y\right) \end{aligned} \tag{5.42}$$

which was studied by [18, 20] recently. It is easy to see that (x^*, y^*), where

$$\begin{aligned} x^* &= \frac{(a_2 - k) + \sqrt{(a_2 - k)^2 + 4ka_3}}{2a_3} \\ y^* &= x^* - 1 \end{aligned}$$

is the only equilibrium point in Ω^+. Let

$$p = (-1 - a_2 + 2k)x^* + (1 - 2k).$$

By Theorems 2.1, 2.2, 3.1, and 3.3, we have

THEOREM A. *All the solutions of system* (5.42) *are bounded for* $t > 0$.

THEOREM B. *If* $a_2 < a_3$*, the equilibrium* $(a_2/a_3, 0)$ *of system* (5.42) *is global asymptotically stable.*

THEOREM C. *If* $a_2 > a_3$ *and* $k \leq 1$, *system* (5.42) *has no limit cycles.*

THEOREM D. *When* $a_2 > a_3$ *and* $k > 1$, *the necessary and sufficient condition for there to exist one and only one limit cycle in system* (5.42) *is* $p > 0$.

Obviously, these theorems are much easier to derive than the ones in [18, 20], and the conditions for the uniqueness of limit cycle are simpler than the ones in [18, 20].

Moreover, there are no any results reported in the literature regarding the relative position of limit cycles of system (5.42), but they can be easily derived as special cases in our Theorems 4.2-4.4.

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