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NON-TRIVIALITY OF TWO HOMOTOPY ELEMENTS IN π_*S

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ABSTRACT. Let A be the mod p Steenrod algebra for p an arbitrary odd prime and S the sphere spectrum localized at p. In this paper, some useful propositions about the May spectral sequence are first given, and then, two new nontrivial homotopy elements $\alpha_1 j \xi_n$ ($p \geq 5, n \geq 3$) and $\gamma_s \alpha_1 j \xi_n$ ($p \geq 7$, $n \geq 4$) are detected in the stable homotopy groups of spheres, where $\xi_n \in \pi_{p^n q + pq - 2} M$ is obtained in [2]. The new ones are of degree $2(p-1)(p^n + p + 1) - 4$ and $2(p-1)(p^n + sp^2 + sp + (s-1)) - 7$ and are represented up to nonzero scalar by $b_0 h_0 h_n$, $b_0 h_0 h_n \tilde{\gamma}_s \neq 0 \in \mathrm{Ext}_A^{*,*}(Z_p, Z_p)$ in the Adams spectral sequence respectively, where $3 \leq s < p-2$.

1. Introduction and statement of the main results

Let A be the mod p Steenrod algebra and S be the sphere spectrum localized at an arbitrary odd prime p. To determine the stable homotopy groups of spheres π_*S is one of the central problems in homotopy theory. One of the main tools to reach it is the Adams spectral sequence $E_2^{s,t} = \mathrm{Ext}_A^{s,t}(Z_p, Z_p) \Rightarrow \pi_{t-s}S$, where the $E_2^{s,t}$-term is the cohomology of A.

If a family of homotopy generators x_i in $E_2^{s,*}$ converges nontrivially in the Adams spectral sequence, then we get a family of homotopy elements f_i in π_*S and we say that f_i is represented by $x_i \in E_2^{s,*}$ and has filtration s in the ASS. So far, not so many families of homotopy elements in π_*S have been detected. Recently, Lin got a series of results and detected some new families in π_*S.

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We set $q = 2(p-1)$.

In [2], Lin detected a new family of filtration 3 in the stable homotopy groups of spheres. Lin's family is constructed using the Cohen family ς_n and he obtained the following theorem.

THEOREM 1.1. *Let $p \geq 5$ and $n \geq 3$. Then*

(1) $i_*(h_1h_n) \in \mathrm{Ext}_A^{2,p^nq+pq}(H^*M, Z_p)$ *is a permanent cycle in the Adams spectral sequence and converges to a nontrivial element* $\xi_n \in \pi_{p^nq+pq-2}M$.

(2) *For* $\xi_n \in \pi_{p^nq+pq-2}M$ *obtained in* (1), $j\xi_n \in \pi_{p^nq+pq-3}S$ *is a nontrivial element of order p which is represented* (up to nonzero scalar) *by* $(b_0h_n + h_1b_{n-1}) \in \mathrm{Ext}_A^{3,p^nq+pq}(Z_p, Z_p)$ *in the Adams spectral sequence.*

In [4], Lin and Zheng obtained the following theorem and detected a new family of filtration 7 in the stable homotopy groups of spheres.

THEOREM 1.2. *Let $p \geq 7$, $n \geq 4$. Then the product*

$$b_{n-1}g_0\tilde{\gamma}_3 \neq 0 \in \mathrm{Ext}_A^{7,p^nq+3(p^2+p+1)q}(Z_p, Z_p)$$

and it converges in the Adams spectral sequence to a nontrivial element in $\pi_{p^nq+3(p^2+p+1)q-7}S$ *of order p.*

Lin [3] detected a new family in π_*S of filtration 6 in the stable homotopy groups of spheres and proved the following theorem.

THEOREM 1.3. *Let $p \geq 7$, $n \geq 4$. Then the product*

$$h_ng_0\tilde{\gamma}_3 \neq 0 \in \mathrm{Ext}_A^{6,p^nq+3(p^2+p+1)q}(Z_p, Z_p)$$

and it converges in the Adams spectral sequence to a nontrivial element in $\pi_{p^nq+3(p^2+p+1)q-6}S$ *of order p.*

In this paper, we will use Lin's results in [2] to detect two new families of filtration 4 and $s+4$ in the stable homotopy groups of spheres. Our main results can be stated as follows.

THEOREM 1.4. *Let $p \geq 5$, $n \geq 3$. Then the product*

$$b_0h_0h_n \neq 0 \in \mathrm{Ext}_A^{4,p^nq+pq+q}(Z_p, Z_p)$$

is a permanent cycle in the Adams spectral sequence and converges to a nontrivial element $\alpha_1 j\xi_n \in \pi_{p^nq+pq+q-4}S$.

THEOREM 1.5. *Let $p \geq 7$, $n \geq 4$ and $3 \leq s < p-2$. Then the product*

$$b_0h_0h_n\tilde{\gamma}_s \neq 0 \in \mathrm{Ext}_A^{s+4,p^nq+sp^2q+spq+(s-1)q+s-3}(Z_p, Z_p)$$

is a permanent cycle in the Adams spectral sequence and converges to a nontrivial element $\gamma_s\alpha_1 j\xi_n \in \pi_{p^nq+sp^2q+spq+(s-1)q-7}S$.

Our main methods are the Adams spectral sequence and the May spectral sequence, especially the May spectral sequence.

The paper is arranged as follows: after giving some useful propositions on the May spectral sequence in Section 2, we will make use of the May spectral sequence and the Adams spectral sequence to prove our main theorems in Section 3.

2. Some results on the May spectral sequence

From [6], $\mathrm{Ext}_A^{1,*}(Z_p, Z_p)$ has Z_p-bases consisting of $a_0 \in \mathrm{Ext}_A^{1,1}(Z_p, Z_p)$, $h_i \in \mathrm{Ext}_A^{1,p^iq}(Z_p, Z_p)$ for all $i \geq 0$ and $\mathrm{Ext}_A^{2,*}(Z_p, Z_p)$ has Z_p-bases consisting of α_2, a_0^2, $a_0h_i(i > 0)$, $g_i(i \geq 0)$, $k_i(i \geq 0)$, $b_i(i \geq 0)$, and $h_ih_j(j \geq i+2, i \geq 0)$ whose internal degrees are $2q+1$, 2, $p^iq+1, p^{i+1}q+2p^iq$, $2p^{i+1}q+p^iq$, $p^{i+1}q$ and p^iq+p^jq respectively.

From [8], there is a May spectral sequence $\{E_r^{s,t,*}, d_r\}$ which converges to $\mathrm{Ext}_A^{s,t}(Z_p, Z_p)$ with E_1-term

$$E_1^{*,*,*} = E(h_{m,i}|m > 0, i \geq 0) \bigotimes P(b_{m,i}|m > 0, i \geq 0) \bigotimes P(a_n|n \geq 0), \tag{2.1}$$

where E is the exterior algebra, P is the polynomial algebra, and

$$h_{m,i} \in E_1^{1,2(p^m-1)p^i,2m-1},$$

$$b_{m,i} \in E_1^{2,2(p^m-1)p^{i+1},p(2m-1)},$$

$$a_n \in E_1^{1,2p^n-1,2n+1}.$$

One has $d_r : E_r^{s,t,u} \to E_r^{s+1,t,u-r}$ and if $x \in E_r^{s,t,*}$ and $y \in E_r^{s',t',*}$, then $d_r(x \cdot y) = d_r(x) \cdot y + (-1)^s x \cdot d_r(y)$, $x \cdot y = (-1)^{ss'+tt'} y \cdot x$ for $x, y = h_{m,i}, b_{m,i}$ or a_n. The first May differential d_1 is given by

$$\begin{cases} d_1(h_{i,j}) = \sum\limits_{0<k<i} h_{i-k,k+j}h_{k,j}, \\ d_1(a_i) = \sum\limits_{0\leq k<i} h_{i-k,k}a_k, \\ d_1(b_{i,j}) = 0. \end{cases} \tag{2.2}$$

For an element $x \in E_1^{s,t,*}$, define dim $x = s$, deg $x = t$. Then we have:

$$\begin{cases} \dim h_{i,j} = \dim a_i = 1, \ \dim b_{i,j} = 2, \\ \deg h_{i,j} = 2(p^i - 1)p^j = 2(p-1)(p^{i+j-1} + \cdots + p^j), \\ \deg b_{i,j} = 2(p^i - 1)p^{j+1} = 2(p-1)(p^{i+j} + \cdots + p^{j+1}), \\ \deg a_i = 2p^i - 1 = 2(p-1)(p^{i-1} + \cdots + 1) + 1, \\ \deg a_0 = 1, \end{cases}$$

where $i \geq 1$, $j \geq 0$.

PROPOSITION 2.3. *Let $t_1 = q(c_n p^n + c_{n-1}p^{n-1} + \cdots + c_1 p + c_0) + e$ be a positive integer with $0 \leq c_i < p$ $(0 \leq i \leq n)$, $0 \leq e < q$, and s a positive integer with $0 < s < p$. If for some j $(0 \leq j \leq n)$, $s < c_j$, then in the May spectral sequence* (see (2.1)) *we have*

$$E_1^{s,t_1,*} = 0.$$

Proof. See [5, Proposition 1.1]. □

PROPOSITION 2.4. *Let $t_2 = q(c_n p^n + c_{n-1}p^{n-1} + \cdots + c_1 p + c_0)$ be a positive integer with $0 \leq c_i \leq c_n < p$ $(0 \leq i \leq n)$. If $c_0 > 1$, then in the May spectral sequence* (see (2.1)) *it follows that*

$$E_1^{c_n,t_2,*} = 0.$$

Proof. Consider $h = x_1 x_2 \cdots x_m \in E_1^{c_n,t_2,*}$ in the May spectral sequence, where x_i is one of a_k, $h_{l,j}$ or $b_{u,z}$, $0 \leq k \leq n+1$, $0 \leq l+j \leq n+1$, $0 \leq u+z \leq n$, $l > 0$, $j \geq 0$, $u > 0$, $z \geq 0$. Assume that deg $x_i = q(c_{i,n}p^n + c_{i,n-1}p^{n-1} + \cdots + c_{i,0}) + e_i$, where $c_{i,j} = 0$ or 1, $e_i = 1$ if $x_i = a_{k_i}$, or $e_i = 0$. Then

$$\begin{aligned} \deg h &= \sum_{i=1}^m \deg x_i \\ &= q((\sum_{i=1}^m c_{i,n})p^n + \cdots + (\sum_{i=1}^m c_{i,2})p^2 \\ &\quad + (\sum_{i=1}^m c_{i,1})p + (\sum_{i=1}^m c_{i,0})) + (\sum_{i=1}^m e_i) \\ &= q(c_n p^n + \cdots + c_0) + 0, \\ \dim h &= \sum_{i=1}^m \dim x_i = c_n. \end{aligned}$$

Note that $0 \le c_i < p$. Using knowledge of the p-adic expression in number theory, we have that

$$\begin{cases} \sum_{i=1}^{m} e_i = 0 + \lambda_{-1} q, & \lambda_{-1} \ge 0; \\ \sum_{i=1}^{m} c_{i,0} + \lambda_{-1} = c_0 + \lambda_0 p, & \lambda_0 \ge 0; \\ \cdots & \cdots \\ \sum_{i=1}^{m} c_{i,n-1} + \lambda_{n-2} = c_{n-1} + \lambda_{n-1} p, & \lambda_{n-1} \ge 0; \\ \sum_{i=1}^{m} c_{i,n} + \lambda_{n-1} = c_n. \end{cases}$$

Because $\dim h_{i,j} = \dim a_i = 1$ and $\dim b_{i,j} = 2$, we know that $0 < m \le c_n < p$ from $\dim h = \sum_{i=1}^{m} \dim x_i = c_n$. Note that $0 \le c_i < p$, $0 \le \sum_{i=1}^{m} c_{i,j} \le m < p$ and $\sum_{i=1}^{m} e_i \le m < q$, we easily see that the sequence $(\lambda_{-1}, \lambda_0, \lambda_1, \lambda_2, \cdots, \lambda_{n-2}, \lambda_{n-1})$ must equal the sequence $(0, 0, 0, 0, \cdots, 0, 0)$. And then we have

$$\begin{cases} \sum_{i=1}^{m} e_i = 0, & \sum_{i=1}^{m} c_{i,0} = c_0, \\ \sum_{i=1}^{m} c_{i,1} = c_1, & \sum_{i=1}^{m} c_{i,2} = c_2, \\ \cdots & \cdots \\ \sum_{i=1}^{m} c_{i,n-1} = c_{n-1}, & \sum_{i=1}^{m} c_{i,n} = c_n. \end{cases}$$

Since $\sum_{i=1}^{m} e_i = 0$, $\deg h_{i,j} \equiv 0 \,(\mathrm{mod}\ q)$ $(i > 0, j \ge 0)$, $\deg a_i \equiv 1 \,(\mathrm{mod}\ q)$ $(i \ge 0)$ and $\deg b_{i,j} \equiv 0 \,(\mathrm{mod}\ q)$ $(i > 0, j \ge 0)$, then $h = x_1 \cdots x_m \in E(h_{m,i} | m > 0, i \ge 0) \bigotimes p(b_{m,i} | m > 0, i \ge 0)$. Meanwhile, from the equality $\sum_{i=1}^{m} c_{i,n} = c_n$, we have that $m \ge c_n$. Thus $m = c_n$. Note that $\dim h = c_n$. Thus $\dim h = 1$. It follows that $h = x_1 \cdots x_m \in E(h_{m,i} | m > 0, i \ge 0)$. From $\sum_{i=1}^{c_n} c_{i,n} = c_n$, we know that $\deg x_i = p^n q +$lower terms, $1 \le i \le c_n$. Meanwhile, since $\sum_{i=1}^{m} c_{i,0} = c_0$ and $\deg h_{i,j} \equiv q \,(\mathrm{mod}\ pq)$ $(i > 0, j \ge 0)$, then by the graded commutativity of $E_1^{*,*,*}$ there would be a factor $h_{i_1,0} h_{i_2,0} \cdots h_{i_{c_0},0}$ $(0 < i_1 \le i_2 \cdots i_{c_0} \le n+1)$ in h such that deg $h_{i_j,0}$ =higher terms+q ($i_1 \le j \le i_{c_0}$). Note that deg $h_{i,0} = p^{i-1}q + \cdots + pq + q$, $i > 0$. Thus there would be c_0 $h_{n+1,0}$'s in h with deg $h_{n+1,0} = p^n q + \cdots + pq + q$. Note that $c_0 \ge 2$. By (2.1), h would equal 0. The proposition is proved. □

3. Proofs of the main theorems

Let M be the Moore spectrum modulo a prime $p \ge 5$ given by the cofibration

$$S \xrightarrow{p} S \xrightarrow{i} M \xrightarrow{j} \Sigma S.$$

Let $\alpha : \Sigma^q M \to M$ be the Adams map and K be its cofibre given by the cofibration

$$\Sigma^q M \xrightarrow{\alpha} M \xrightarrow{i'} K \xrightarrow{j'} \Sigma^{q+1} M,$$

where $q = 2(p-1)$. This spectrum which we briefly write as K is known to be the Toda-Smith spectrum $V(1)$. Let $V(2)$ be the cofibre of $\beta : \Sigma^{(p+1)q} K \to K$ given by the cofibration

$$\Sigma^{(p+1)q} K \xrightarrow{\beta} K \xrightarrow{\bar{i}} V(2) \xrightarrow{\bar{j}} \Sigma^{(p+1)q+1} K.$$

Let $\gamma : \Sigma^{q(p^2+p+1)} V(2) \to V(2)$ be the v_3-map. As we know, in the Adams spectral sequence, for $p \ge 7$ the γ-element $\gamma_t = jj'\bar{j}\gamma^t \bar{i}i'i$ is a nontrivial element of order p in $\pi_{tq(p^2+p+1)-q(p+2)-3}S$ (see [7, Theorem 2.12]).

From [5, Theorem 1.1], we have the following.

Theorem 3.1. *Let $p \ge 7$, $0 \le s < p-3$. Then the element*

$$a_3^s h_{3,0} h_{2,1} h_{1,2} \in E_1^{s+3,t,*}$$

converges to the third Greek letter family element $\tilde{\gamma}_{s+3} \in \mathrm{Ext}_A^{s+3,t}(Z_p, Z_p)$ in the May spectral sequence, where $t = (s+3)p^2q + (s+2)pq + (s+1)q + s$ and $\tilde{\gamma}_{s+3}$ converges to the γ-element $\gamma_{s+3} \in \pi_{(s+3)p^2q+(s+2)pq+(s+1)q-3}S$ in the Adams spectral sequence, where $\gamma_{s+3} = jj'\bar{j}\gamma^{s+3}\bar{i}i'i \in \pi_{t-s-3}S$.

Lemma 3.2. *Let $p \ge 7$, $n \ge 4$, $0 \le s < p-5$. Then in the May spectral sequence, the group $E_1^{s+6,p^nq+(s+3)p^2q+(s+3)pq+(s+2)q+s,*}$ has the following generators:*

$$\begin{aligned}
&a_3^s h_{1,0} h_{3,0} h_{2,1} b_{2,0} h_{1,n},\\
&a_3^s h_{1,0} h_{3,0} h_{1,2} h_{2,1} h_{1,1} h_{1,n},\\
&a_3^s h_{2,0} h_{3,0} h_{2,1} b_{1,1} h_{1,n},\\
&a_3^s h_{2,0} h_{3,0} h_{1,2} b_{2,0} h_{1,n},\\
&a_3^s h_{3,0} h_{2,0} h_{1,2} h_{2,1} b_{1,n-1}.
\end{aligned}$$

Proof. For convenience, we let $t' = p^n q + (s+3)p^2 q + (s+3)pq + (s+2)q + s$. Consider $h = x_1 x_2 \cdots x_m \in E_1^{s+6,t',*}$ in the May spectral sequence, where x_i is one of a_k, $h_{l,j}$ or $b_{u,z}$, $0 \le k \le n+1$, $0 \le l+j \le n+1$, $0 \le u+z \le n$, $l > 0$, $j \ge 0$, $u > 0$, $z \ge 0$. Assume that $\deg x_i = q(c_{i,n}p^n + c_{i,n-1}p^{n-1} + \cdots + c_{i,0}) + e_i$, where $c_{i,j} = 0$ or 1, $e_i = 1$ if $x_i = a_{k_i}$, or $e_i = 0$. Then we have

$$\begin{aligned} \deg h &= \sum_{i=1}^m \deg x_i \\ &= q((\sum_{i=1}^m c_{i,n})p^n + \cdots + (\sum_{i=1}^m c_{i,2})p^2 + (\sum_{i=1}^m c_{i,1})p \\ &\quad + (\sum_{i=1}^m c_{i,0})) + (\sum_{i=1}^m e_i) \\ &= q(p^n + (s+3)p^2 + (s+3)p + (s+2)) + s, \\ \dim h &= \sum_{i=1}^m \dim x_i = s+6. \end{aligned} \tag{3.3}$$

Note that $0 \le s, s+2, s+3 < p$, so from (3.3) we have

$$\begin{cases} \sum_{i=1}^m e_i = s + \lambda_{-1} q, & \lambda_{-1} \ge 0; \\ \sum_{i=1}^m c_{i,0} + \lambda_{-1} = s + 2 + \lambda_0 p, & \lambda_0 \ge 0; \\ \sum_{i=1}^m c_{i,1} + \lambda_0 = s + 3 + \lambda_1 p, & \lambda_1 \ge 0; \\ \sum_{i=1}^m c_{i,2} + \lambda_1 = s + 3 + \lambda_2 p, & \lambda_2 \ge 0; \\ \sum_{i=1}^m c_{i,3} + \lambda_2 = 0 + \lambda_3 p, & \lambda_3 \ge 0; \\ \cdots & \cdots \\ \sum_{i=1}^m c_{i,n-1} + \lambda_{n-2} = 0 + \lambda_{n-1} p, & \lambda_{n-1} \ge 0; \\ \sum_{i=1}^m c_{i,n} + \lambda_{n-1} = 1. \end{cases} \tag{3.4}$$

Case 1. $0 \le s < p-6$.

By the facts that $\dim h_{i,j} = \dim a_i = 1$ and $\dim b_{i,j} = 2$, we know that $0 < m \le s+6 < p$ from $\dim h = \sum_{i=1}^m \dim x_i = s +$

6. Note that $0 \le \sum_{i=1}^{m} e_i, \sum_{i=1}^{m} c_{i,j} \le m < p$, it is easy to see that the sequence $(\lambda_{-1}, \lambda_0, \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_{n-2}, \lambda_{n-1})$ must equal the sequence $(0, 0, 0, 0, 0, \ldots, 0, 0)$. Thus, from (3.4), we get that

$$\begin{cases} \sum_{i=1}^{m} e_i = s, & \sum_{i=1}^{m} c_{i,0} = s+2, \\ \sum_{i=1}^{m} c_{i,1} = s+3, & \sum_{i=1}^{m} c_{i,2} = s+3, \\ \sum_{i=1}^{m} c_{i,3} = \cdots = \sum_{i=1}^{m} c_{i,n-1} = 0, & \sum_{i=1}^{m} c_{i,n} = 1. \end{cases}$$

It is easy to see that there exists a factor $h_{1,n}$ or $b_{1,n-1}$ in h. Note the graded commutativity of $E_1^{*,*,*}$, we can denote the factor $h_{1,n}$ or $b_{1,n-1}$ by x_m. Then $h' = x_1 x_2 \cdots x_{m-1} \in E_1^{l,t'-p^n q,*}$, where $l = s+5$ or $s+4$ and we have

$$\begin{cases} \sum_{i=1}^{m-1} e_i = s, & \sum_{i=1}^{m-1} c_{i,0} = s+2, \\ \sum_{i=1}^{m-1} c_{i,1} = s+3, & \sum_{i=1}^{m-1} c_{i,2} = s+3. \end{cases} \tag{3.5}$$

We can get that $m \ge s+4$ from $\sum_{i=1}^{m-1} c_{i,1} = s+3$. Meanwhile, we know that $m \le s+6$, so $m = s+4$, $m = s+5$ or $m = s+6$. Since $\sum_{i=1}^{m-1} e_i = s$, $\deg h_{i,j} \equiv 0 (\mathrm{mod}\ q)$ $(i > 0, j \ge 0)$, $\deg a_i \equiv 1 (\mathrm{mod}\ q)$ $(i \ge 0)$ and $\deg b_{i,j} \equiv 0 (\mathrm{mod}\ q)$ $(i > 0, j \ge 0)$, then by the graded commutativity of $E_1^{*,*,*}$, h' must have a factor $a_{j_1} a_{j_2} \cdots a_{j_s}$ $(0 \le j_1 \le j_2 \le \cdots \le j_s)$ up to sign. Note the degrees of a_i's, suppose that $h' = a_0^x a_1^y a_2^z a_3^k x_{s+1} \cdots x_{m-1}$, where $0 \le x, y, z, k \le s$, $x+y+z+k = s$ and $m = s+4, s+5$ or $s+6$. From (3.5) we have

$$\begin{cases} x+y+z+k+\sum_{i=s+1}^{m-1} e_i = s, & y+z+k+\sum_{i=s+1}^{m-1} c_{i,0} = s+2, \\ z+k+\sum_{i=s+1}^{m-1} c_{i,1} = s+3, & k+\sum_{i=s+1}^{m-1} c_{i,2} = s+3. \end{cases}$$

It follows that $h'' = x_{s+1}\cdots x_{m-1} \in E_1^{l-s,t'',*}$, $t'' = (s+3-k)p^2q + (s+3-z-k)pq + (s+2-y-z-k)q$ and

$$(3.6)\quad \begin{cases} \sum\limits_{i=s+1}^{m-1} e_i = 0, & \sum\limits_{i=s+1}^{m-1} c_{i,0} = s+2-y-z-k, \\ \sum\limits_{i=s+1}^{m-1} c_{i,1} = s+3-z-k, & \sum\limits_{i=s+1}^{m-1} c_{i,2} = s+3-k. \end{cases}$$

Subcase 1.1. If $h = x_1x_2\cdots x_{m-1}h_{1,n}$, then $h' = a_0^x a_1^y a_2^z a_3^k x_{s+1}\cdots x_{m-1} \in E_1^{s+5,t'-p^nq,*}$ and $h'' = x_{s+1}\cdots x_{m-1} \in E_1^{5,t'',*}$.

When $m = s+4$, (3.6) becomes

$$\begin{cases} \sum\limits_{i=s+1}^{s+3} e_i = 0, & \sum\limits_{i=s+1}^{s+3} c_{i,0} = s+2-y-z-k, \\ \sum\limits_{i=s+1}^{s+3} c_{i,1} = s+3-z-k, & \sum\limits_{i=s+1}^{s+3} c_{i,2} = s+3-k. \end{cases}$$

We can get that $k \geq s+3-\sum\limits_{i=s+1}^{s+3} c_{i,2} \geq s+3-3 = s$ from the equality $\sum\limits_{i=s+1}^{s+3} c_{i,2} = s+3-k$. Note that $x+y+z+k = s$ and $0 \leq x,y,z,k \leq s$. Thus we have that $k = s, x = y = z = 0$. Then $h' = a_3^s x_{s+1}x_{s+2}x_{s+3}$ with $h'' = x_{s+1}x_{s+2}x_{s+3} \in E_1^{5,3p^2q+3pq+2q,*} = Z_p\{h_{1,0}h_{3,0}h_{2,1}b_{2,0}, h_{1,0}h_{3,0}h_{1,2}h_{2,1}h_{1,1}, h_{2,0}h_{3,0}h_{2,1}b_{1,1}, h_{2,0}h_{3,0}h_{1,2}b_{2,0}\}$. Thus at this time $h'' = x_{s+1}x_{s+2}x_{s+3}$ cannot exist, and consequently h' cannot exist.

When $m = s+5$, (3.6) becomes

$$\begin{cases} \sum\limits_{i=s+1}^{s+4} e_i = 0, & \sum\limits_{i=s+1}^{s+4} c_{i,0} = s+2-y-z-k, \\ \sum\limits_{i=s+1}^{s+4} c_{i,1} = s+3-z-k, & \sum\limits_{i=s+1}^{s+4} c_{i,2} = s+3-k. \end{cases}$$

We can get that $k \geq s+3-\sum\limits_{i=s+1}^{s+4} c_{i,2} \geq s+3-4 = s-1$ from the equality $\sum\limits_{i=s+1}^{s+4} c_{i,2} = s+3-k$. Note that $x+y+z+k = s$ and $0 \leq x,y,z,k \leq s$. Thus there are four possibilities that satisfy the two conditions. If $k = s, x = y = z = 0$, then $h' = a_3^s x_{s+1}\cdots x_{s+4}$ with $h'' = x_{s+1}\cdots x_{s+4} \in E_1^{5,3p^2q+3pq+2q,*} = Z_p\{h_{1,0}h_{3,0}h_{2,1}b_{2,0}, h_{1,0}h_{3,0}h_{1,2}h_{2,1}h_{1,1}, h_{2,0}h_{3,0}h_{2,1}b_{1,1}, h_{2,0}h_{3,0}h_{1,2}\, b_{2,0}\}$. Thus up to sign h' can equal $a_3^s h_{1,0}h_{3,0}$

$h_{2,1}b_{2,0}$, $a_3^s h_{2,0}h_{3,0}h_{2,1}b_{1,1}$ and $a_3^s h_{2,0}h_{3,0}h_{1,2}b_{2,0}$. From the facts that $E_1^{5,4p^2q+r_1pq+r_2q,*} = 0$ for $(r_1,r_2) = (4,3)$, $(4,2)$ and $(3,2)$, it follows that when $k = s-1, x = 1, y = z = 0$, $k = s-1, y = 1, x = z = 0$ and $k = s-1, z = 1, x = y = 0$, h'' cannot exist respectively. Consequently, h' cannot exist.

When $m = s+6$, (3.6) becomes

$$\begin{cases} \sum_{i=s+1}^{s+5} e_i = 0, & \sum_{i=s+1}^{s+5} c_{i,0} = s+2-y-z-k, \\ \sum_{i=s+1}^{s+5} c_{i,1} = s+3-z-k, & \sum_{i=s+1}^{s+5} c_{i,2} = s+3-k. \end{cases}$$

We can get that $k \geq s+3-\sum_{i=s+1}^{s+5} c_{i,2} \geq s+3-5 = s-2$ from the equality $\sum_{i=s+1}^{s+5} c_{i,2} = s+3-k$. Note that $k \leq s$. Thus $k = s-2$, $s-1$ or s. If $k = s-2$, we would have that $2 = s+2-s \leq \sum_{i=s+1}^{s+5} c_{i,0} = s+2-y-z-k = 4-y-z < 5$ and $\sum_{i=s+1}^{s+5} c_{i,2} = s+3-k = 5$. By Proposition 2.4, it follows that $h'' = x_{s+1}\cdots x_{s+5} \in E_1^{5,5p^2q+(5-z)pq+(4-y-z)q,*} = 0$. Thus when $k = s-2$, h'' cannot exist. Because $E_1^{5,4p^2q+r_1pq+r_2q,*} = 0$ for $(r_1,r_2) = (4,3)$, $(4,2)$ and $(3,2)$, it follows that when $k = s-1, x = 1, y = z = 0$, $k = s-1, y = 1, x = z = 0$ and $k = s-1, z = 1, x = y = 0$, h' cannot exist respectively.

If $k = s, x = y = z = 0$, then $h' = a_3^s x_{s+1}x_{s+2}x_{s+3}x_{s+4}x_{s+5}$ with $h'' = x_{s+1}\cdots x_{s+5} \in E_1^{5,3p^2q+3pq+2q,*} = Z_p\{h_{1,0}h_{3,0}h_{2,1}b_{2,0}, h_{1,0}h_{3,0}h_{1,2}h_{2,1}h_{1,1}, h_{2,0}h_{3,0}h_{2,1}b_{1,1}, h_{2,0}h_{3,0}h_{1,2}b_{2,0}\}$. Thus h' can be $a_3^s h_{1,0}h_{3,0}h_{1,2}h_{2,1}h_{1,1}$ up to sign.

From the above argument, we have that $h = x_1\cdots x_{m-1}h_{1,n}$ exists, and up to sign h can equal $a_3^s h_{1,0}h_{3,0}h_{2,1}b_{2,0}h_{1,n}$, $a_3^s h_{2,0}h_{3,0}h_{2,1}b_{1,1}h_{1,n}$, $a_3^s h_{2,0}h_{3,0}h_{1,2}b_{2,0}h_{1,n}$ and $a_3^s h_{1,0}h_{3,0}h_{1,2}h_{2,1}h_{1,1}h_{1,n}$.

Subcase 1.2. If $h = x_1x_2\cdots x_{m-1}b_{1,n-1}$, then $h' = a_0^x a_1^y a_2^z a_3^k x_{s+1}\cdots x_{m-1} \in E_1^{s+4,t'-p^nq,*}$ and $h'' = x_{s+1}\cdots x_{m-1} \in E_1^{4,t'',*}$.

When $m = s+4$, (3.6) becomes

$$\begin{cases} \sum\limits_{i=s+1}^{s+3} e_i = 0, & \sum\limits_{i=s+1}^{s+3} c_{i,0} = s+2-y-z-k, \\ \sum\limits_{i=s+1}^{s+3} c_{i,1} = s+3-z-k, & \sum\limits_{i=s+1}^{s+3} c_{i,2} = s+3-k. \end{cases}$$

We can get that $k \geq s+3-\sum\limits_{i=s+1}^{s+3} c_{i,2} \geq s+3-3 = s$ from the equality $\sum\limits_{i=s+1}^{s+3} c_{i,2} = s+3-k$. Note that $x+y+z+k = s$ and $0 \leq x,y,z,k \leq s$. Thus we have that $k = s, x = y = z = 0$. Then $h' = a_3^s x_{s+1}x_{s+2}x_{s+3}$ with $h'' = x_{s+1}x_{s+2}x_{s+3} \in E_1^{4,3p^2q+3pq+2q,*} = Z_p\{h_{2,0}h_{3,0}h_{1,2}h_{2,1}\}$. Thus h'' cannot exist, and then h' cannot exist.

When $m = s+5$, (3.6) becomes

$$\begin{cases} \sum\limits_{i=s+1}^{s+4} e_i = 0, & \sum\limits_{i=s+1}^{s+4} c_{i,0} = s+2-y-z-k, \\ \sum\limits_{i=s+1}^{s+4} c_{i,1} = s+3-z-k, & \sum\limits_{i=s+1}^{s+4} c_{i,2} = s+3-k. \end{cases}$$

We can get that $k \geq s+3-\sum\limits_{i=s+1}^{s+4} c_{i,2} \geq s+3-4 = s-1$ from the equality $\sum\limits_{i=s+1}^{s+4} c_{i,2} = s+3-k$. Note that $x+y+z+k = s$ and $0 \leq x,y,z,k \leq s$. Thus there are four possibilities that satisfy the two conditions. If $k = s, x = y = z = 0$, then $h' = a_3^s x_{s+1}\cdots x_{s+4}$ with $h'' = x_{s+1}\cdots x_{s+4} \in E_1^{4,3p^2q+3pq+2q,*} = Z_p\{h_{2,0}h_{3,0}h_{1,2}h_{2,1}\}$. Thus up to sign h' can equal $a_3^s h_{2,0}h_{3,0}h_{1,2}h_{2,1}$. By Proposition 2.4 we can get that $E_1^{4,4p^2q+r_1pq+r_2q,*} = 0$ for $(r_1,r_2) = (4,3)$, $(4,2)$, $(3,2)$. It follows that when $k = s-1, x = 1, y = z = 0$, $k = s-1, y = 1, x = z = 0$ and $k = s-1, z = 1, x = y = 0$, h'' cannot exist respectively, and then h' cannot exist.

When $m = s+6$, we would have that $h' = x_1x_2\cdots x_{m-1} = x_1x_2\cdots x_{s+5} \in E_1^{s+4,t'-p^nq,*}$. Note that $\dim x_i = 1$ or 2. It is easy to see that m is impossible to equal $s+6$.

From the above argument, we get that $h = x_1x_2\cdots x_{m-1}b_{1,n-1}$ can exist, and up to sign $h = a_3^s h_{2,0}h_{3,0}h_{1,2}h_{2,1}b_{1,n-1}$.

From Subcase 1.1 and Subcase 1.2, we see that when $0 \leq s < p-6$, h exists, and up to sign h can be $a_3^s h_{1,0}h_{3,0}h_{2,1}b_{2,0}h_{1,n}$, $a_3^s h_{1,0}h_{3,0}h_{1,2}h_{2,1}$

$h_{1,1}\ h_{1,n}$, $a_3^s h_{2,0}h_{3,0}h_{2,1}b_{1,1}h_{1,n}$, $a_3^s h_{2,0}h_{3,0}h_{1,2}b_{2,0}h_{1,n}$ and $a_3^s h_{2,0}h_{3,0}h_{1,2}$ $h_{2,1}\ b_{1,n-1}$.

Case 2. $s = p - 6$.

Then $m \le s + 6 = p - 6 + 6 = p$. From $0 \le \sum_{i=1}^{m} e_i, \sum_{i=1}^{m} c_{i,j} \le m \le p$, it is easy to see that the sequence $(\lambda_{-1}, \lambda_0, \lambda_1, \lambda_2,)$ must equal the sequence $(0,0,0,0)$. From (3.4) we have that $\sum_{i=1}^{m} c_{i,3} = \lambda_3 p$. Note that $0 \le \sum_{i=1}^{m} c_{i,3} \le m \le p$. Thus we have that λ_3 may equal 0 or 1.

Subcase 2.1 If $\lambda_3 = 0$, then $\sum_{i=1}^{m} c_{i,3} = 0$.

When $n = 4$, we have that $\sum_{i=1}^{m} c_{i,4} = 1$. From the above results, it follows that there exist a factor $h_{1,4}$ or $b_{1,3}$ among x_i's.

When $n > 4$, we can similarly discuss and obtain that λ_4 may equal 0 or 1. We claim that $\lambda_4 = 0$, for otherwise, we would have that $\lambda_4 = 1$ and $\sum_{i=1}^{m} c_{i,4} = p$, then $m = p$. For each $1 \le i \le m$, deg x_i = higher terms $+p^4q$+lower terms. Since $\sum_{i=1}^{p} e_i = p - 6$, deg $b_{i,j} \equiv 0 (\text{mod } q)$ $(i > 0,\ j \ge 0)$, deg $a_i \equiv 1 (\text{mod } q)$ $(i \ge 0)$ and deg $h_{i,j} \equiv 0 (\text{mod } q)$ $(i > 0, j \ge 0)$, then by the graded commutativity of $E_1^{*,*,*}$, there would exist a factor $a_{j_1}a_{j_2}\cdots a_{j_{p-6}}$ $(0 \le j_1 \le j_2 \le \cdots \le j_{p-6} \le n+1)$ among x_i's such that for any $1 \le i \le p-6, j_i \ge 5$ and deg a_{j_i} =higher terms $+p^4q + p^3q + p^2q + pq + q + 1$. It is obvious that $\sum_{i=1}^{m} c_{i,3} \ge p - 6$ which contradicts $\sum_{i=1}^{m} c_{i,3} = 0$, thus the claim is proved. By induction on j we can get $\lambda_j = 0$ $(4 \le j \le n-1)$, so $\sum_{i=1}^{m} c_{i,n} = 1$, that is to say, there is a factor $h_{1,n}$ or $b_{1,n-1}$ in h.

In all, for $n \ge 4$, there is a factor $h_{1,n}$ or $b_{1,n-1}$ in h. Note the graded commutativity of $E_1^{*,*,*}$, we can denote the factor $h_{1,n}$ or $b_{1,n-1}$ by x_m. By an argument similar to that used in the proof in Case 1, we can show that h exists, and up to sign h can be $a_3^{p-6}h_{1,0}h_{3,0}h_{2,1}b_{2,0}h_{1,n}$, $a_3^{p-6}h_{1,0}h_{3,0}h_{1,2}h_{2,1}h_{1,1}h_{1,n}$, $a_3^{p-6}h_{2,0}h_{3,0}h_{2,1}b_{1,1}h_{1,n}$, $a_3^{p-6}h_{2,0}h_{3,0}h_{1,2}b_{2,0}$ $h_{1,n}$ and $a_3^{p-6}h_{2,0}h_{3,0}h_{1,2}h_{2,1}b_{1,n-1}$.

Subcase 2.2. If $\lambda_3 = 1$, then $\sum_{i=1}^{m} c_{i,3} = p$.

Note that $c_{i,3} = 0$ or 1 and $m \leq p$. It is easy to get that $m = p$. Note that dim $h = p$, we can easily see that for any i, dim $x_i = 1$ and

$$h = x_1 x_2 \cdots x_p \in E(h_{m,i} | m > 0, i \geq 0) \bigotimes P(a_n | n \geq 0).$$

When $n = 4$, it is easy to see that $\sum_{i=1}^{p} c_{i,4} = \sum_{i=1}^{p} c_{i,n} = 0$.

For $n > 4$, from (3.4) we have that

$$\sum_{i=1}^{p} c_{i,4} + 1 = 0 + \lambda_4 p.$$

By the fact that $c_{1,4} = 0$ or 1, we have that $\lambda_4 = 1$. By induction on j, we have that $\lambda_j = 1$, $4 \leq j \leq n-1$. And then we have that $\sum_{i=1}^{p} c_{i,3} = p$, $\sum_{i=1}^{p} c_{i,4} = \cdots = \sum_{i=1}^{p} c_{i,n-1} = p-1$, and $\sum_{i=1}^{p} c_{i,n} = 0$.

When $n = 4$, From $\sum_{i=1}^{p} e_i = p-6$, $\sum_{i=1}^{p} c_{i,0} = p-4$, $\sum_{i=1}^{p} c_{i,1} = p-3$, $\sum_{i=1}^{p} c_{i,2} = p-3$ and $\sum_{i=1}^{p} c_{i,3} = p$, we can prove that $h = x_1 x_2 \cdots x_p$ cannot exist by an argument similar to that used in the proof of Theorem 3.1 (cf. [5]).

When $n > 4$, by the facts that $\sum_{i=1}^{p} c_{i,3} = p$, $\sum_{i=1}^{p} c_{i,4} = \cdots = \sum_{i=1}^{p} c_{i,n-1} = p-1$, deg $h_{k,j} = q(p^{k+j-1} + \cdots + p^j)$ $(k \geq 1, j \geq 0)$ and deg $a_i = q(p^{i-1} + \cdots + p + 1) + 1$ $(i > 0)$, we can divide the p x_i's into two disjoint classes S_1 and S_2. The two disjoint classes are given by

$$\begin{cases} S_1 = \{x | \deg x = q(p^{n-1} + p^{n-2} + \cdots + p^3) + \text{lower terms}\}, \\ S_2 = \{x | \deg x = qp^3 + \text{lower terms}\}. \end{cases}$$

For a class S in this paper, denote the number of elements in S by $N(S)$, then we can get $N(S_1) = p-1$ and $N(S_2) = 1$. Similarly, by the facts that $\sum_{i=1}^{p} e_i = p-6$, $\sum_{i=1}^{p} c_{i,0} = p-4$, $\sum_{i=1}^{p} c_{i,1} = p-3$, $\sum_{i=1}^{p} c_{i,2} = p-3$, $\sum_{i=1}^{p} c_{i,3} = p$, deg $h_{k,j} = q(p^{k+j-1} + \cdots + p^j)$ $(k \geq 1, j \geq 0)$ and deg $a_i = q(p^{i-1} + \cdots + p + 1) + 1$ $(i > 0)$, we can also divide the p x_i's into four disjoint classes. The four classes are given by

$$\begin{cases} S_3 = \{x|\deg x = q(\text{higher terms} + p^3 + p^2 + p + 1) + 1\}, \\ \qquad\qquad\qquad\qquad\qquad\qquad N(S_3) = p - 6; \\ S_4 = \{x|\deg x = q(\text{higher terms} + p^3 + p^2 + p + 1)\}, N(S_4) = 2; \\ S_5 = \{x|\deg x = q(\text{higher terms} + p^3 + p^2 + p)\}, N(S_5) = 1; \\ S_6 = \{x|\deg x = q(\text{higher terms} + p^3)\}, N(S_6) = 3. \end{cases}$$

Since $S_1 \bigcup S_2 = S_3 \bigcup S_4 \bigcup S_5 \bigcup S_6$, then at least two elements of S_6 must be in S_1. Then there would be at least two $h_{n-3,3}$'s in h with $\deg h_{n-3,3} = q(p^{n-1} + \cdots + p^3)$. This is impossible since $h_{i,j}^2 = 0$, $i > 0$, $j \geq 0$.

From Subcase 2.1 and Subcase 2.2, we get that when $s = p-6$, $E_1^{p,t',*}$ has the following generators: $a_3^{p-6}h_{1,0}h_{3,0}h_{2,1}b_{2,0}h_{1,n}$, $a_3^{p-6}h_{1,0}h_{3,0}h_{1,2}h_{2,1}h_{1,1}h_{1,n}$, $a_3^{p-6}h_{2,0}h_{3,0}h_{2,1}b_{1,1}h_{1,n}$, $a_3^{p-6}h_{2,0}h_{3,0}h_{1,2}b_{2,0}h_{1,n}$ and $a_3^{p-6}h_{2,0}h_{3,0}h_{1,2}h_{2,1}b_{1,n-1}$.

From Case 1 and Case 2, the lemma follows.

LEMMA 3.7. *Let $p \geq 7$, $n \geq 4$, $0 \leq s < p-5$. Then the product*

$$b_0h_0h_n\tilde{\gamma}_{s+3} \neq 0 \in \mathrm{Ext}_A^{s+7,p^nq+(s+3)p^2q+(s+3)pq+(s+2)q+s}(Z_p, Z_p).$$

Proof. It is known that $h_{1,n}$, $b_{1,n}$ and $a_3^sh_{3,0}h_{2,1}h_{1,2} \in E_1^{*,*,*}$ are permanent cycles in the May spectral sequence and converge nontrivially to h_n, b_n, $\tilde{\gamma}_{s+3} \in \mathrm{Ext}_A^{*,*}(Z_p, Z_p)$ for $n \geq 0$ respectively (see Theorem 3.1).

From Lemma 3.2, we know that $E_1^{s+6,t',*}$ has the generators $a_3^sh_{1,0}h_{3,0}h_{2,1}b_{2,0}h_{1,n}$, $a_3^sh_{1,0}h_{3,0}h_{1,2}h_{2,1}h_{1,1}h_{1,n}$, $a_3^sh_{2,0}h_{3,0}h_{2,1}b_{1,1}h_{1,n}$, $a_3^s h_{2,0}h_{3,0}h_{1,2}b_{2,0}h_{1,n}$, and $a_3^sh_{2,0}h_{3,0}h_{1,2}h_{2,1}b_{1,n-1}$, where $t' = p^nq + (s+3)p^2q + (s+3)pq + (s+2)q + s$.

By induction on s and (2.2), we can show that

$$\begin{aligned} & d_1(a_3^sh_{3,0}h_{2,1}h_{1,0}h_{1,n}b_{2,0}) \\ =& (-1)^s s a_3^{s-1}a_2h_{1,2}h_{3,0}h_{2,1}h_{1,0}h_{1,n}b_{2,0} + \\ & (-1)^{s+1}a_3^sh_{2,0}h_{1,2}h_{2,1}h_{1,0}h_{1,n}b_{2,0} + (-1)^s a_3^sh_{3,0}h_{1,1}h_{1,2}h_{1,0}h_{1,n}b_{2,0} \\ \neq& 0, \\ & d_1(a_3^sh_{3,0}h_{2,0}h_{1,2}h_{1,n}b_{2,0}) \\ =& (-1)^s s a_3^{s-1}a_1h_{2,1}h_{3,0}h_{2,0}h_{1,2}h_{1,n}b_{2,0} \\ & + (-1)^s a_3^sh_{2,0}h_{1,2}h_{2,1}h_{1,0}h_{1,n}b_{2,0} + (-1)^s a_3^sh_{3,0}h_{1,1}h_{1,2}h_{1,0}h_{1,n}b_{2,0} \\ \neq& 0, \end{aligned}$$

$d_1(a_3^s h_{3,0}h_{2,0}h_{2,1}h_{1,n}b_{1,1})$
$=(-1)^s s a_3^{s-1}a_2 h_{1,2}h_{3,0}h_{2,0}h_{2,1}h_{1,n}b_{1,1}+$
$(-1)^s a_3^s h_{3,0}h_{1,0}h_{1,1}h_{2,1}h_{1,n}b_{1,1} + (-1)^{s+1}a_3^s h_{3,0}h_{2,0}h_{1,1}h_{1,2}h_{1,n}b_{1,1}$
$\neq 0,$
$d_1(a_3^s h_{3,0}h_{2,1}h_{1,2}b_{1,n-1}h_{2,0})$
$=(-1)^s a_3^s h_{3,0}h_{2,1}h_{1,2}b_{1,n-1}h_{1,0}h_{1,1}$
$\neq 0.$

From the above results, we see that the four differentials of generators are impossible to equal $b_{1,0}h_{1,0}h_{1,n}a_3^s h_{3,0}h_{2,1}h_{1,2}$ up to nonzero scalar and are linearly independent.

Meanwhile, we have that for $r \geq 1$, $d_r(a_3^s h_{1,0}h_{3,0}h_{1,2}h_{2,1}h_{1,1}h_{1,n}) = 0$. Thus $E_r^{s+6,t',*} \subseteq Z_p\{a_3^s h_{1,0}h_{3,0}h_{1,2}h_{2,1}h_{1,1}h_{1,n}\}$ for all $r \geq 2$. Thus the permanent cycle $b_{1,0}h_{1,0}h_{1,n}a_3^s h_{3,0}h_{2,1}h_{1,2} \in E_r^{s+7,t',*}$ does not bound. That is to say, $b_{1,0}h_{1,0}h_{1,n}a_3^s h_{3,0}h_{2,1}h_{1,2} \in E_r^{s+7,t',*}$ is a permanent cycle in the May spectral sequence and converges nontrivially to $b_0 h_0 h_n \tilde{\gamma}_{s+3} \in \mathrm{Ext}_A^{s+7,t'}(Z_p, Z_p)$. It follows that $b_0 h_0 h_n \tilde{\gamma}_{s+3} \neq 0 \in \mathrm{Ext}_A^{s+7,t'}(Z_p, Z_p)$. The lemma is proved. □

LEMMA 3.8. *Let $p \geq 7$, $n \geq 4$, $0 \leq s < p-5$, $2 \leq r \leq s+7$. Then the group* $\mathrm{Ext}_A^{s+7-r,q(p^n+(s+3)p^2+(s+3)p+(s+2))+(s-r+1)}(Z_p, Z_p) = 0$.

Proof. The proof is divided into two cases.

Case 1. $r = s+7$.

Since $q(p^n + (s+3)p^2 + (s+3)p + (s+2)) + s + 1 - r = q(p^n + (s+3)p^2 + (s+3)p + (s+2)) + s + 1 - s - 7 = q(p^n + (s+3)p^2 + (s+3)p + (s+1)) + q - 6 > 0$, then we have that when $r = s+7$, $\mathrm{Ext}_A^{s+7-r,q(p^n+(s+3)p^2+(s+3)p+(s+2))+(s-r+1)}(Z_p, Z_p) = 0$.

Case 2. $2 \leq r < s+7$.

Now we prove that when $2 \leq r < s+7$, $\mathrm{Ext}_A^{s+7-r,t'''}(Z_p, Z_p) = 0$. It suffices to prove that in the May spectral sequence $E_1^{s+7-r,t''',*} = 0$, where $t''' = q(p^n + (s+3)p^2 + (s+3)p + (s+2)) + (s-r+1)$.

Consider $h = x_1 x_2 \cdots x_m \in E_1^{s+7-r,t''',*}$, where x_i is one of a_k, $h_{l,j}$ or $b_{u,z}$, $0 \leq k \leq n+1$, $0 \leq l+j \leq n+1$, $0 \leq u+z \leq n$, $l > 0$, $j \geq 0$, $u > 0$, $z \geq 0$. Assume that deg $x_i = q(c_{i,n}p^n + c_{i,n-1}p^{n-1} + \cdots + c_{i,0}) + e_i$, where $c_{i,j} = 0$ or 1, $e_i = 1$ if $x_i = a_{k_i}$, or $e_i = 0$. Then we have

$$\begin{aligned}\deg h &= \sum_{i=1}^{m} \deg x_i \\ &= q((\sum_{i=1}^{m} c_{i,n})p^n + \cdots + (\sum_{i=1}^{m} c_{i,2})p^2 + (\sum_{i=1}^{m} c_{i,1})p \\ &\quad + (\sum_{i=1}^{m} c_{i,0})) + (\sum_{i=1}^{m} e_i) \\ &= q(p^n + (s+3)p^2 + (s+3)p + (s+2)) + (s-r+1),\end{aligned}$$

$$\dim h = \sum_{i=1}^{m} \dim x_i = s + 7 - r.$$

Note that $\dim x_i = 1$ or 2 and $2 \leq r < s+7$, we can get that $m \leq s+7-r \leq s+7-2 = s+5 < p$ from the equality $\dim h = \sum_{i=1}^{m} \dim x_i = s+7-r$.

We claim that $s - r + 1 \geq 0$. For otherwise, we would have $p > \sum_{i=1}^{m} e_i = q + (s+1-r) \geq q - 6 \geq p$ by $p \geq 7$. That is impossible. Then the claim follows.

By an argument similar to that used in Case 1 of Lemma 3.2, we can get that

$$\begin{cases} \sum_{i=1}^{m} e_i = s-r+1, & \sum_{i=1}^{m} c_{i,0} = s+2, \qquad\qquad \sum_{i=1}^{m} c_{i,1} = s+3, \\ \sum_{i=1}^{m} c_{i,2} = s+3, & \sum_{i=1}^{m} c_{i,3} = \cdots = \sum_{i=1}^{m} c_{i,n-1} = 0, \quad \sum_{i=1}^{m} c_{i,n} = 1. \end{cases}$$

It is easy to see that there exists a factor $h_{1,n}$ or $b_{1,n-1}$ in h. We can denote $h_{1,n}$ or $b_{1,n-1}$ by x_m, then $h' = x_1 \cdots x_{m-1} \in E_1^{l,t'''-p^n q,*}$, where $l = s+6-r$ or $s+5-r$. And we have that

$$\begin{cases} \sum_{i=1}^{m-1} e_i = s-r+1, & \sum_{i=1}^{m-1} c_{i,0} = s+2, \\ \sum_{i=1}^{m-1} c_{i,1} = s+3, & \sum_{i=1}^{m-1} c_{i,2} = s+3. \end{cases}$$

Subcase 2.1. If

$$h = x_1 x_2 \cdots x_{m-1} h_{1,n}, h' = x_1 x_2 \cdots x_{m-1} \in E_1^{s+6-r,t'''-p^n q,*}.$$

When $r > 3$, from $s+6-r < s+3 = \sum_{i=1}^{m-1} c_{i,1}$ we can get that $E_1^{s+6-r,t'''-p^nq,*} = 0$ by Proposition 2.3. When $2 \leq r \leq 3$, we can easily show that $E_1^{s+6-r,t'''-p^nq,*} = 0$ by an argument similar to that used in the proof of Lemma 3.2.

Subcase 2.2. If

$$h = x_1x_2\cdots x_{m-1}b_{1,n-1}, h' = x_1x_2\cdots x_{m-1} \in E_1^{s+5-r,t'''-p^nq,*}.$$

If $2 < r < s+7$, from $s+5-r < s+3 = \sum_{i=1}^{m-1} c_{i,1}$, we have that $E_1^{s+5-r,t'''-p^nq,*} = 0$ by Proposition 2.3. If $r = 2$, we can easily show that $E_1^{s+5-r,t'''-p^nq,*} = 0$ by an argument similar to that used in the proof of Lemma 3.2.

From Case 1 and Case 2, it follows that $E_1^{s+7-r,t''',*} = 0$. Thus $\mathrm{Ext}_A^{s+7-r,t'''}(Z_p, Z_p) = 0$. This finishes the proof of Lemma 3.8. □

Now, we give the proofs of the main theorems in this paper.

Proof of Theorem 1.4. It is known that $h_0 \in \mathrm{Ext}_A^{1,q}(Z_p, Z_p)$ is a permanent cycle and converges nontrivially to the α-element $\alpha_1 = j\alpha i \in \pi_{q-1}S$ in the Adams spectral sequence. By virtue of the second part of Theorem 1.1, we have that the homotopy element $\alpha_1 j\xi_n \in \pi_*S$ is represented up to nonzero scalar by $h_0(b_0h_n + h_1b_{n-1}) = h_0b_0h_n \in \mathrm{Ext}_A^{4,p^nq+pq+q}(Z_p, Z_p)$ in the Adams spectral sequence.

From [9], we see that

$$h_0b_0h_n \neq 0 \in \mathrm{Ext}_A^{4,p^nq+pq+q}(Z_p, Z_p).$$

Meanwhile, from [6], we have that groups

$$\mathrm{Ext}_A^{4-r,p^nq+pq+q-r+1}(Z_p, Z_p) = 0,$$

where $2 \leq r$. Thus it follows that $h_0b_0h_n \neq 0 \in \mathrm{Ext}_A^{4,p^nq+pq+q}(Z_p, Z_p)$ converges nontrivially to the homotopy element $\alpha_1 j\xi_n \in \pi_*S$ in the Adams spectral sequence. This completes the proof of Theorem 1.4. □

To prove Theorem 1.5, it is equivalent to prove the following.

THEOREM 3.9. *Let $p \geq 7$, $n \geq 4$. Then the product*

$$b_0h_0h_n\tilde{\gamma}_{s+3} \neq 0 \in \mathrm{Ext}_A^{s+7,p^nq+(s+3)p^2q+(s+3)pq+(s+2)q+s}(Z_p, Z_p)$$

is a permanent cycle in the Adams spectral sequence and converges to a nontrivial element $\gamma_{s+3}\alpha_1 j\xi_n$ of order p in $\pi_{p^nq+(s+3)p^2q+(s+3)pq+(s+2)q-7}S$, where $0 \leq s < p-5$, $q = 2(p-1)$.

Proof. From [2], $(b_0h_n + h_1b_{n-1}) \in \mathrm{Ext}_A^{3,p^nq+pq}(Z_p, Z_p)$ is a permanent cycle in the Adams spectral sequence and converges to a nontrivial element $j\xi_n \in \pi_{p^nq+pq-3}S$ for $n \geq 4$. Let $\gamma : \Sigma^{q(p^2+p+1)}V(2) \to V(2)$ be the v_3-map and consider the following composition of maps

$$\bar{f} = \gamma_{s+3}\alpha_1 j\xi_n : \quad \Sigma^{p^nq+pq-3}S \xrightarrow{j\xi_n} S \xrightarrow{j\alpha i} \Sigma^{-q+1}S \xrightarrow{\bar{i}i'i} \Sigma^{-q+1}V(2)$$
$$\xrightarrow{\gamma^{s+3}} \Sigma^{-(s+3)(p^2+p+1)q-q+1}V(2) \xrightarrow{jj'\bar{j}} \Sigma^{-(s+3)(p^2+p+1)q+(p+1)q+4}S.$$

Since up to nonzero scaler ξ_n and $\alpha_1 = j\alpha i$ are represented by $(b_0h_n + h_1b_{n-1}), h_0 \in \mathrm{Ext}_A^{*,*}(Z_p, Z_p)$ respectively, then the above $\bar{f}$ is represented up to nonzero scalar by

$$\begin{aligned}\bar{c} &= (jj'\bar{j})_*(\gamma_*)^{s+3}(\bar{i}i'i)_*(h_0(b_0h_n + h_1b_{n-1}))\\ &= (jj'\bar{j}\gamma^{s+3}\bar{i}i'i)_*(b_0h_0h_n).\end{aligned}$$

From Theorem 3.1 and the knowledge of Yoneda products we know that the composition

$$\mathrm{Ext}_A^{0,0}(Z_p, Z_p) \xrightarrow{(\bar{i}i'i)_*} \mathrm{Ext}_A^{0,0}(H^*V(2), Z_p)$$
$$\xrightarrow{(jj'\bar{j})_*(\gamma_*)^{s+3}} \mathrm{Ext}_A^{s+3,(s+3)p^2q+(s+2)pq+(s+1)q+s}(Z_p, Z_p)$$

is a multiplication (up to nonzero scalar) by

$$\tilde{\gamma}_{s+3} \in \mathrm{Ext}_A^{s+3,(s+3)p^2q+(s+2)pq+(s+1)q+s}(Z_p, Z_p).$$

Hence, $\bar{f}$ is represented (up to nonzero scalar) by

$$\bar{c} = \tilde{\gamma}_{s+3}b_0h_0h_n \neq 0 \in \mathrm{Ext}_A^{s+7,p^nq+(s+3)p^2q+(s+3)pq+(s+2)q+s}(Z_p, Z_p)$$

in the Adams spectral sequence (see Lemma 3.7).

Moreover, from Lemma 3.8, we can see that $b_0h_0h_n\tilde{\gamma}_{s+3}$ cannot be hit by the differentials in the Adams spectral sequence and so the corresponding homotopy element $\bar{f} = \gamma_{s+3}\alpha_1 j\xi_n \in \pi_*S$ is nontrivial and of order p. This finishes the proof of the theorem. □

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