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THE OPERATIONAL CALCULUS FOR A MEASURE-VALUED DYSON SERIES

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ABSTRACT. Recently, we proved the existence theorem of measure-valued Feynman-Kac formula and showed that it satisfied Volterra's integral equation. In this paper, we establish the operational calculus for a measure-valued Dyson series and give some examples related to the measure-valued Dyson series.

1. Introduction

In 1951, for the sake of a solution of Schrödinger's wave equation, Feynman suggested a special integral, so called the Feynman integral ([2]) and he gave heuristic formulation for an operational calculus for noncommuting operators ([3]). Because his integral was not perfect from the mathematical point of view, Cameron and Storvick introduced an operator-valued Feynman integral which was make sense mathematically and they investigated some properties for the integral ([1]). In 1988, Johnson and Lapidus established the operational calculus for the Cameron and Storvick's Feynman integral ([4]).

In 2002, we presented some definitions and theories for an analogue of Wiener measure which was a kind of generalization of the classical Wiener measure and we derived a measure-valued measure V_φ from the concept of the analogue of Wiener. We also established a measure-valued Feynman-Kac formula using the concept of Bartle integral with respect to V_φ ([5]). Recently, we proved the existence theorem of a measure-valued Dyson series, the generalized Feynman-Kac formula and its stability theorems, a kind of convergence theorem for the Bartle integral

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with respect to V_φ ([6]).

In this paper, we establish the operational calculus for a measure-valued Dyson series and give some examples related to the measure-valued Dyson series. In the next section, we introduce some notations, definitions, and basic facts which are needed to understand the last section.

2. Preliminaries

Here, we introduce some notations, definitions and facts which are needed to understand this note.

(A) Let $\mathbb{R}$ be the real number system. For a natural number n, let $\mathbb{R}^n$ be the n-times product space of $\mathbb{R}$. Let $\mathcal{B}(\mathbb{R})$ be the set of all Borel subsets of $\mathbb{R}$ and let m_L be the Lebesgue measure on the measurable space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

(B) For τ in $\mathbb{R}$, let δ_τ be the Dirac measure concentrated at τ with total mass one. Let $\mathcal{M}(\mathbb{R})$ be the space of all finite complex-valued countable additive measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. For μ in $\mathcal{M}(\mathbb{R})$ and for E in $\mathcal{B}(\mathbb{R})$, the total variation $|\mu|(E)$ on E is defined by

$$|\mu|(E) = \sup \sum_{i=1}^{n} |\mu(E_i)|, \tag{2.1}$$

where the supremum is taken over all finite sequences $\langle E_i \rangle$ of disjoint sets in $\mathcal{B}(\mathbb{R})$ with $\cup_{i=1}^{n} E_i = E$. Let $\mathbb{B}$ be a complex Banach space and let $\mathbb{B}^*$ be the dual space of $\mathbb{B}$. For a $\mathbb{B}$-valued countably additive measure ν on $(X, \mathcal{B})$ and for E in $\mathcal{B}$, the semivariation $\|\nu\|(E)$ of ν on E is given by

$$\|\nu\|(E) = \sup\{|x^*\nu|(E) \mid x^*\nu \text{ is in } \mathbb{B}^* \text{ and } \|x^*\|_{\mathbb{B}^*} \leq 1\}, \tag{2.2}$$

where $|x^*\nu|(E)$ is the total variation on E of $x^*\nu$.

(C) For two real numbers a and b with $a < b$, let $C[a, b]$ be the space of all real-valued continuous functions on a closed bounded interval $[a, b]$ with the supremum norm $\|\cdot\|_\infty$. For $\vec{s} = (s_0, s_1, \ldots, s_n)$ with $a = s_0 < s_1 < s_2 < \cdots < s_n = b$, let $I_{\vec{s}} : C[a, b] \to \mathbb{R}^{n+1}$ be a function with

$$I_{\vec{s}}(x) = (x(s_0), x(s_1), \ldots, x(s_n)). \tag{2.3}$$

For B_j $(j = 0, 1, 2, \ldots, n)$ in $\mathcal{B}(\mathbb{R})$, the subset $I_{\vec{s}}^{-1}(\prod_{j=0}^{n} B_j)$ of $C[a, b]$ is called an interval. Let $\mathcal{I}$ be the set of all intervals. For φ in $\mathcal{M}(\mathbb{R})$, we let

$$m_\varphi^{a,b}(I_{\vec{s}}^{-1}(\prod_{j=0}^{n} B_j)) = \int_{B_0}\Big[\int_{\prod_{j=1}^{n} B_j} W(n+1;\vec{s};u_0,u_1,\ldots,u_n) \, d\prod_{j=1}^{n} m_L(u_1,u_2,\ldots,u_n)\Big]\, d\varphi(u_0), \tag{2.4}$$

where

$$\begin{aligned} &W(n+1;\vec{s};u_0,u_1,\ldots,u_n)\\ &=\Big(\prod_{j=1}^{n}\frac{1}{\sqrt{2\pi(s_j-s_{j-1})}}\Big)\exp\Big\{-\frac{1}{2}\sum_{j=1}^{n}\frac{(u_j-u_{j-1})^2}{s_j-s_{j-1}}\Big\}. \end{aligned}\tag{2.5}$$

Then there exists a unique complex-valued measure $\omega_\varphi^{a,b}$ on $(C[a,b], \mathcal{B}(C[a,b]))$ such that $\omega_\varphi^{a,b}(I) = m_\varphi^{a,b}(I)$ for all I in $\mathcal{I}$.

By the change of variable formula, we have the following theorem.

THEOREM 2.1 (The Wiener integration formula). *If f is a complex-valued Borel measurable function on $\mathbb{R}^{n+1}$ and $\vec{s} = (s_0, s_1, \ldots, s_n)$ is a vector in $\mathbb{R}^{n+1}$ with $a = s_0 < s_1 < \cdots < s_n = b$ then the following equality holds.*

$$\begin{aligned} &\int_{C[a,b]} f(x(s_0),x(s_1),\ldots,x(s_n))\, d\omega_\varphi^{a,b}(x)\\ &\overset{*}{=}\int_{\mathbb{R}^{n+1}} f(u_0,u_1,\ldots,u_n)W(n+1;\vec{s};u_0,u_1,\ldots,u_n)\\ &\quad d(\prod_{j=1}^{n} m_L\times\varphi)((u_1,u_2,\ldots,u_n),u_0), \end{aligned}\tag{2.6}$$

where $\overset{*}{=}$ means that if one side exists then both side exist and two values are equal.

(D) Let $X : C[a,b] \to \mathbb{R}$ be a function with $X(x) = x(b)$. For φ in $\mathcal{M}(\mathbb{R})$ and for B in $\mathcal{B}(C[a,b])$, we let

$$[V_\varphi^{a,b}(B)](E) = \omega_\varphi^{a,b}(B\cap X^{-1}(E)) \tag{2.7}$$

for E in $\mathcal{B}(\mathbb{R})$. Then $V_\varphi^{a,b}$ is a measure-valued measure on $(C[a,b], \mathcal{B}(C[a,b]))$ in the total variation norm sense, for B in $\mathcal{B}(C[a,b])$, $\|V_\varphi^{a,b}(B)\| \le 4|\varphi|(\mathbb{R})$ and for B in $\mathcal{B}(C[a,b])$ with $|\omega_\varphi^{a,b}(B)| = 0$, $V_\varphi^{a,b}(B)$ is a zero measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. For B in $\mathcal{B}(C[a,b])$, letting $V^{a,b}(B) : \mathcal{M}(\mathbb{R}) \to \mathcal{M}(\mathbb{R})$ with $[V^{a,b}(B)](\varphi) = V_\varphi^{a,b}(B)$, $V^{a,b}(B)$ is a bounded linear operator.

From [5, Theorem 4.4, p.4934], we can find the following theorem.

THEOREM 2.2. *Let f be a complex-valued Borel measurable function on $\mathbb{R}^{n+1}$ and let $\vec{s} = (s_0, s_1, \ldots, s_n)$ be a vector in $\mathbb{R}^{n+1}$ with $a = s_0 < s_1 < s_2 < \cdots < s_n = b$. Let φ be in $\mathcal{M}(\mathbb{R})$. If $f(u_0, u_1, u_2, \ldots, u_n)W(n+1; \vec{s}; u_0, u_1, u_2, \ldots, u_n)$ is $|\varphi| \times \prod_{j=1}^{n} m_L$-integrable then a function $F(x) = f(x(s_0), x(s_1), \ldots, x(s_n))$ is $V_\varphi^{a,b}$-Bartle integrable on $C[a,b]$ and for E in $\mathcal{B}(\mathbb{R})$,*

(2.8)

$$\Big[(Ba) - \int_{C[a,b]} F(x)\, dV_\varphi^{a,b}(x)\Big](E)$$
$$= \int_E \Big\{ \int_{\mathbb{R}^{n-1}} \Big(\int_{\mathbb{R}} f(u_0, u_1, \ldots, u_n) W(n+1; \vec{s}; u_0, u_1, \ldots, u_n)\, d\varphi(u_0) \Big)$$
$$d \prod_{j=1}^{n-1} m_L(u_1, u_2, \ldots, u_{n-1}) \Big\} dm_L(u_n).$$

(E) Let $\mathcal{RM}(\mathbb{R})$ be the space of all finite complex-valued measures μ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ that are absolutely continuous with respect to m_L. For θ in $L^\infty(\mathbb{R}, m_L)$, we let

$$[M_\theta(\mu)](E) = \int_E \frac{d\mu}{dm_L}(\xi)\theta(\xi)\, dm_L(\xi), \tag{2.9}$$

for E in $\mathcal{B}(\mathbb{R})$ and for μ in $\mathcal{RM}(\mathbb{R})$. Then M_θ is an operator from $\mathcal{RM}(\mathbb{R})$ into itself by the bound $\|\theta\|_\infty$. For $s > 0$, let $S_s : \mathcal{RM}(\mathbb{R}) \to \mathcal{RM}(\mathbb{R})$ be an operator such that

$$[S_s(\mu)](E) = \frac{1}{\sqrt{2\pi s}} \int_{\mathbb{R}} \Big\{ \int_E \exp\Big(-\frac{(u-v)^2}{2} \Big)\, dm_L(u) \Big\} d\mu(v) \tag{2.10}$$

for E in $\mathcal{B}(\mathbb{R})$. It is not hard to show that S_s is a bounded linear operator and the operator norm $\|S_s\|$ of S_s is less that or equal to one.

For $s > 0$, for φ in $\mathcal{M}(\mathbb{R})$, for a Borel measurable $|\varphi|$-essentially bounded function θ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and for E in $\mathcal{B}(\mathbb{R})$, we let

(2.11)

$$[T(s, \varphi, \theta)](E) = \frac{1}{\sqrt{2\pi s}} \int_{\mathbb{R}} \Big[\int_E \theta(v) \exp\Big\{ -\frac{(u-v)^2}{2} \Big\}\, dm_L(u) \Big] d\varphi(v).$$

Let φ be in $\mathcal{M}(\mathbb{R})$ and let $\eta = \mu + \nu$ be a complex-valued Borel measure on $[a,b]$ such that μ is a continuous part of η and $\nu = \sum_{p=0}^{n} c_p \delta_{\tau_p}$, where $a = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_n = b$ and $c_p (p = 0, 1, 2, \ldots, n)$ are complex numbers. Let θ be a complex-valued Borel measurable function on $[a,b]\times\mathbb{R}$ such that

$$\|\theta\|_{\varphi;\infty,1;\eta} \equiv \int_{[a,b]} \|\theta(s,\cdot)\|_{\varphi;\infty}\, d|\eta|(s) \tag{2.12}$$

is finite where $\|\theta(a,\cdot)\|_{\varphi;\infty}$ is the $|\varphi|$-essentially supremum norm for a function $\theta(a,\xi)$ of ξ and $\|\theta(s,\cdot)\|_{\varphi;\infty}$ $(a < s \le b)$ is the m_L-essentially supremum norm for a function $\theta(s,\xi)$ of ξ. By [6, Theorem 3.2.], we have the following theorem.

THEOREM 2.3. *Under the notation in above, let g be an analytic function with the radius of convergence less than $\|\theta\|_{\varphi;\infty,1;\eta}$, say $g(z) = \sum_{m=0}^{\infty} a_m z^m$. Then $g(\int_{[a,b]}\theta(s,x(s))\,d\eta(s))$ is $V_\varphi^{a,b}$-Bartle integrable on $C[a,b]$ and for E in $\mathcal{B}(\mathbb{R})$,*

$$\begin{aligned}
&\Big[(Ba) - \int_{C[a,b]} g\Big(\int_{[a,b]} \theta(s,x(s))\,d\eta(s)\Big) dV_\varphi^{a,b}(x)\Big](E) \\
&= \sum_{m=0}^{\infty} a_m \frac{1}{m!} \sum_{q_0+q_1+\cdots+q_{n+1}=m} \frac{\prod_{p=0}^{n} c_p^{q_p}}{\prod_{p=0}^{n} q_p!} \sum_{j_1+j_2+\cdots+j_n=q_{n+1}} \\
&\int_{\triangle^{a,b}_{q_{n+1};j_1,j_2,\ldots,j_n}} [(L_n \circ L_{n-1}\circ\cdots\circ L_1)(T(s_{1,1},\varphi,\theta(a,\cdot)^{q_0}))](E) \\
&d\Big(\prod_{i=1}^{n}\prod_{j=1}^{j_i}\mu\Big)(s_{1,1},\ldots,s_{n,j_n}).
\end{aligned}\tag{2.13}$$

Moreover,

$$\begin{aligned}
&\Big|(Ba) - \int_{C[a,b]} g\Big(\int_{[a,b]} \theta(s,x(s))\,d\eta(s)\Big)\, dV_\varphi(x)\Big|(\mathbb{R}) \\
&\le 4|\varphi|(\mathbb{R})\Big[\sum_{m=0}^{\infty} |a_m|\|\theta\|^m_{\varphi;\infty,1;\eta}\Big].
\end{aligned}\tag{2.14}$$

Here, for $k = 2,\ldots,n$,

$$L_k = M_{\theta(\tau_k)^{q_k}}\circ S_{\tau_k - s_{k,j_k}}\circ M_{\theta(s_{k,j_k})}\circ S_{s_{k,j_k}-s_{k,j_k-1}}\circ\cdots\circ M_{\theta(s_{k,1})}\circ S_{s_{k,1}-s_{k,0}}, \tag{2.15}$$

and

$$L_1 = M_{\theta(\tau_1)^{q_1}}\circ S_{\tau_1 - s_{1,j_1}}\circ M_{\theta(s_{1,j_1})}\circ S_{s_{1,j_1}-s_{1,j_1-1}}\circ\cdots\circ M_{\theta(s_{1,1})}. \tag{2.16}$$

And, for nonnegative integers q and $j_1, j_2, \dots, j_n$ with $q = j_1 + j_2 + \cdots + j_n$, let

(2.17)

$$\begin{aligned}\triangle^{a,b}_{q;j_1,j_2,\dots,j_n} =\{(&s_{1,1}, s_{1,2}, \dots, s_{1,j_1}, s_{2,1}, \dots, s_{n-1,j_{n-1}}, s_{n,1}, \dots, s_{n,j_n})| \\ &\tau_0 = a < s_{1,1} < \cdots < s_{1,j_1} < \tau_2 < s_{2,1} < \cdots \\ &\cdots < \tau_{n-1} < s_{n,1} < \cdots < s_{n,j_n} < \tau_n = b\}.\end{aligned}$$

3. The operational calculus for a measure-valued Dyson series

In this section, we investigate Feynman's operational calculus for a measure-valued Dyson series.

Throughout this section, let t_1 and t_2 be two real numbers with $0 < t_1 < t_2$ and let φ be in $\mathcal{M}(\mathbb{R})$.

THEOREM 3.1. *Let $(s_0, s_1, s_2, \dots, s_{m+n})$ be in $\mathbb{R}^{m+n+1}$ with $0 < s_0 < s_1 < \cdots < s_m = t_1 < s_{m+1} < \cdots < s_{m+n} = t_2$. Let f_1 and f_2 be two complex-valued Borel measurable functions on $\mathbb{R}^{m+1}$ and $\mathbb{R}^n$, respectively such that*

$$f_1(u_0, u_1, \dots, u_m) W(m+1; (s_0, s_1, \dots, s_m); u_0, u_1, \dots, u_m) \tag{3.1}$$

is $|\varphi| \times \prod_{j=1}^{m} m_L$-integrable on $\mathbb{R}^{m+1}$ and

$$\begin{aligned} &f_1(u_0, \dots, u_m) f_2(u_{m+1}, \dots, u_{m+n}) \\ &\times W(m+n+1; (s_0, \dots, s_{m+n+1}); u_0, u_1, \dots, u_{m+n}) \end{aligned} \tag{3.2}$$

is $|\varphi| \times \prod_{j=1}^{m+n} m_L$-integrable on $\mathbb{R}^{m+n+1}$. Then

$$F_1(x) = f_1(x(s_0), x(s_1), \dots, x(s_m)) \tag{3.3}$$

is V_φ^{0,t_1}-Bartle integrable on $C[0, t_1]$,

$$F(x) = f_1(x(s_0), x(s_1), \dots, x(s_m)) f_2(x(s_{m+1}), \dots, x(s_{m+n})) \tag{3.4}$$

is V_φ^{0,t_2}-Bartle integrable on $C[0, t_2]$ and

$$F_2(x) = f_2(x(s_{m+1}), x(s_{m+2}), \dots, x(s_{m+n})) \tag{3.5}$$

is $V_{\tilde{\varphi}}^{t_1,t_2}$-Bartle integrable on $C[t_1, t_2]$, where

$$\tilde{\varphi}(E) = \left[(Ba) - \int_{C[0,t_1]} F_1(x)\, dV_\varphi^{0,t_1}(x)\right](E) \tag{3.6}$$

for E in $\mathcal{B}(\mathbb{R})$. Moreover,

$$(3.7)\quad (Ba)-\int_{C[0,t_2]} F(x)\,dV_{\varphi}^{0,t_2}(x) = (Ba)-\int_{C[t_1,t_2]} F_2(x)\,dV_{\tilde{\varphi}}^{t_1,t_2}(x).$$

Proof. From Theorem 2.2, F_1 and F are V_{φ}^{0,t_1} and V_{φ}^{0,t_2}-Bartle integrable on $C[0,t_1]$ and $C[0,t_2]$, respectively. By Theorem 2.2, the Radon-Nikodym derivative $d\tilde{\varphi}/dm_L$ exists and

(3.8)
$$\frac{d\tilde{\varphi}}{dm_L}(u_m) = \int_{\mathbb{R}^{m-1}}\Big(\int_{\mathbb{R}} f(u_0,u_1,\dots,u_m)W(m+1;(s_0,s_1,\dots,s_m); u_0,u_1,\dots,u_m)\,d\varphi(u_0)\Big)\,d\prod_{j=1}^{m-1} m_L(u_1,u_2,\dots,u_{m-1}).$$

So, for E in $\mathcal{B}(\mathbb{R})$,

$$\begin{aligned}
&\Big[(Ba)-\int_{C[0,t_2]} F(x)\,dV_{\varphi}^{0,t_2}(x)\Big](E)\\
&\overset{(1)}{=}\int_E\Big\{\int_{\mathbb{R}^{m+n-1}}\Big(\int_{\mathbb{R}} f_1(u_0,u_1,\dots,u_m)f_2(u_{m+1},u_{m+2},\dots,u_{m+n})\\
&\quad W(m+n+1;(s_0,s_1,\dots,s_{m+n});u_0,u_1,\dots,u_{m+n})\,d\varphi(u_0)\Big)\\
&\quad d\prod_{j=1}^{m+n-1} m_L(u_1,u_2,\dots,u_{m+n-1})\Big\}\,dm_L(u_{m+n})\\
&\overset{(2)}{=}\int_E\Big[\int_{\mathbb{R}^n} f_2(u_{m+1},u_{m+2},\dots,u_{m+n})\\
(3.9)\quad &\quad W(n+1;(s_m,s_{m+1},\dots,s_{m+n});u_m,u_{m+1},\dots,u_{m+n})\\
&\quad \Big\{\int_{\mathbb{R}^{m-1}}\Big(\int_{\mathbb{R}} f_1(u_0,u_1,\dots,u_m)W(m+1;(s_0,s_1,\dots,s_m);\\
&\quad u_0,u_1,\dots,u_m)\,d\varphi(u_0)\Big)\,d\prod_{j=1}^{m-1} m_L(u_1,u_2,\dots,u_{m-1})\Big\}\\
&\quad d\prod_{j=m}^{m+n-1} m_L(u_m,u_{m+1},\dots,u_{m+n-1})\Big]\,dm_L(u_{m+n})\\
&\overset{(3)}{=}\int_E\Big\{\int_{\mathbb{R}^{n-1}}\Big(\int_{\mathbb{R}} f_2(u_{m+1},u_{m+2},\dots,u_{m+n})
\end{aligned}$$

$$W(n+1;(s_m,s_{m+1},\dots,s_{m+n});u_m,u_{m+1},\dots,u_{m+n})\,d\tilde{\varphi}(u_m)\Big)$$
$$d\prod_{j=m+1}^{m+n-1} m_L(u_{m+1},\dots,u_{m+n-1})\Big\}\,dm_L(u_{m+n}).$$

Step (1) results from Theorem 2.2. From Fubini's theorem, we have Step (2). Using the Radon-Nikodym derivative $d\tilde{\varphi}/dm_L$, we obtain Step (3).

Since the integral $[(Ba)-\int_{C[0,t_2]}F(x)\,dV_\varphi^{0,t_2}(x)](E)$ exists, $f_2(u_{m+1},\dots,u_{m+n})W(n+1;(s_m,\dots,s_{m+n});u_m,\dots,u_{m+n})$ is $|\tilde{\varphi}|\times\prod_{j=m}^{m+n}m_L$-integrable, so F_2 is $V_{\tilde{\varphi}}^{t_1,t_2}$-Bartle integrable on $C[t_1,t_2]$ and

$$(Ba)-\int_{C[0,t_2]}F(x)\,dV_\varphi^{0,t_2}(x)=(Ba)-\int_{C[t_1,t_2]}F_2(x)\,dV_{\tilde{\varphi}}^{t_1,t_2}(x) \tag{3.10}$$

holds. □

REMARK 3.2. Let φ be in $\mathcal{M}(\mathbb{R})$, let $P_1:C[0,t_2]\to C[0,t_1]$ be a function with $[P_1(x)](s)=x(s)$ for $0\le s\le t_1$ and let $P_2:C[0,t_2]\to C[t_1,t_2]$ be a function with $[P_2(x)](s)=x(s)$ for $t_1\le s\le t_2$. Then by Theorem 3.1, $V_\varphi^{0,t_2}(I)=V_{\tilde{\varphi}}^{t_1,t_2}(P_2(I))$ for I in $\mathcal{I}$ where $\tilde{\varphi}=V_\varphi^{0,t_1}(P_1(I))$. But it is not true always that $V_\varphi^{0,t_2}(B)=V_{\tilde{\varphi}}^{t_1,t_2}(P_2(B))$ for B in $\mathcal{B}(C[0,t_2])$ where $\tilde{\varphi}=V_\varphi^{0,t_1}(P_1(B))$. Because, putting $\varphi=\delta_0$, $t_1=1,t_2=2$ and $B=\{x$ in $C[0,t_2]\mid$ either $x(1)\ge 0$ and $x(2)\ge 0$ or $x(1)<0$ and $x(2)<0$ holds$\}$, B in $\mathcal{B}(C[0,2])$, $[V_\varphi^{0,2}(B)](\mathbb{R})=\frac{1}{2}$, $V_\varphi^{0,1}(P_1(B))=V_\varphi^{0,1}(C[0,1])=S_1(\delta_0)$, the standard normal distribution, and

$$[V_{S_1(\delta_0)}^{1,2}(P_2(B))](\mathbb{R})=[S_1\circ S_1(\delta_0)](\mathbb{R})=[S_2(\delta_0)](\mathbb{R})=1. \tag{3.11}$$

Here, we establish the operational calculus for a measure-valued Dyson series, the main theorem in this note.

THEOREM 3.3. *Under the assumptions in Theorem 3.1, let* $g(z)=\exp(z)$ *and let* $\eta=\mu+\nu$ *be a complex-valued Borel measure on* $[0,t_2]$ *such that* μ *is a continuous part of* η *and* $\nu=\sum_{p=0}^{m+n}c_p\delta_{\tau_p}$ *where* $0=\tau_0<\tau_1<\cdots<\tau_m=t_1<\tau_{m+1}<\cdots<\tau_{m+n}=t_2$ *and* c_p $(p=0,1,2,\dots,m+n)$ *are complex numbers. Then* $\exp(\int_{[0,t_1]}\theta(s,x(s))\,d\eta(s))$, $\exp(\int_{(t_1,t_2]}\theta(s,x(s))\,d\eta(s))$ *and* $\exp(\int_{[0,t_2]}\theta(s,x(s))\,d\eta(s))$ *are all* V_φ^{0,t_1}-, $V_{\tilde{\varphi}}^{t_1,t_2}$- *and* V_φ^{0,t_2}-*Bartle integrable of* x *on* $C[0,t_1]$, $C[t_1,t_2]$ *and* $C[0,t_2]$, *respectively, where*

$$\tilde{\varphi}=(Ba)-\int_{C[0,t_1]}\exp\Big(\int_{[0,t_1]}\theta(s,x(s))\,d\eta(s)\Big)\,dV_\varphi^{0,t_1}(x). \tag{3.12}$$

Moreover,

$$
\begin{aligned}
&(Ba) - \int_{C[0,t_2]} \exp\Big(\int_{[0,t_2]} \theta(s,x(s))\,d\eta(s)\Big)\,dV_{\varphi}^{0,t_2}(x)\\
=&(Ba) - \int_{C[t_1,t_2]} \exp\Big(\int_{(t_1,t_2]} \theta(s,x(s))\,d\eta(s)\Big)\,dV_{\tilde{\varphi}}^{t_1,t_2}(x).
\end{aligned}
\tag{3.13}
$$

Proof. From Theorem 2.3, we have that $\exp(\int_{[0,t_1]}\theta(s,x(s))\,d\eta(s))$ and $\exp(\int_{[0,t_2]}\theta(s,x(s))\,d\eta(s))$ are V_{φ}^{0,t_1}- and V_{φ}^{0,t_2}-Bartle integrable on $C[0,t_1]$ and $C[0,t_2]$, respectively. So, putting

$$
\tilde{\varphi} = (Ba) - \int_{C[0,t_1]} \exp\Big(\int_{[0,t_1]} \theta(s,x(s))\,d\eta(s)\Big)\,dV_{\varphi}^{0,t_1}(x),
\tag{3.14}
$$

$\tilde{\varphi}$ is well defined in $\mathcal{M}(\mathbb{R})$ and $\exp(\int_{(t_1,t_2]}\theta(s,x(s))\,d\eta(s))$ is $V_{\tilde{\varphi}}^{t_1,t_2}$-Bartle integrable on $C[t_1,t_2]$. For nonnegative integers q, k and $j_1, j_2, \dots, j_{m+n}$ with $q = j_1 + j_2 + \cdots + j_m$ and $k = j_{m+1} + j_{m+2} + \cdots + j_{m+n}$, let

$$
\begin{aligned}
&\triangle_{q;j_1,j_2,\dots,j_m}^{0,t_1}\\
=&\{(s_{1,1}, s_{1,2}, \dots, s_{1,j_1}, s_{2,1}, \dots, s_{m-1,j_{m-1}}, s_{m,1}, \dots, s_{m,j_m})|\\
&\tau_0 \equiv 0 \equiv s_{1,0} < s_{1,1} < s_{1,2} < \cdots < s_{1,j_1} < s_{1,j_1+1} \equiv \tau_1\\
&\equiv s_{2,0} < s_{2,1} < s_{2,2} < \cdots < s_{2,j_2} < s_{2,j_2+1} \equiv \tau_2 \equiv s_{3,0}\\
&< \cdots < s_{m,j_m} < s_{m,j_m+1} \equiv \tau_m \equiv t_1\},
\end{aligned}
\tag{3.15}
$$

$$
\begin{aligned}
&\triangle_{k;j_{m+1},\dots,j_{m+n}}^{t_1,t_2}\\
=&\{(s_{m+1,1}, \dots, s_{m+1,j_{m+1}}, s_{m+2,1}, \dots, s_{m+2,j_{m+2}}, \dots,\\
&s_{m+n,j_{m+n}})|t_1 \equiv \tau_m \equiv s_{m+1,0} < s_{m+1,1} < \cdots\\
&< s_{m+1,j_{m+1}} < s_{m+1,j_{m+1}+1} \equiv \tau_{m+1} \equiv s_{m+2,0}\\
&< s_{m+2,1} < \cdots < s_{m+n,j_{m+n}}\\
&< s_{m+n,j_{m+n}+1} \equiv \tau_{m+n} \equiv t_2\}
\end{aligned}
\tag{3.16}
$$

and

$$
\triangle_{q+k;j_1,j_2,\dots,j_{m+n}}^{0,t_2} = \triangle_{q;j_1,j_2,\dots,j_m}^{0,t_1} \times \triangle_{k;j_{m+1},\dots,j_{m+n}}^{t_1,t_2}.
\tag{3.17}
$$

Then by Theorem 3.1. and Fubini's theorem,

$$
\begin{aligned}
&(Ba) - \int_{C[0,t_2]} \exp\Big(\int_{[0,t_2]} \theta(s,x(s))\,d\eta(s)\Big)\,dV_{\varphi}^{0,t_2}(x)\\
=&\sum_{l=0}^{\infty} \sum_{q_0+q_1+\cdots+q_{m+n+1}=l} \frac{\prod_{j=0}^{m+n} c_j^{q_j}}{\prod_{j=0}^{m+n} q_j!} \sum_{j_1+j_2+\cdots+j_{m+n}=q_{m+n+1}}
\end{aligned}
\tag{3.18}
$$

$$(Ba)-\int_{C[0,t_2]}\int_{\triangle^{0,t_2}_{q_{m+n+1};j_1,\dots,j_{m+n}}}\theta(\tau_0,x(\tau_0))^{q_0}\Big\{\prod_{i=1}^{m+n}\Big(\prod_{j=1}^{j_i}\theta(s_{i,j},x(s_{i,j}))\Big)$$

$$\theta(s_{i,j_i+1},x(s_{i,j_i+1}))^{q_i}\Big\}\,d\Big(\prod_{i=1}^{m+n}\prod_{j=1}^{j_i}\mu\Big)(s_{1,1},\dots,s_{m+n,j_{m+n}})\,dV^{0,t_2}_{\varphi}(x)$$

$$=\sum_{l_1=0}^{\infty}\sum_{l_2=0}^{\infty}\sum_{q_0+q_1+\dots+q_m+q=l_1}\sum_{q_{m+1}+\dots+q_{m+n}+k=l_2}\frac{\prod_{j=0}^{m+n}c_j^{q_j}}{\prod_{j=0}^{m+n}q_j!}$$

$$\sum_{j_1+j_2+\dots+j_m=q}\sum_{j_{m+1}+\dots+j_{m+n}=k}(Ba)-\int_{C[0,t_2]}\int_{\triangle^{0,t_1}_{q;j_1,\dots,j_m}}\theta(\tau_0,x(\tau_0))^{q_0}$$

$$\Big\{\prod_{i=1}^{m}\Big(\prod_{j=1}^{j_i}\theta(s_{i,j},x(s_{i,j}))\Big)\theta(s_{i,j_i+1},x(s_{i,j_i+1}))^{q_i}\Big\}$$

$$d\Big(\prod_{i=1}^{m}\prod_{j=1}^{j_i}\mu\Big)(s_{1,1},\dots,s_{m,j_m})$$

$$\int_{\triangle^{t_1,t_2}_{k;j_{m+1},\dots,j_{m+n}}}\Big\{\prod_{i=m+1}^{m+n}\Big(\prod_{j=1}^{j_i}\theta(s_{i,j},x(s_{i,j}))\Big)\theta(s_{i,j_i+1},x(s_{i,j_i+1}))^{q_i}\Big\}$$

$$d\Big(\prod_{i=m+1}^{m+n}\prod_{j=1}^{j_i}\mu\Big)(s_{m+1,1},\dots,s_{m+n,j_{m+n}})\,dV^{0,t_2}_{\varphi}(x)$$

$$=\sum_{l_1=0}^{\infty}\sum_{l_2=0}^{\infty}\sum_{q_0+q_1+\dots+q_m+q=l_1}\sum_{q_{m+1}+\dots+q_{m+n}+k=l_2}\frac{\prod_{j=0}^{m+n}c_j^{q_j}}{\prod_{j=0}^{m+n}q_j!}$$

$$\sum_{j_1+j_2+\dots+j_m=q}\sum_{j_{m+1}+\dots+j_{m+n}=k}(Ba)-\int_{C[t_1,t_2]}\int_{\triangle^{t_1,t_2}_{k;j_{m+1},\dots,j_{m+n}}}$$

$$\Big\{\prod_{i=m+1}^{m+n}\Big(\prod_{j=1}^{j_i}\theta(s_{i,j},x(s_{i,j}))\Big)\theta(s_{i,j_i+1},x(s_{i,j_i+1}))^{q_i}\Big\}$$

$$d\Big(\prod_{i=m+1}^{m+n}\prod_{j=1}^{j_i}\mu\Big)(s_{m+1,1},\dots,s_{m+n,j_{m+n}})\,dV^{t_1,t_2}_{\psi}(x).$$

Here

$$\psi(q;j_1,\dots,j_m)=(Ba)-\int_{C[0,t_1]}\int_{\triangle^{0,t_1}_{q;j_1,\dots,j_m}}\theta(\tau_0,x(\tau_0))^{q_0} \tag{3.19}$$

$$\Big\{\prod_{i=1}^{m}\Big(\prod_{j=1}^{j_i}\theta(s_{i,j},x(s_{i,j}))\Big)\theta(s_{i,j_i+1},x(s_{i,j_i+1}))^{q_i}\Big\}$$

$$d\Big(\prod_{i=1}^{m}\prod_{j=1}^{j_i}\mu\Big)(s_{1,1},\ldots,s_{m,j_m})\,dV_{\varphi}^{0,t_1}(x).$$

By the essentially same evaluation as in the proof of Theorem 5.5 in [5], (3.20)

$$\tilde{\varphi}=\sum_{l_1=0}^{\infty}\sum_{q_0+q_1+\cdots+q_m+q=l_1}\frac{\prod_{j=0}^{m}c_j^{q_j}}{\prod_{j=0}^{m}q_j!}\sum_{j_1+j_2+\cdots+j_m=q}\psi(q;j_1,\ldots,j_m)$$

and

$$(Ba)-\int_{C[t_1,t_2]}\exp\Big(\int_{(t_1,t_2]}\theta(s,x(s))\,d\eta(s)\Big)\,dV_{\tilde{\varphi}}^{t_1,t_2}(x)$$

$$=\sum_{l_2=0}^{\infty}\sum_{q_{m+1}+\cdots+q_{m+n}+k=l_2}\frac{\prod_{j=m+1}^{m+n}c_j^{q_j}}{\prod_{j=m+1}^{m+n}q_j!}\sum_{j_{m+1}+\cdots+j_{m+n}=k}$$

$$(3.21)\qquad (Ba)-\int_{C[t_1,t_2]}\int_{\triangle_{k;j_{m+1},\ldots,j_{m+n}}^{t_1,t_2}}\Big\{\prod_{i=m+1}^{m+n}\Big(\prod_{j=1}^{j_i}\theta(s_{i,j},x(s_{i,j}))\Big)$$

$$\theta(s_{i,j_i+1},x(s_{i,j_i+1}))^{q_i}\Big\}$$

$$d\Big(\prod_{i=m+1}^{m+n}\prod_{j=1}^{j_i}\mu\Big)(s_{m+1,1},\ldots,s_{m+n,j_{m+n}})\,dV_{\tilde{\varphi}}^{t_1,t_2}(x).$$

Hence, from (D) in Section 2, we have

$$(Ba)-\int_{C[0,t_2]}\exp\Big(\int_{[0,t_2]}\theta(s,x(s))\,d\eta(s)\Big)\,dV_{\varphi}^{0,t_2}(x)$$

$$=\sum_{l_2=0}^{\infty}\sum_{q_{m+1}+\cdots+q_{m+n}+k=l_2}\frac{\prod_{j=m+1}^{m+n}c_j^{q_j}}{\prod_{j=m+1}^{m+n}q_j!}\sum_{j_{m+1}+\cdots+j_{m+n}=k}$$

$$(3.22)\qquad (Ba)-\int_{C[t_1,t_2]}\int_{\triangle_{k;j_{m+1},\ldots,j_{m+n}}^{t_1,t_2}}\Big\{\prod_{i=m+1}^{m+n}\Big(\prod_{j=1}^{j_i}\theta(s_{i,j},x(s_{i,j}))\Big)$$

$$\theta(s_{i,j_i+1},x(s_{i,j_i+1}))^{q_i}\Big\}$$

$$d\Big(\prod_{i=m+1}^{m+n}\prod_{j=1}^{j_i}\mu\Big)(s_{m+1,1},\ldots,s_{m+n,j_{m+n}})\,dV_{\tilde{\varphi}}^{t_1,t_2}(x)$$

$$=(Ba) - \int_{C[t_1,t_2]} \exp\Big(\int_{(t_1,t_2]} \theta(s,x(s))\,d\eta(s)\Big)\,dV_{\tilde{\varphi}}^{t_1,t_2}(x),$$

as desired. □

REMARK 3.4. (a) Let θ be a constant function on $[0,t_2]\times\mathbb{R}$, say $\theta(s,u)=c$, let η be the Lebesgue measure on $[0,t_2]$ and let $\varphi=\delta_0$. Then

(3.23)
$$\int_{C[0,t_1]} \exp\Big(\int_{[0,t_1]} \theta(s,x(s))\,d\eta(s)\Big)\,dV_{\varphi}^{0,t_1}(x) = \exp(ct_1)S_{t_1}(\delta_0) \equiv \tilde{\varphi}$$

and

$$\begin{aligned}(3.24)\qquad &\int_{C[t_1,t_2]} \exp\Big(\int_{(t_1,t_2]} \theta(s,x(s))\,d\eta(s)\Big)\,dV_{\tilde{\varphi}}^{t_1,t_2}(x)\\ &=\exp(c(t_2-t_1))S_{t_2-t_1}(\tilde{\varphi})\\ &=\exp(ct_2)S_{t_2}(\delta_0)\\ &=\int_{C[0,t_2]} \exp\Big(\int_{[0,t_2]} \theta(s,x(s))\,d\eta(s)\Big)\,dV_{\varphi}^{0,t_2}(x),\end{aligned}$$

so, a formula, given in Theorem 3.3. holds.

(b) Taking $g(z)=z$ in (a)

$$(3.25)\qquad \int_{C[0,t_1]} g\Big(\int_{[0,t_1]} \theta(s,x(s))\,d\eta(s)\Big)\,dV_{\varphi}^{0,t_1}(x) = ct_1S_{t_1}(\delta_0) \equiv \psi$$

and

(3.26)
$$\begin{aligned}\int_{C[t_1,t_2]} g\Big(\int_{(t_1,t_2]} \theta(s,x(s))\,d\eta(s)\Big)\,dV_{\psi}^{t_1,t_2}(x) &= c(t_2-t_1)S_{t_2-t_1}(\psi)\\ &= c^2t_1(t_2-t_1)S_{t_2}(\delta_0).\end{aligned}$$

Since $\int_{C[0,t_2]} g(\int_{[0,t_2]} \theta(s,x(s))\,d\eta(s))\,dV_{\varphi}^{0,t_2}(x) = ct_2S_{t_2}(\delta_0)$, a formula, given in Theorem 3.3. doesn't hold for the general function g.

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