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Algebraic semantics for some weak Boolean logics* †

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This paper investigates algebraic semantics for some weak Boolean (wB) logics, which may be regarded as left-continuous t-norm based logics (or monoidal t-norm based logics (MTLs)). We investigate as infinite-valued logics each of wB-LC and wB-sKD, and each corresponding first order extension wB-LC∀ and wB-sKD∀. We give algebraic completeness for each of them.

【주요어】 weak Boolean, infinite-valued, algebraic semantics, wB-LC, wB-sKD

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1. Introduction

Yang [8] has recently investigated several weak Boolean (wB) logics, i.e., provided Routley-Meyer semantics for them, based on wB algebras (see section 3). The wB complementation $-$ (of a wB algebra) rejects Heyting complementation for the intuitionistic propositional calculus H of Heyting (and its extension the Dummett logic LC (or the Gödel logic G)), and yet instead accepts its dual one. Consider the conditions (1), (2) below for the wB complementation in contrast with the corresponding properties (1′) and (2′) below of the Heyting complementation.

(1) $--a \wedge a = --a$, i.e., $--a \leq a$, for all $a \in A$;
(2) $a \vee -a = 1$ for all $a \in A$,
(1′) $a \wedge \neg\neg a = a$, i.e., $a \leq \neg\neg a$, for all $a \in A$;
(2′) $a \wedge \neg a = 0$ for all $a \in A$.

Each (1) and (2) is dual to (1′) and (2′), respectively. To make clear this, let L be a bounded lattice with bounds 0 and 1, $x \mapsto n(x)$ be a complementation on L (in the sense that it satisfies either (3) if $a \leq b$ then $n(b) \leq n(a)$ and (1) or (3) and (1′)), and let a and b be elements of L. Then, Hasse diagrams in [8] show this fact very well.

As he noted in it, wB logics work paraconsistently because of the property of wB-negation, i.e., the rejection of (2′). This paper shows that wB-logics may also behave many-valuedly, not merely paraconsistently, by giving well-known algebraic semantics for some wB logics. As infinite-valued propositional logics we algebraically investigate wB-LC and wB-sKD: wB-LC is the LC with — in place of ¬, and wB-sKD is the sKD with — as well as the involutive negation ~of KD (see [9] for sKD).

Dunn and Meyer [3] algebraically investigated G (and its extensions). Baaz [1] extended G to the G with (so called) Baaz's projection △ ($G_{\triangle}$) and quantifiers. Hájek [6] especially investigated as an extension of the basic fuzzy logic BL, which is the residuated many-valued logic capturing the tautologies of *continuous* t-norms and their residua, G (together with ŁC (the Łukasiewicz logic) and Π (the product logic)) and its extensions with △ and quantifiers. He, Esteva, Godo, and Navara [5] extended his investigation to residuated logics with ~. Esteva and Godo [4] moreover investigated as a system weaker than BL the Monoidal t-norm based logic MTL, which copes with the tautologies of *left-continuous* t-norms and their residua, and its extensions such as the Nilpotent Minimal logic NM and the NM with △ ($NM_{\triangle}$).

Very interestingly, wB-LC and wB-sKD we shall investigate can be regarded as extensions of MTL, i.e, as kinds of MTLs, and thus as logics capturing the tautologies of left-continuous t-norms and their residua. We, however, investigate these as logics capturing the tautologies of extensions of distributive lattices. We give algebraic completeness results for each of them and each first order extension wB-LC∀ and wB-sKD∀.

For convenience, by *wB-L*, we shall ambiguously express wB-LC and wB-sKD together, if we do not need distinguish them, but context should determine which system is intended, and by *wB-L*∀, wB-LC∀ and wB-sKD∀. Also we shall adopt the notation and terminology similar to those in [4, 5, 6], and assume familiarity with them (together with results found in [4, 5, 6]).

2. Axiom Schemes and Rules for wB-L

For convenience, we present the axiomatic systems for wB-L using the following axiom schemes and rules of inference. We shall use the biconditional ↔, where A ↔ B = (A → B) ∧ (B → A), and the falsity f. For the remainder we shall follow the customary notation and terminology. We use the axiom systems to provide a consequence relation.

AXIOM SCHEMES[1]

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1) Note that in wB-L A14 and the first of A15 are redundant: we can obtain them by CP together with other axioms. We prove the first of A14 as example: 1. —A → —A ∨ —B and —B → —A ∨ —B (A7), 2. —(—A ∨ —B) → A and —(—A ∨ —B) → B (1, CP, A12, transitivity), 3. —(—A ∨ —B) → (A ∧ B) (AD, A6, MP), 4. —(A ∧ B) → (—A ∨ —B) (3, CP, A12, transitivity).

A1. $A \to A$ (self-implication)

A2. $(A \to B) \to ((C \to A) \to (C \to B))$ (prefixing)

A3. $(A \to (B \to C)) \to (B \to (A \to C))$ (permutation)

A4. $(A \to (A \to B)) \to (A \to B)$ (contraction)

A5. $(A \wedge B) \to A$, $(A \wedge B) \to B$ ($\wedge$-elimination)

A6. $((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$ ($\wedge$-introduction)

A7. $A \to (A \vee B)$, $B \to (A \vee B)$ ($\vee$-introduction)

A8. $((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$ ($\vee$-elimination)

A9. $(A \wedge (B \vee C)) \to ((A \wedge B) \vee (A \wedge C))$ (distributive law)

A10. $A \to (B \to A)$ (positive paradox)

A11. $(A \to B) \vee (B \to A)$ (chain)

A12. $--A \to A$ (classical double negation)

A13. $A \vee -A$ (excluded middle)

A14. $-(A \wedge B) \to (-A \vee -B)$, $(-A \vee -B) \to -(A \wedge B)$ (negated conjunction)

A15. $-(A \vee B) \to (-A \wedge -B)$, $(-A \wedge -B) \to -(A \vee B)$ (negated disjunction)

A16. $\sim\sim A \leftrightarrow A$ (double negation)

A17. $(A \to B) \to (\sim B \to \sim A)$ (contraposition)

A18. $(\sim A \vee B) \to (A \to B)$

A19. $(A \to B) \vee ((A \to B) \to (\sim A \vee B))$

A20. $(A \to (A \to \sim A)) \to (A \to \sim A)$ (special contraction)

RULES

$A \to B, A \vdash B$ (modus ponens (MP))

$A, B \vdash A \wedge B$ (adjunction (AD))

From $\vdash A \to B$ derive $\vdash -B \to -A$ (contraposition (CP))

DEFINITIONS

df1. $A \vee B := ((A \to B) \to B) \wedge ((B \to A) \to A)$

df2. $\sim A := A \to f$

df3. $A \;\&\; B := \sim(A \to \sim B)$.

SYSTEMS

wB-LC: A1 to A15; MP, AD, CP; df1.

wB-sKD: A1 to A3, A5 to A20; MP, AD, CP; df1 to df3.

By df1, we may concern ourselves with $\to$, $\wedge$, and $-$ as propositional connectives for wB-LC; and by df1 and df2, $\to$, $\wedge$, $-$, and f for wB-sKD. By df3, we can obtain

(R) $(A \to (B \to C)) \leftrightarrow ((A \;\&\; B) \to C)$ (residuation)

as a theorem of wB-sKD.[2)]

Note first that in wB-sKD (as well as wB-LC) $\triangle$ can be defined as $\triangle A := --A$ (df4), but it can not by $\neg$ and $\sim$ as in SBL (the strict basic (fuzzy) logic) of [5]. Note second that in wB-sKD $\wedge$ can not be defined as (D) $A \wedge B := A \;\&\; (A \to B)$ and thus the axiom $(A \;\&\; (A \to B)) \to (B \;\&\; (B \to A))$ of BL (the basic logic for residuated fuzzy logics) in [5, 6] is not valid in it. wB-sKD instead satisfies the axiom schemes of both Monoidal Logic ML introduced

2) We can easily prove this. We show left to right as example: let $A \to (B \to C)$. Then by A17, transitivity, and MP, $A \to (\sim C \to \sim B)$. Thus by A3 and MP, $\sim C \to (A \to \sim B)$, and so by A17, transitivity, and MP, $\sim(A \to \sim B) \to C$. Hence by df3 $(A \;\&\; B) \to C$.

by Höhle [7] and its extension NM suggested by Esteva and Godo [4], each of which is a weakening of BL in the sense that (D) does not hold in it. Note thirdly that in wB-LC ∧ can be taken in place of & in BL but the former does not satisfies df3 and thus it can not be defined by df3.

Note that "—", "~", "∧", and "∨" are used ambiguously as propositional connectives and as algebraic operators, but context should make their meaning clear. Note also that with respect to any wB negated formula of the form —B, the customary definitions of connectives, e.g., A → B = —A (or ~A) ∨ B, etc., in CPL can be applied to wB-L since such formulas have Boolean properties (see T1 in section 4).

3. rpc-wB, skd, and skd-wB algebras

To prove algebraic completeness for wB-L, we must define an algebra, more exactly a matrix, that will characterize wB-L. We shall call it *a wB-L algebra*; more exactly, an *rpc-wB* algebra for wB-LC, and an *skd-wB* algebra for wB-sKD. Note that, for convenience, by an *wB-L* (algebra), we shall ambiguously express an rpc-wB and an skd-wB (algebra) together.

First, we define a *wB algebra* to be a structure (A, ⊤, ⊥, ∧, ∨, —) such that[3]

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3) Note that by using wB4 we can get (3) in section 1, and conversely by (3) (together with wB3) wB4: (wB4 to (3)) Let wB4 hold and a

wB1. (A, $\wedge$, $\vee$) is a distributive lattice;
wB2. a $\wedge$ $\top$ = a and a $\vee$ $\bot$ = a for all a $\in$ A;
wB3. $-$$-$a $\wedge$ a = $-$$-$a, i.e., $-$$-$a $\leq$ a, for all a $\in$ A;
wB4. $-$(a $\wedge$ b) = $-$a $\vee$ $-$b for all a, b $\in$ A;
wB5. a $\vee$ $-$a = $\top$ for all a $\in$ A,

and call $-$, which satisfies wB3 to wB5, *wB complementation.*

We next define an *rpc-wB algebra* whose class will characterize wB-LC. An rpc-wB algebra is a structure (A, $\top$, $\bot$, $\wedge$, $\vee$, $\rightarrow$, $-$), where (A, $\top$, $\bot$, $\wedge$, $\vee$, $\rightarrow$) is a relatively pseudo-complemented (rpc) lattice satisfying

(4) (a $\rightarrow$ b) $\vee$ (b $\rightarrow$ a) = $\top$,

called "prelinearity axiom" by Hájek [6], and $-$ is a unary operation on A which satisfies wB3 to wB5 above, and

wB6. $-$(a $\vee$ b) = $-$a $\wedge$ $-$b for all a, b $\in$ A.

That is, (A, $\top$, $\bot$, $\wedge$, $\vee$, $\rightarrow$) is an rpc lattice satisfying (4) and (A, $\top$, $\bot$, $\wedge$, $\vee$, $-$) is a wB algebra satisfying wB6. (Note that when we especially need to mention this algebra in distinction from a wB algebra that just satisfies wB1 to wB5, we call it a *de Morgan wB (dM-wB) algebra,*

$\leq$ b, i.e., a $\wedge$ b = a. Then, $-$a = $-$a $\vee$ $-$b, i.e., $-$b $\leq$ $-$a. ((3) to wB4) (i) a $\wedge$ b $\leq$ a and a $\wedge$ b $\leq$ b. Then, by (3), $-$a $\leq$ $-$(a $\wedge$ b) and $-$b $\leq$ $-$(a $\wedge$ b). Thus, $-$a $\vee$ $-$b $\leq$ $-$(a $\wedge$ b). (ii) $-$($-$a $\vee$ $-$b) $\leq$ $-$$-$a $\wedge$ $-$$-$b, and $-$$-$a $\wedge$ $-$$-$b $\leq$ a $\wedge$ b. Then, by (3), $-$(a $\wedge$ b) $\leq$ $-$$-$($-$a $\vee$ $-$b), and so by wB3, $-$$-$($-$a $\vee$ $-$b) $\leq$ $-$a $\vee$ $-$b. Hence, $-$(a $\wedge$ b) $\leq$ $-$a $\vee$ $-$b. Equationally to define a wB algebra, we take wB4 in place of (3).

and corresponding complementation, i.e., the complementation satisfying wB3 to wB6, *dM-wB complementation.*)

Now we define an *skd algebra* whose class characterizes sKD. An skd algebra is a structure **A** = (A, ⊤, ⊥, ~, ∧, ∨, →) such that

(i) (A, ⊤, ⊥, ~, ∧, ∨) is a bounded de Morgan (b-DM) lattice, i.e., (A, ∧, ∨) is a distributive lattice with the greatest element ⊤ and the least ⊥, and ~is a unary operation on A which is an involution.
(ii) let a ↔ b := (a → b) ∧ (b → a). The following conditions (together with (4)) hold for all a, b, c: (with respect to lattice ordering ≤)
(5) (a → b) ∨ ((a → b) → (~a ∨ b)) = ⊤
(6) (a → b) ≤ ((a → b) ↔ ⊤)
(7) ((a → b) → (~a ∨ b)) ≤ ((a → b) ↔ (~a ∨ b))
(8) (a → (a → ~a)) ≤ (a → ~a).

We shall call the implication satisfying (4) to (8) *strict Kleene-Diense (skd) implication* and a corresponding algebra, i.e., an algebra satisfying (i) and (ii) (4) to (8), an *skd algebra.* Note that since by (df5) a * b := ~(a → ~b) it can be obtained that

(9) b ≤ a → c iff a * b ≤ c (residuation),

An skd algebra can be (regarded as) a *residuated algebra.*

We next define an *skd-wB algebra* whose class will characterize wB-sKD. An skd-wB algebra is a structure (A, ⊤, ⊥, ~, ∧, ∨, →, —) such that (i) (A, ⊤, ⊥, ~, ∧, ∨, →) is an skd algebra (satisfying (df5)) and (ii) (A,

$\top$, $\bot$, $\wedge$, $\vee$, $-$) is a dM-wB algebra.

An wB-L algebra is *linearly ordered* if the ordering of its algebra is linear, i.e., a $\leq$ b or b $\leq$ a (equivalently, a $\wedge$ b = a or a $\wedge$ b = b) for each pair a, b.

We note that in an rpc-wB algebra $\wedge$ is a continuous t-norm and $\rightarrow$ is its residual (see Definition 2.1.1 in [6]), and that in an skd-wB algebra $*$ is a left-continuous t-norm (but not a continuous one) and $\rightarrow$ is its residual, and $\sim$is the precomplement in the sense that $\sim$a can be defined as a $\rightarrow$ $\bot$ (cf. [6]). (A, $\top$, $\bot$, $\wedge$ ($*$), $\vee$, $\rightarrow$) ($\wedge$ in an rpc-wB algebra and $*$ in an skd-wB algebra) is a *residuated lattice* in the sense that it satisfies the definition of a residuated lattice (see Definition 2.3.2 in [6]).

Since $\top$ is the dual of $\bot$, i.e., $\top$ = $-\bot$ (or $\sim\bot$) and join $\vee$ can be defined by using $\rightarrow$ and meet $\wedge$ (see df1) (and in an skd-wB algebra $\sim$ by $\rightarrow$ and $\bot$ (see df2)), an rpc-wB algebra (A, $\top$, $\bot$, $-$, $\wedge$, $\vee$, $\rightarrow$) may be abbreviated to (A, $\bot$, $-$, $\wedge$, $\rightarrow$) (and an skd-wB algebra (A, $\top$, $\bot$, $\sim$, ($-$,) $\wedge$, $\vee$, $\rightarrow$) to (A, $\bot$, ($-$,) $\wedge$, $\rightarrow$)).

Remark 1 The following are the conditions for $\triangle$ in a $\triangle$-algebra extending a BL algebra by adding a unary operation $\triangle$:

($\triangle$1) $\triangle a \vee \neg\triangle a = \top$;
($\triangle$2) $\triangle(a \vee b) \leq (\triangle a \vee \triangle b)$;
($\triangle$3) $\triangle a \leq a$;
($\triangle$4) $\triangle a \leq \triangle\triangle a$;
($\triangle$5) $\triangle a * \triangle(a \rightarrow b) \leq \triangle b$; and

(△6) △⊤ = ⊤.

We recall that △a can be defined as ——a. Interestingly, by this definition, in an wB-L algebra we can get (△2) to (△6), i.e., they hold in it. But (△1) does not because an wB-L algebra does not have Heyting complement ¬ (see [5, 6]). However, by (df6) ⊥ := —(a ∨ —a) (= —a ∧ ——a) and (df7) ¬a := a → ⊥, in an rpc-wB algebra all the conditions above may hold in an rpc-wB algebra. (It ensures that wB-LC can be equivalent to $G_{\triangle}$ because conversely — can be defined as (df8) —a := ¬△a, and corresponding conditions for — hold in an $G_{\triangle}$ algebra.) However, df7 does not hold in an skd-wB algebra.

4. Algebraic completeness for wB-L

We first present the tables for evaluation. An *evaluation* for wB-L is a function v: PV → [0, 1] that is extended to all well-formed formulas of L(—, ~, →, ∧, ∨, p_0, p_1, ⋯) by the following tables: (PV: set of propositional variables, [0, 1]: the unit interval)

TABLES

T1.v(—A) = 0 if v(A) = 1
otherwise,

T2.v(~A) = 1 — v(A),

T3. v(A ∧ B) = min(v(A), v(B)),

T4. $v(A \vee B) = \max(v(A), v(B))$,

T5. $v(A \rightarrow B) = 1$ if $v(A) \leq v(B)$
$v(B)$ otherwise,

T6. $v(A \rightarrow B) = 1$ if $v(A) \leq v(B)$
$\max(v(\sim A), v(B))$ otherwise.

For wB-LC: T1, T3 to T5; and
for wB-sKD: T1 to T4, T6.

Note that, in fact, T2 and T4 are redundant because the former can be defined by T6 and $v(f) = 0$ and the latter by both T3 and T5 (w.r.t wB-LC), and T6 (w.r.t wB-sKD) (see df1 and df2). We define a formula A to be a *1-tautology* of wB-L, briefly a *wB-L-tautology*, if $v(A) = 1$, i.e., $\top$, for each wB-L-evaluation v.

We next define several notions. A *theory* over wB-L is a set T of formulas. A *proof* in a sequence of formulas whose each member is either an axiom of wB-L or a member of T or follows from some preceding members of the sequence using the rules above. $T \vdash A$, more exactly $T \vdash_{wB\text{-}L} A$, means that A is provable in T, i.e., there is a wB-L-proof of A in T. The deduction theorem for wB-L is as follows: (see Theorem 4 [5])

Proposition 1 Let T be a theory and let A, B be formulas. $T \cup \{A\} \vdash_{wB\text{-}L} B$ iff $T \vdash_{wB\text{-}L} —A \rightarrow B$.

Note that (R) ensures that T $\vdash_{sKD}$ $A^2 \rightarrow B$ can be regarded as T $\vdash_{sKD}$ $A \rightarrow (A \rightarrow B)$. A theory is *inconsistent* if T $\vdash$ f; otherwise it is *consistent*.

Let **A** be an wB-L algebra. In an analogy to the above, we define an ***A**-evaluation* of propositional variables to be any mapping v assigning to each propositional variable p an element v(p) of **A**. In the obvious way, this extends to an evaluation of all formulas using the operations on **A** as truth functions, for example, v(A → B) = v(A) → v(B). We define a formula A to be an ***A**-tautology* if v(A) = 1, i.e., ⊤, for each **A**-evaluation v. Then, we can easily show that

Proposition 2 (Soundness) The logic wB-L is sound with respect to wB-L-tautologies: if A is provable in wB-L, then A is an **A**-tautology for each wB-L algebra **A**.

We note that in each wB-L algebra the equations (9) to (13), the (equational) conditions for adjointness (9), of Lemma 2.3.10 in [6] hold. Note also that with respect to an rpc-wB algebra the class of all rpc lattices satisfying (4) forms a variety, and that with respect to an skd-wB algebra each condition (4) to (8) for skd implication has a form of equation or can be defined in equation. Thus, since the class of (bounded) distributive/de Morgan lattices is a variety and each condition wB2 to wB6 has a form of equation, the class of all dM-wB algebras is also a variety. This ensures that

Proposition 3 The class of all wB-L algebras is a variety of algebras.

Next, we show that classes of provably equivalent formulas form an wB-L algebra. Let T be a fixed theory over wB-L. For each formula A, let $[A]_T$ be the set of all formulas B such that $T \vdash A \leftrightarrow B$ (formulas T-provably equivalent to A). A_T is the set of all the classes $[A]_T$. We define that $[A]_T \to [B]_T = [A \to B]_T$, $—[A]_T = [—A]_T$ (w.r.t wB-LC and wB-sKD), $[A]_T * [B]_T = [A \,\&\, B]_T$ (w.r.t wB-sKD), $\sim[A]_T = [\sim A]_T$, i.e., $[A]_T \to [f]_T = [A \to f]_T$, (w.r.t wB-sKD), $[A]_T \wedge [B]_T = [A \wedge B]_T$, $[A]_T \vee [B]_T = [A \vee B]_T$, $0 = [f]_T$, and $1 = [t]_T$.[4] By $\mathcal{A}_T$, we denote this algebra.

Note that to define A_T algebra we need just the definitions of →, ∧, —, and 0 because we can define other operations and special element by using them.

Proposition 4 A_T is an wB-L algebra.

Proof Note that wB-LC has the same positive part as

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4) It can be ensured that this definition is correct due to the provabilities as follows (we just need to check that ↔ is a congruence with respect to —, ∧, and →): we check just one direction. Let $\vdash A \to B$. With respect to —, by CP, we can prove $—B \to —A$ from the assumption; with respect to ∧, by A5 and transitivity, $(A \wedge C) \to B$, and thus $(A \wedge C) \to (B \wedge C)$ by A5, A6, AD, and MP; with respect to →, by transitivity, it is almost immediate that $(B \to C) \to (A \to C)$ and $(C \to A) \to (C \to B)$.

LC in the sense that with respect to the part of negation-free formulas wB-LC may have the same axiom schemes and rules as LC, and thus a formula not containing negation is provable in LC iff it is provable in wB-LC. (This is obvious. We leave its proof to interested readers.) Thus, with respect to wB-LC it suffices to check that — is a dM-wB complementation. Note also that the lattice ordering $\leq$ satisfies the following (see the proof of Lemma 2.3.12 [6]):

$$[A]_T \leq [B]_T \text{ iff } T \vdash A \rightarrow B.$$

The axiom schemes A12, A13, A14, and A15 ensure that — is a dM-wB complementation, i.e., — satisfies wB3, wB5, wB4, and wB6, respectively. Thus, AT (of wB-LC) is an rpc-wB algebra. Moreover, with respect to wB-sKD, A5 to A9, A16, and A17 ensure that $\wedge$, $\vee$, and ~ satisfy de Morgan lattice properties, i.e., (i) in section 3. A11, A19, and A20 together with the theorems (10) $(A \rightarrow B) \rightarrow ((A \rightarrow B) \leftrightarrow t)$, (11) $((A \rightarrow B) \rightarrow (\sim A \vee B)) \rightarrow ((A \rightarrow B) \leftrightarrow (\sim A \vee B))$ ensure that $\rightarrow$ together with $\vee$, $\wedge$, and ~ satisfies (ii) in section 3. That is, A11, A19, A20, (10), and (11) ensure that (4), (5), (8), (6), and (7), respectively, can be satisfied by these operations. Thus A_T (of wB-sKD) is an skd-wB algebra. □

Now we show how filters on residuated lattices determine homomorphisms and characterize homomorphisms to linearly ordered algebras. Let A be a residuated lattice. A *filter* on A is a non-empty set $F \subseteq A$ such that for each $x, y \in A$,

(F1) $x \in F$ and $y \in F$ imply $x \wedge y \in F$,
(F2) $x \in F$ and $x \leq y$ (or $x \rightarrow y \in F$) imply $y \in F$,
(F3) $(x \rightarrow y) \in F$ implies $(-y \rightarrow --x) \in F$.

F is a *prime filter* iff it is a filter and for each $x, y \in A$,

(PF) $(x \rightarrow y) \in F$ or $(y \rightarrow x) \in F$.

Note that with respect to a filter of wB-L algebras (PF) implies the usual definition of a prime filter (and vice versa) as follows:

Lemma 1 A filter F is prime iff (PF′) for each pair of elements x, y such that $x \vee y \in F$, $x \in F$ or $y \in F$.

Proof See Lemma 1 in [9]. □

Corollary 1 Let F be a filter. Then,
(i) $x \in F$ implies $---x \in F$.
(ii) Let F be yet prime. Then, in a linearly ordered wB-L algebra, $---x = 0 \in F$ for all $x \neq 1 \in F$.

Proof (i) By use of (F3), we can prove this. (By using A13 and the primeness (PF′) in Lemma 1, we may also obtain $---x \in F$ from $x \in F$ in case F is prime.)
(ii) Let $x \neq 1$. Then, $-x = 1$ and thus $---x = 0$.

Hence, by (i), if x ($\neq$ 1) $\in$ F, then $---$x = 0 $\in$ F. □

Proposition 5 Let **A** be an wB-L algebra and let F be a filter. Put x $\equiv_F$ y iff (x $\rightarrow$ y) $\in$ F and (y $\rightarrow$ x) $\in$ F. Then,

(i) $\equiv_F$ is a congruence relation over an wB-L algebra.

(ii) The quotient of algebra A/$\equiv_F$ is an wB-L algebra.

(iii) A/$\equiv_F$ is linearly ordered iff F is a prime filter.

(iv) Linearly ordered wB-L algebras **A** are simple, i.e., the only filters of a linearly ordered wB-L algebra **A** are {1} and **A** itself.

Proof For (i), we first observe that $\equiv_F$ is transitive to show that $\equiv_F$ is an equivalence: it follows from the fact that the formula (A $\rightarrow$ B) $\rightarrow$ ((B $\rightarrow$ C) $\rightarrow$ (A $\rightarrow$ C)) is a 1-tautology over **A**, and thus (a $\rightarrow$ b) $\leq$ ((b $\rightarrow$ c) $\rightarrow$ (a $\rightarrow$ c)): let (a $\rightarrow$ b), (b $\rightarrow$ c) $\in$ F. If (a $\rightarrow$ b) $\in$ F, then ((b $\rightarrow$ c) $\rightarrow$ (a $\rightarrow$ c)) $\in$ F. Hence, since (b $\rightarrow$ c) $\in$ F, (a $\rightarrow$ c) $\in$ F. Thus, we may define equivalence classes $[x]_F$ = {y: x $\equiv_F$ y}.

We next verify that $\equiv_F$ is a congruence, i.e., preserves operations. Analogously to the proof of the statement that $[A]_T \leq [B]_T$ iff T $\vdash$ A $\rightarrow$ B in Proposition 4, we can show that $[x]_F \leq [y]_F$ iff x $\rightarrow$ y $\in$ F: we prove right to left as example. Let x $\rightarrow$ y $\in$ F. Then, since x $\rightarrow$ x $\in$ F, A6 ensures that x $\rightarrow$ (x $\wedge$ y) $\in$ F. Thus, since A5 ensures that (x $\wedge$ y) $\rightarrow$ x $\in$ F, (x $\wedge$ y) $\equiv_F$ x. Hence, [x

$\wedge$ $y]_F$ (= $[x]_F \wedge [y]_F$) = $[x]_F$, i.e., $[x]_F \leq [y]_F$.

Note that, in an analogy to the footnote 5, we can verify that $[x]_F = [y]_F$ implies $[-x]_F = [-y]_F$, $[x \wedge z]_F = [y \wedge z]_F$, $[x \rightarrow z]_F = [y \rightarrow z]_F$, and $[z \rightarrow x]_F = [z \rightarrow y]_F$: as an example we show that $[x]_F = [y]_F$ implies $[-x]_F = [-y]_F$. Let $[x]_F \leq [y]_F$. Then, $x \rightarrow y \in F$. Thus, by (F3), $-y \rightarrow -x \in F$, and so $[-y]_F \leq [-x]_F$. Analogously, $[-x]_F \leq [-y]_F$ follows from $[y]_F \leq [x]_F$. Hence, $[x]_F = [y]_F$ implies $[-x]_F = [-y]_F$. Therefore, $\equiv_F$ is a congruence.

(ii) and (iii) are analogous to those of Lemma 2.3.14 and those in proofs of Theorem 2.4.12 in [6].

The proof of (iv) reduces to showing that the only filters of a linearly ordered wB-L algebra **A** are {1} and the full algebra **A** itself. This is true because if a filter F has an element $x \neq 1$, then $0 \in F$ by Corollary 1 (ii) and thus F = A. □

Proposition 6 Let **A** be an wB-L algebra and let $a \in A$, $a \neq 1$. Then there is a prime filter F on **A** not containing a.

Proof Note that we can use the primeness (PF′) in place of (PF) by Lemma 1, and thus by Prime Filter Separation Principle in a distributive lattice, it is immediate. The proof is very analogous to that of Lemma 8.6.2 in Dunn and Hardegree [2].

Let A, a, 1 be as in the hypothesis. Then, $F_0 = [1), = \{x \in A: 1 \leq x\}$, is a filter separating 1 from a. We let F be the family of filters of A, which have 1 as a member but not a. E is non-empty, since it contains F_0. Now let C be any non-empty chain of E. Then, $\cup C \in$ E. For clearly 1 $\in \cup C$, and a $\notin \cup C$. Then, it remains to show that $\cup C$ is a filter. Suppose b, c $\in \cup C$, but then there are F′, F″ $\in$ C such that b $\in$ F′ and c $\in$ F″. But either F′ $\subseteq$ F″ or F″ $\subseteq$ F′, and so either (b $\wedge$ c) $\in$ F″ or (b $\wedge$ c) $\in$ F′ because both F′ and F″ are filters. But then in either case (b $\wedge$ c) $\in \cup C$, and thus (F1) is satisfied. Similarly, we can show that (F2) is satisfied. The interesting point to check is that (F3) can be satisfied (with respect to an wB-L algebra): let (b $\rightarrow$ c) $\in \cup C$. Then, there is F′ $\in$ C such that (b $\rightarrow$ c) $\in$ F′. So, since F′ is a filter, (—c $\rightarrow$ —b) $\in$ F′. Hence, (—c $\rightarrow$ —b) $\in \cup C$, as desired.

By Zorn's Lemma, we may conclude that E has some maximal member F, which is a filter such that 1 $\in$ F and a $\notin$ F. It remains to show that F is prime. Its proof is as usual (see the proof of Lemma 8.6.2 in [2] for it). □

Proposition 7 Each wB-L algebra is a subdirect product of linearly ordered wB-L algebras.

Proof Its proof is as usual (see the proof of Lemma 2.3.16 in [6]). □

Note that with respect to wB-algebras, i.e., rpc-wB algebras and skd-wB algebras, this theorem is a subdirect decomposition theorem because by Proposition 5 (iv) linearly ordered wB algebras are simple, and thus subdirectly irreducible, which is not the case for skd algebras.

Let us associate with each formula A of wB-L a term A^t of the language of wB-L algebras by replacing the connectives and constants —, ~, →, &, ∧, ∨, f, t by function symbols and special elements —, ~, →, *, ∧, ∨, 0 (or ⊥), 1 (or ⊤), respectively, and replacing each propositional variable p_i by a corresponding object variable x_i.

Proposition 8 (i) Each formula which is an **A**-tautology for all linearly ordered wB-L algebras is an **A**-tautology for all wB-L algebras.

(ii) A is an **A**-tautology iff the identity $A^t = 1$ is true in **A**.

Proof (i) follows from (ii) and the subdirect product representation. (ii) is evident since the value of the term A^t given by an evaluation v is $v_A(A)$. □

Theorem 1 (Weak completeness) wB-L is complete with respect to the class of wB-L algebras, i.e., for each formula A the following are equivalent:

(i) A is provable in wB-L, i.e., $\vdash_{wB\text{-}L} A$,

(ii) For each linearly ordered wB-L algebra **A**, A is an **A**-tautology;

(iii) For each wB-L algebra **A**, A is an **A**-tautology.

Proof The implications of (i) to (ii) and (ii) to (iii) have been established. Thus it suffices to show that (iii) to (i) holds:

Note that Proposition 4 says that the algebra $A_{wB\text{-}L}$ of classes of equivalent formulas of wB-L is an wB-L algebra. Thus, an A satisfying (iii) is an $A_{wB\text{-}L}$-tautology. Now let $v(p_i) = [p_i]_{wB\text{-}L}$ for all propositional variables. Then $v(A) = [A]_{wB\text{-}L} = [t]_{wB\text{-}L}$, and thus $\vdash_{wB\text{-}L} A \leftrightarrow t$. Hence, $\vdash_{wB\text{-}L} A$. □

To achieve strong completeness for wB-L, we add more definitions on a theory T to the definitions above. Let A be an wB-L algebra. Note that elements of T are axioms of T. An A-evaluation v is an *A-model* of T if $v_A(\alpha) = 1_A$ for each axioms $\alpha \in T$. T is *complete* if for each pair A, B of formulas, $T \vdash A \rightarrow B$ or $T \vdash B \rightarrow A$. Note that corresponding to Lemma 1 it can be ensured that T is complete iff for each pair of A, B such that $T \vdash A \vee B$, $T \vdash A$ or $T \vdash B$ (see Lemma 5.2.3 in [6]). We call this, i.e., the T of the second statement, also complete.

Proposition 9 (i) T is complete iff the wB-L algebra A_T is linearly ordered.

(ii) If T is a theory and $T \nvdash A$, then there is a consistent complete supertheory $T' \supseteq T$ such that $T' \nvdash A$.

Proof (i) Left to right. Let T be complete and A, B be the pair of formulas of its language. We note that (*) $[A]_T \le [B]_T$ iff $T \vdash A \rightarrow B$. Since T is complete, either $T \vdash A \rightarrow B$ and thus by (*) $[A]_T \le [B]_T$, or $T \vdash B \rightarrow A$ and thus by (*) $[B]_T \le [A]_T$. Hence $\le$ is linear and thus A_T is linearly ordered.

Right to left. Let A_T be linearly ordered and A, B be as above. Then, either $[A]_T \le [B]_T$ and $T \vdash A \rightarrow B$, or $[B]_T \le [A]_T$ and $T \vdash B \rightarrow A$. Hence, T is complete.

(ii) We shall use the completeness property of T, which corresponds to (PF′) in Lemma 1. Where Δ is a set of formulas not necessarily a theory, $\Delta \vdash A$ can be thought of as saying that A is deducible from the 'axioms' Δ. The set of $\{A: \Delta \vdash A\}$ is intuitively the smallest theory containing the axioms Δ, and we shall label it as *Th(Δ)*.

Now take an enumeration $\{A_n: n \in \omega\}$ of the well-formed formulas of wB-L. We define a sequence of sets by induction as follows:

$$T_0 = \{A': T \vdash_{wB\text{-}L} A'\}.$$
$$T_{i+1} = \begin{cases} Th(T_i \cup \{A_{i+1}\}) & \text{if it is not the case that } T_i, A_{i+1} \vdash_{wB\text{-}L} A, \\ T_i & \text{otherwise.} \end{cases}$$

Let T′ be the union of all these T_n's. It is easy to see that T′ is a theory not containing A (and thus it is consistent). Also we can show that it is complete.

Suppose toward contradiction that $B \vee C \in T'$ and $B, C \notin T'$. Then the theories obtained from $T' \cup B$ and $T' \cup C$ must both contain A. It follows that there is a conjunction of members of T′ T″ such that T″ ∧ B $\vdash_{wB\text{-}L}$ A and T″ ∧ C $\vdash_{wB\text{-}L}$ A. Then, by A8, Proposition 1 (i.e., Deduction Theorem), AD, and MP, $\vdash_{wB\text{-}L}$ (——(T″ ∧ B) ∨ ——(T″ ∧ C)) → A. Note that A14 and A15 ensure that we can get ——(A ∧ B) ↔ (——A ∧ ——B) and ——(A ∨ B) ↔ (——A ∨ ——B) as theorems. Thus, we can indistinguishably use these properties. Hence, $\vdash_{wB\text{-}L}$ ((——T″ ∧ ——B) ∨ (——T″ ∧ ——C)) → A. Then, we obtain $\vdash_{wB\text{-}L}$ (——T″ ∧ (——B ∨ ——C)) → A by A2, A9, and MP, and thus $\vdash_{wB\text{-}L}$ ——(T″ ∧ (B ∨ C)) → A. Hence, T″ ∧ (B ∨ C) $\vdash_{wB\text{-}L}$ A by Proposition 1. From this we get that T′ ⊢ A, which is contrary to our supposition.

(Note that in place of T_0, T_{i+1}, B, and C above, we may use the completeness of T corresponding to (PF) and get the same result, i.e., the completeness of T′, just by taking T_0 = T,

T_{i+1} = T ∪ {$\alpha_i \rightarrow \beta_i$} if it is not the case that T, $\alpha_i \rightarrow \beta_i \vdash$ A;
T ∪ {$\beta_i \rightarrow \alpha_i$} otherwise, i.e., T, $\beta_i \rightarrow \alpha_i \nvdash$ A,

B, = $\alpha_i \rightarrow \beta_i$, and C, = $\beta_i \rightarrow \alpha_i$, respectively (cf. see Lemma 2.4.2 in [6]).) □

By using Proposition 9 (and Soundness as usual), we can easily show that

Theorem 2 (Strong completeness) Let T be a theory over wB-L and let A be a formula. Then the following are equivalent:

(i) $T \vdash_{wB\text{-}L} A$.

(ii) For each linearly ordered wB-L algebra A and each A-model v of T, $v_A(A) = 1_A$.

(iii) For each wB-L algebra A and each A-model v of T, $v_A(A) = 1_A$.

5. wB-L∀: the first order extension of wB-L

The completeness theorems for fuzzy predicate logics presented in [5, 6] may generalize for the present situation.

A trivial generalization of those of section 6 in [5] and Chapter V in [6] gives the notions of a language, its interpretations, and formulas for wB-L∀ as follows:

Given a linearly ordered wB-L algebra A, an *A-interpretation*, i.e., an *A-structure*, of a language consisting of some predicates $P \in Pred$ and constants $c \in Const$ is a structure $\mathbf{M} = (M, (r_P)_{P \in Pred}, (m_c)_{c \in Const})$, where $M \neq \varnothing$, $r_P : M^{ar(P)} \to A$, and $m_c \in M$ (for each $P \in Pred$, $c \in Const$).

Let L be a predicate language and let **M** be an A-structure for L. An *M-evaluation* of object variables is a mapping e assigning to each object variable x an element $e(x) \in M$. Let e, e′ be two evaluations. $e \equiv_x e'$ means that $e(y) = e'(y)$ for each variable y distinct from x.

The value of a term given by **M**,e is defined as follows:

$|x|_{M,e}$ = e(x) and $|c|_{M,e}$ = m_c. The *(truth) value* $|A|^A_{M,e}$ of a formula (where e(x) ∈ M for each variable x) is defined inductively: for A being P(x, ⋯ , c, ⋯),

$|P(x, \cdots, c, \cdots)|^A_{M,e} = r_P(e(x), \cdots, m_c, \cdots)$,

the value commutes with connectives, and

$|(\forall x)A|^A_{M,e} = \inf\{|A|^A_{M,e'}: e \equiv_x e'\}$

if this infimum exists, otherwise undefined, and similarly for ∃x and sup. M is *A-safe* if all infs and sups needed for definition of the value of any formula exist in A, i.e., $|A|^A_{M,e}$ is defined for all A, e.

Let A be a formula of a language L and let M be a safe A-structure for L. The *truth value* of A in M is

$|A|^A_M = \inf\{|A|^A_{M,e}: e\ M\text{-evaluation}\}$.

A formula A of a language L is an *A-tautology* if $|A|_M$ = 1_A for each safe A-structure M, i.e., $|A|^A_{M,e}$ = 1 for each safe A-structure M and each M-evaluation of object variables.

The axioms of wB-L∀ are those of wB-L plus the following set of axioms for quantifiers (taken by Hájek [6] as those of the basic predicate logic BL∀):

(∀1) $(\forall x)A(x) \to A(t)$ (t substitutable for x in A(x))
(∃1) $A(t) \to (\exists x)A(x)$ (t substitutable for x in A(x))
(∀2) $(\forall x)(A \to B) \to (A \to (\forall x)B)$ (x not free in A)
(∃2) $(\forall x)(A \to B) \to ((\exists x)A \to B)$ (x not free in B)
(∀3) $(\forall x)(A \vee B) \to ((\forall x)A \vee B)$ (x not free in B)

Rules of inference for wB-L∀ are MP, AD, CP, and generalization (GN), i.e., from A infer $(\forall x)A$. (Note that in wB-sKD$\forall$ one quantifier is definable from the other one and the negation $\sim$, for instance, $(\exists x)A := \sim(\forall x)\sim A$. Thus the above set of axioms for quantifiers could be simplified, i.e., $(\forall 3)$, $(\exists 1)$, and $(\exists 2)$ become provable as in the Łukasiewicz predicate logic Ł$\forall$ (cf. see Remark 5.4.2 in [3]).

Proposition 12 (i) The axioms $(\forall 1)$, $(\forall 2)$, $(\forall 3)$, $(\exists 1)$, and $(\exists 2)$ are **A**-tautologies for each linearly ordered wB-L algebra **A**. (ii) The rules MP, AD, CP, and GN preserve **A**-tautologyhood.

Proof (i) By Lemmas 5.1.9 in [6].

(ii) MP and GN are by Lemma 5.1.10 in [3]. AD is by Proposition 12 in [9]. Thus, for wB-L$\forall$ we need just to consider that the rule CP preserves **A**-tautologyhood. For CP, we show that

(1) for any formulas A, B, safe **A**-structure **M**, and evaluation e,

if $|A \rightarrow B|^{\mathbf{A}}_{\mathbf{M},e} = 1_{\mathbf{A}}$, then $|—B \rightarrow —A|^{\mathbf{A}}_{\mathbf{M},e} = 1_{\mathbf{A}}$, and

(2) consequently,

if $|A \rightarrow B|^{\mathbf{A}}_{\mathbf{M}} = 1_{\mathbf{A}}$, then $|—B \rightarrow —A|^{\mathbf{A}}_{\mathbf{M}} = 1_{\mathbf{A}}$;

thus if $A \rightarrow B$ is $1_{\mathbf{A}}$-true in **M**, then $—B \rightarrow —A$ is.

(1) is as in propositional calculus. To prove (2) put $|A|_w = a_w$ and $|B|_w = b_w$. We have to show that

if $\inf_w(a_w \Rightarrow b_w) = 1$, then $\inf_w(—b_w \Rightarrow —a_w) = 1$

(indices **A**, **M** deleted, w runs over all evaluations $\equiv_x$ e). To do this, it suffices to derive $(\forall x)(-B \to -A)$ from $(\forall x)(A \to B)$ in wB-L$\forall$. Let $\vdash (\forall x)(A \to B)$. Then, by ($\forall$1) and MP, $\vdash A \to B$. Thus, by CP $\vdash -B \to -A$, and so $\vdash (\forall x)(-B \to -A)$ by GN. □

Definitions of a theory T over wB-L$\forall$ are almost the same as wB-L. We need just to consider such definitions in **M**. Let **A** be a linearly ordered wB-L algebra and let **M** be a safe **A**-structure for the language of T. **M** is an *A-model* of T if all axioms of T are 1_A-true in **M**, i.e., $|A|^A_M = 1_A$ in each A ∈ T. Then, Proposition 12 ensures that wB-L$\forall$ is sound with respect to linearly ordered wB-L algebras as follows.

Proposition 13 (Soundness) Let T be a theory in the language of T over wB-L$\forall$ and let A be a formula of T. If $T \vdash A$, then $|A|^A_M = 1_A$ for each linearly ordered wB-L algebra **A** and each **A**-model **M** of T.

Proof By induction on the length of a proof. □

To investigate completeness for wB-L$\forall$, we have the same definitions on "consistency" and "completeness" of a theory T as in wB-L. We moreover define the Henkinness of T (over wB-L$\forall$) as follows: T is *Henkin* if for each closed formula of the form $(\forall x)A(x)$ unprovable in T, i.e., T $\nvdash$ $(\forall x)A(x)$, there is a constant c in the language of T such that A(c) is unprovable in T, i.e., T $\nvdash$ A(c).

For each theory T over wB-L$\forall$, let A_T be the algebra of classes of T-equivalent closed formulas with the usual operations. It is clear that A_T is an wB-L algebra.

Lemma 5 For each theory T and each closed formula A, if T $\nvdash$ A, then there is a complete Henkin supertheory T′ of T such that T′ $\nvdash$ A.

Proof See the proofs of Proposition 9 (ii) above and Lemma 5.2.7 in [6]. □

Lemma 6 For each complete Henkin theory T and each closed formula A, if T $\nvdash$ A, then there is a linearly ordered wB-L algebra **A** and **A**-model **M** of T such that $|A|^{\mathbf{A}}_{\mathbf{M}} < 1_T$.

Proof By Lemma 5.2.8 in [6]. □

By using Lemmas 5 and 6, we can show the completeness for wB-L$\forall$ as follows.

Theorem 4 (Completeness) Let T be a theory over wB-L $\forall$ and let A be a formula. T proves A over wB-L$\forall$ iff $|A|^{\mathbf{A}}_{\mathbf{M}} = 1_{\mathbf{A}}$ for each linearly ordered wB-L algebra **A**, each safe **A**-model **M** of T.

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