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SELF-ADJOINT INTERPOLATION ON $Ax = y$ IN A TRIDIAGONAL ALGEBRA ALG$\mathcal{L}$

DONGWAN PARK AND JAE HYUN PARK

Abstract. Given vectors x and y in a separable Hilbert space $\mathcal{H}$, an interpolating operator is a bounded operator A such that $Ax = y$. In this article, we investigate self-adjoint interpolation problems for vectors in a tridiagonal algebra : Let Alg$\mathcal{L}$ be a tridiagonal algebra on a separable complex Hilbert space $\mathcal{H}$ and let $x = (x_i)$ and $y = (y_i)$ be vectors in $\mathcal{H}$.Then the following are equivalent:

(1) There exists a self-adjoint operator $A = (a_{ij})$ in Alg$\mathcal{L}$ such that $Ax = y$.

(2) There is a bounded real sequence $\{\alpha_n\}$ such that $y_i = \alpha_i x_i$ for $i \in \mathbb{N}$.

1. Introduction

Let $\mathcal{C}$ be a subalgebra of the algebra $\mathcal{B}(\mathcal{H})$ of all operators acting on a Hilbert space $\mathcal{H}$ and let x and y be vectors in $\mathcal{H}$. An *interpolation question* for $\mathcal{C}$ asks for which x and y is there a bounded operator $A \in \mathcal{C}$ such that $Ax = y$. A variation, the 'n-vector interpolation problem', asks for an operator A such that $Ax_i = y_i$ for fixed finite collections $\{x_1, x_2, \cdots, x_n\}$ and $\{y_1, y_2, \cdots, y_n\}$. The n-vector interpolation problem was considered for a C^*-algebra $\mathcal{U}$ by Kadison[8]. In case $\mathcal{U}$ is a nest algebra, the (one-vector) interpolation problem was solved by Lance[9]:

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his result was extended by Hopenwasser[2] to the case that $\mathcal{U}$ is a CSL-algebra. Munch[10] obtained conditions for interpolation in case A is required to lie in the ideal of Hilbert-Schmidt operators in a nest algebra. Hopenwasser[3] once again extended the interpolation condition to the ideal of Hilbert-Schmidt operators in a CSL-algebra. Hopenwasser's paper also contains a sufficient condition for interpolation n-vectors, although necessity was not proved in that paper.

We establish some notations and conventions. A commutative subspace lattice $\mathcal{L}$, or CSL $\mathcal{L}$ is a strongly closed lattice of pairwise-commuting projections acting on a Hilbert space $\mathcal{H}$. We assume that the projections 0 and I lie in $\mathcal{L}$. We usually identify projections and their ranges, so that it makes sense to speak of an operator as leaving a projection invariant. If $\mathcal{L}$ is CSL, Alg$\mathcal{L}$ is called a CSL-algebra. The symbol Alg$\mathcal{L}$ is the algebra of all bounded operators on $\mathcal{H}$ that leave invariant all the projections in $\mathcal{L}$. Let x and y be two vectors in a Hilbert space $\mathcal{H}$. Then $\langle x, y\rangle$ means the inner product of the vectors x and y. Let M be a subset of a Hilbert space $\mathcal{H}$. Then $\overline{M}$ means the closure of M and $\overline{M}^{\perp}$ the orthogonal complement of $\overline{M}$. Let $\mathbb{N}$ be the set of all natural numbers and let $\mathbb{C}$ be the set of all complex numbers. For each $z \in \mathbb{C}$, $\overline{z}$ is the complex conjugation of z.

2. Results

Let $\mathcal{H}$ be a separable complex Hilbert space with a fixed orthonormal basis $\{e_1, e_2, \cdots\}$. Let $x_1, x_2, \cdots, x_n$ be vectors in $\mathcal{H}$. Then $[x_1, x_2, \cdots, x_n]$ means the closed subspace generated by the vectors $x_1, x_2, \cdots, x_n$. Let $\mathcal{L}$ be the subspace lattice generated by the subspaces $[e_{2k-1}]$, $[e_{2k-1}, e_{2k}, e_{2k+1}]$ $(k = 1, 2, \cdots)$. Then the algebra Alg$\mathcal{L}$ is called a tridiagonal algebra which was introduced by F. Gilfeather and D. Larson[1]. These algebras have been found to be useful counterexample to a number of plausible conjectures.

Let $\mathcal{A}$ be the algebra consisting of all bounded operators acting on $\mathcal{H}$ of the form

$$\begin{pmatrix} * & * & & & \\ & * & & & \\ & * & * & * & \\ & & & * & \\ & & & * & \ddots \end{pmatrix}$$

with respect to the orthonormal basis $\{e_1, e_2, \cdots\}$, where all non-starred entries are zero. It is easy to see that Alg$\mathcal{L}$=$\mathcal{A}$.

We consider interpolation problems for the above tridiagonal algebra Alg$\mathcal{L}$.

Lemma 1. *Let $A = (a_{ij})$ be an operator in the tridiagonal algebra Alg$\mathcal{L}$. Then the following are equivalent:*

(1) A is self-adjoint.

(2) A is diagonal and a_{ii} is real for all $i \in \mathbb{N}$.

Proof. Suppose that A is self-adjoint. Since $A = A^*$,

$$\begin{pmatrix} a_{11} & a_{12} & & & & \\ & a_{22} & & & & \\ & a_{32} & a_{33} & a_{34} & & \\ & & & a_{44} & & \\ & & & a_{54} & & \\ & & & & \ddots & \end{pmatrix} = \begin{pmatrix} \overline{a_{11}} & & & & & \\ \overline{a_{12}} & \overline{a_{22}} & \overline{a_{32}} & & & \\ & & \overline{a_{33}} & & & \\ & & \overline{a_{34}} & \overline{a_{44}} & \overline{a_{54}} & \\ & & & & \overline{a_{55}} & \\ & & & & \ddots & \end{pmatrix}.$$

Hence $a_{ij} = 0$ for all $i \neq j$ and a_{ii} is real. So A is a real diagonal matrix.

Conversely, it is clear. □

Theorem 2. *Let Alg$\mathcal{L}$ be the tridiagonal algebra on a separable complex Hilbert space $\mathcal{H}$ and let $x = (x_i)$ and $y = (y_i)$ be vectors in $\mathcal{H}$. Then the following are equivalent:*

(1) There exists a self-adjoint operator $A = (a_{kt})$ in Alg$\mathcal{L}$ such that $Ax = y$.

(2) There is a bounded sequence $\{\alpha_n\}$ of real numbers such that $y_i = \alpha_i x_i$ for all $i \in \mathbb{N}$.

Proof. Suppose that A is a self-adjoint operator $A = (a_{kt})$ in Alg$\mathcal{L}$ such that $Ax = y$. By Lemma 1, A is diagonal and a_{kk} is real for all $k \in \mathbb{N}$. Let $\alpha_k = a_{kk}$ for $k = 1, 2, \cdots$. Since $Ax = y$, $y_i = a_{ii}x_i = \alpha_i x_i$ for $i = 1, 2, \cdots$.

Conversely, assume that there is a bounded sequence $\{\alpha_n\}$ of real numbers such that $y_i = \alpha_i x_i$ for $i = 1, 2, \cdots$. Let $A = (a_{ii})$ be a diagonal matrix such that $\alpha_n = a_{nn}$. Since $\{\alpha_n\}$ is bounded, A is a bounded operator. Also A is self-adjoint and $Ax = y$. $\square$

Theorem 3. *Let Alg$\mathcal{L}$ be the tridiagonal algebra on a separable complex Hilbert space $\mathcal{H}$ and let $x_i = (x_j^{(i)})$ and $y_i = (y_j^{(i)})$ be vectors in $\mathcal{H}$ for $i = 1, 2, \cdots, n$. Where n is a fixed natural number. Then the following are equivalent:*

(1) There exists a self-adjoint operator $A = (a_{kt})$ in Alg$\mathcal{L}$ such that $Ax_i = y_i$ for $i = 1, 2, \cdots, n$.

(2) There is a bounded sequence $\{\alpha_m\}$ of real numbers such that $y_j^{(i)} = \alpha_j x_j^{(i)}$ for $i = 1, 2, \cdots, n$ and $j \in \mathbb{N}$.

Proof. Suppose that A is a self-adjoint operator $A = (a_{kt})$ in Alg$\mathcal{L}$ such that $Ax_i = y_i$ for $i = 1, 2, \cdots, n$. Then A is diagonal and a_{kk} is real for each $k \in \mathbb{N}$ by Lemma 1. Let $\alpha_m = a_{mm}$ for $m = 1, 2, \cdots$. Then $\{\alpha_m\}$ is bounded. Since $Ax_i = y_i$, $y_j^{(i)} = a_{jj}x_j^{(i)} = \alpha_j x_j^{(i)}$ for $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots$.

Conversely, assume that there is a bounded sequence $\{\alpha_m\}$ of real numbers such that $y_j^{(i)} = \alpha_i x_j^{(i)}$ for $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots$. Let A be a diagonal matrix with diagonal $\{\alpha_m\}$. Since $\{\alpha_m\}$ is bounded, A is a bounded operator. Also A is self-adjoint and $Ax_i = y_i$ for $i = 1, 2, \cdots, n$. $\square$

By the similar way with the above, we have the following.

Theorem 4. *Let $Alg\mathcal{L}$ be the tridiagonal algebra on a separable complex Hilbert space $\mathcal{H}$ and let $x_i = (x_j^{(i)})$ and $y_i = (y_j^{(i)})$ be vectors in $\mathcal{H}$ for $i = 1, 2, \cdots$. Then the following are equivalent:*

(1) There exists a self-adjoint operator $A = (a_{kl})$ in $Alg\mathcal{L}$ such that $Ax_i = y_i$ for $i = 1, 2, \cdots$.

(2) There is a bounded sequence $\{\alpha_m\}$ of real numbers such that $y_j^{(i)} = \alpha_j x_j^{(i)}$ for each $i \in \mathbb{N}$.

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Dongwan Park and Jae Hyun Park
Department of Mathematics
Keimyung University Daegu, Korea
Email : dpark@kmu.ac.kr