Honam Mathematical J. **28** (2006), No. 1, pp. 57–67

# HYPER $BCC$-ALGEBRAS

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**Abstract.** we apply the hyperstructures to $BCC$-algebras, and introduce the concept of a hyper $BCC$-algebra which is a generalization of a $BCC$-algebra, and investigates some related properties. We also introduce the notion of a hyper $BCC$-subalgebra, $BCC$-scalar element and a hyper $BCC$-ideal, and discuss related properties.

## 1. Introduction

The study of $BCK$-algebras was initiated by K. Iséki in 1966 as a generalization of the concept of set-theoretic difference and propositional calculus. Since then a great deal of literature has been produced on the theory of $BCK$-algebras, In particular, emphasis seems to have been put on the ideal theory of $BCK$-algebras. In [9], Y. Komori introduced a notion of $BCC$-algebra which is a generalization of a $BCK$-algebra and proved that the class of all $BCC$-algebras is not a variety. The hyperstructure theory (called also multialgebras) is introduced in 1934 by F. Marty ([7]) at the 8th congress of Scandinavian Mathematiciens. Around the 40's, several authors worked on hypergroups, especially in France and in the United States, but also in Italy, Russia and Japan. Over the following decades, many important results appeared, but above

Received December 27, 2005. Revised February 7, 2006.

**2000 Mathematics Subject Classification** : 06F35, 03G25, 20N20, 03B52.

**Key words and phrases** : Hyper $BCC$-algebra, hyper $BCC$-subalgebra, $BCC$-scalar element, hyper $BCC$-ideal.

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all since the 70's onwards the most luxuriant flourishing of hyperstructures has been seen. Hyperstructures have many applications to several sectors of both pure and applied sciences. In [3], Y. B. Jun et al. applied the hyperstructures to $BCK$-algebras, and introduced the concept of a hyperBCK-algebra which is a generalization of a $BCK$-algebra, and investigated some related properties. They also introduced the notion of a hyper $BCK$-ideal and a weak hyper $BCK$-ideal, and gave relations between hyper $BCK$-ideals and weak hyper $BCK$-ideals. In this paper we apply the hyperstructures to $BCC$-algebras, and introduce the concept of a hyper $BCC$-algebra which is a generalization of a $BCC$-algebra, and investigates some related properties. We also introduce the notion of a hyper $BCC$-subalgebra, $BCC$-scalar element and a hyper $BCC$-ideal, and discuss related properties.

## 2. Preliminaries

Recall that a *BCC-algebra* is an algebra $(X, *, 0)$ of type (2,0) satisfying the following axioms: for every $x, y, z \in X$,

(C1) $((x * y) * (z * y)) * (x * z) = 0$,

(C2) $0 * x = 0$,

(C3) $x * 0 = x$,

(C4) $x * y = 0$ and $y * x = 0$ imply $x = y$.

In a $BCC$-algebra $X$, the identity

(p1) $x * x = 0$,

(p2) $x * y \leq x$

hold for all $x, y \in X$, where $x \leq y$ is defined by $x * y = 0$.

Let $G$ be a nonempty set endowed with a hyper operation "$\circ$", that is, $\circ$ is a function from $G \times G$ to $\mathcal{P}^*(G) = \mathcal{P}(G) \setminus \{\emptyset\}$. For two subsets $A$ and $B$ of $G$, denote by $A \circ B$ the set $\bigcup_{a \in A, b \in B} a \circ b$. We shall use $x \circ y$ instead of $x \circ \{y\}$, $\{x\} \circ y$, or $\{x\} \circ \{y\}$.

By a *hyper $BCK$-algebra* we mean a non-empty set $H$ endowed with a hyper operation "$\circ$" and a constant 0 satisfying the following axioms:

(A1) $(x \circ z) \circ (y \circ z) \ll x \circ y$,
(A2) $(x \circ y) \circ z = (x \circ z) \circ y$,
(A3) $x \circ H \ll \{x\}$,
(A4) $x \ll y$ and $y \ll x$ imply $x = y$

for all $x, y, z \in H$, where $x \ll y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A \ll B$ is defined by $\forall a \in A$, $\exists b \in B$ such that $a \ll b$(see [3]).

A non-empty subset $I$ of a hyper $BCK$-algebra $H$ is called a *hyper $BCK$-ideal* of $H$ (see [3]) if it satisfies

(i) $0 \in I$,
(ii) $(\forall x, y \in H)$ $(x \circ y \ll I,\ y \in I \ \Rightarrow\ x \in I)$.

## 3. Hyper $BCC$-algebras

**Definition 3.1.** By a *hyper $BCC$-algebra* we mean a nonempty subset $G$ endowed with a hyper operation "$\diamond$" and a constant 0 satisfying the following axioms:

(H1) $(x \diamond y) \diamond (z \diamond y) \ll x \diamond z$,
(H2) $x \ll x$,
(H3) $x \diamond y \ll x$,
(H4) $x \ll y$ and $y \ll x$ imply $x = y$

for all $x, y, z \in G$, where $x \ll y$ is defined by $0 \in x \diamond y$ and for every $A, B \subseteq G$, $A \ll B$ is defined by $\forall a \in A$, $\exists b \in B$ such that $a \ll b$.

**Example 3.2.** (1) Let $(G; *, 0)$ be a $BCC$-algebra and define a hyper operation "$\diamond$" on $G$ by $x \diamond y = \{x * y\}$ for all $x, y \in G$. Then $(G, \diamond)$ is a hyper $BCC$-algebra.

(2) Let $G = \{0, a, b\}$. Consider the following table:

| $\diamond$ | $0$ | $a$ | $b$ |
|---|---|---|---|
| $0$ | $\{0\}$ | $\{0\}$ | $\{0\}$ |
| $a$ | $\{a\}$ | $\{0, a\}$ | $\{0, a\}$ |
| $b$ | $\{b\}$ | $\{a, b\}$ | $\{0, a, b\}$ |

Then $(G, \diamond)$ is a hyper $BCC$-algebra.

(3) Define a hyper operation "$\diamond$" on $G := [0, \infty)$ by

$$x \diamond y := \begin{cases} [0, x] & \text{if } x \leq y \\ (0, y] & \text{if } x > y \neq 0 \\ \{x\} & \text{if } y = 0 \end{cases}$$

for all $x, y \in G$. Then $(G, \diamond)$ is a hyper $BCC$-algebra.

(4) Let $G = \{0, a, b, c\}$. Consider the following Cayley tables:

| $\diamond_1$ | $0$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $0$ | $\{0\}$ | $\{0\}$ | $\{0\}$ | $\{0\}$ |
| $a$ | $\{a\}$ | $\{0, a\}$ | $\{0, a\}$ | $\{a, c\}$ |
| $b$ | $\{b\}$ | $\{b\}$ | $\{0, a\}$ | $\{b\}$ |
| $c$ | $\{c\}$ | $\{0, c\}$ | $\{0, c\}$ | $\{0, c\}$ |

| $\diamond_2$ | $0$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $0$ | $\{0\}$ | $\{0\}$ | $\{0\}$ | $\{0\}$ |
| $a$ | $\{c\}$ | $\{0, c\}$ | $\{0, c\}$ | $\{c\}$ |
| $b$ | $\{a\}$ | $\{a\}$ | $\{0, a\}$ | $\{0, a\}$ |
| $c$ | $\{b\}$ | $\{0, b\}$ | $\{b\}$ | $\{0, b\}$ |

Then $(G, \diamond_1)$ and $(G, \diamond_2)$ are hyper $BCC$-algebras, which are not hyper $BCK$-algebras since $(a \diamond_1 a) \diamond_1 b \neq (a \diamond_1 b) \diamond_1 a$ and $(a \diamond_2 b) \diamond_2 c \neq (a \diamond_2 c) \diamond_2 b$.

**Proposition 3.3.** *In a hyper $BCC$-algebra $(G, \diamond)$, the following hold.*

(1) $x \ll y$ *and* $x \neq y$ *imply* $y \not\ll x$,

(2) $x \diamond 0 \ll x$, $0 \diamond x \ll 0$, *and* $0 \diamond 0 \ll 0$,

(3) $0 \diamond x = \{0\}$,

(4) $0 \ll x$,

(5) $0 \diamond A = \{0\}$,

(6) $A \ll A$,

(7) $A \subseteq B$ *implies* $A \ll B$,

(8) $0 \diamond 0 = \{0\}$,

(9) $A \diamond B \ll A$,

(10) $A \ll 0$ *implies* $A = \{0\}$,

(11) $x \diamond z = \{0\}$ *implies* $(x \diamond y) \diamond (z \diamond y) = \{0\}$ *and* $x \diamond y \ll z \diamond y$,

*for all* $x, y, z \in G$ *and* $A, B \in \mathcal{P}^*(G)$.

*Proof.* (1) is by (H4).

(2) It is straightforward by (H3).

(3) Let $a \in 0 \diamond x$. Then $a \ll 0$ because $0 \diamond x \ll 0$. It follows from $a \ll 0$ that $0 \in a \diamond 0$. Then by (2), we have $0 \ll a$. Hence $a = 0$ by (H4), which shows $0 \diamond x = \{0\}$.

(4) It is straightforward by (3).

(5) By (3), we have $0 \diamond A = \cup\{0 \diamond a \mid a \in A\} = \{0\}$.

(6) It is by (H2).

(7) Assume $A \subseteq B$ and let $a \in A$. Taking $b = a$ implies $b = a \in A \subseteq B$. Since $a \ll b$ by (H2), it follows that $A \ll B$.

(8) It is by (3).

(9) Let $x \in A \diamond B$. Then $x \in a \diamond b$ for some $a \in A$ and $b \in B$. It follows from (H3) that $x \ll a$ so that $A \diamond B \ll A$.

(10) Assume that $A \ll 0$ and let $a \in A$. Then $a \ll 0$, and so $a = 0$. Hence $A = \{0\}$.

(11) Assume that $x \diamond z = \{0\}$. Then $(x \diamond y) \diamond (z \diamond y) \ll x \diamond z = \{0\}$ and so $(x \diamond y) \diamond (z \diamond y) = \{0\}$ by (10), which implies that $x \diamond y \ll z \diamond y$. □

Note that a hyper $BCC$-algebra may not satisfy the equality

$$(x \diamond y) \diamond z = (x \diamond z) \diamond y$$

in general (see Example 3.2(4)).

**Proposition 3.4.** *Every hyper $BCC$-algebra $(G, \diamond)$ satisfying the equality*

$$(x \diamond y) \diamond z = (x \diamond z) \diamond y, \forall x, y, z \in G$$

*is a hyper $BCK$-algebra.*

*Proof.* The proof is straightforward. □

We know that there is a hyper $BCC$-algebra $G$ in which the condition $x \in x \diamond 0$ is not true in general (see Example 3.2(4)).

**Proposition 3.5.** *Let $(G, \diamond)$ be a hyper $BCC$-algebra such that $x \in x \diamond 0$ for all $x \in G$. Then*

(i) *$x \diamond 0 \ll y$ implies $x \ll y$,*
(ii) *$x \ll y$ implies $z \diamond y \ll z \diamond x$.*

*Proof.* (i) It is straightforward.

(ii) Let $x, y \in X$ be such that $x \ll y$. Then $(z \diamond y) \diamond 0 \subseteq (z \diamond y) \diamond (x \diamond y) \ll z \diamond x$ and so $(z \diamond y) \diamond 0 \ll z \diamond x$. Thus for each $a \in z \diamond y$ there exists $b \in z \diamond x$ such that $a \diamond 0 \ll b$. It follows from (i) that $a \ll b$ so that $z \diamond y \ll z \diamond x$. □

**Proposition 3.6.** *Let $A$ be a subset of a hyper $BCC$-algebra $G$ and let $x, y, z \in G$. If $(x \diamond y) \diamond z \ll A$, then $a \diamond z \ll A$ for all $a \in x \diamond y$.*

*Proof.* The proof is straightforward. □

**Definition 3.7.** Let $(G, \diamond)$ be a hyper $BCC$-algebra and let $S$ be a subset of $G$ containing 0. If $S$ is a hyper $BCC$-algebra with respect to the hyper operation "$\diamond$" on $G$, we say that $S$ is a *hyper $BCC$-subalgebra* of $G$.

**Proposition 3.8.** *Let $S$ be a subset of a hyper $BCC$-algebra $(G, \diamond)$. If $x \diamond y \subseteq S$ for all $x, y \in S$, then $0 \in S$.*

*Proof.* Assume that $x \diamond y \subseteq S$ for all $x, y \in S$. Let $a \in S$. Since $a \diamond a \ll a$ by (H3), it follows that

$$0 \in (a \diamond a) \diamond a = \cup\{x \diamond a \mid x \in a \diamond a\} \subseteq S.$$

This completes the proof. □

**Theorem 3.9.** *Let $S$ be a non-empty subset of a hyper $BCC$-algebra $(G, \diamond)$. Then $S$ is a hyper $BCC$-subalgebra of $G$ if and only if $x \diamond y \subseteq S$ for all $x, y \in S$.*

*Proof.* ($\Rightarrow$) Obvious.

($\Leftarrow$) Assume that $x \diamond y \subseteq S$ for all $x, y \in S$. Then $0 \in S$ by Proposition 3.8. For any $x, y, z \in S$, we have $z \diamond y \subseteq S$ and $x \diamond z \subseteq S$. Hence

$$(x \diamond y) \diamond (z \diamond y) = \cup\{a \diamond b \mid a \in x \diamond y,\ b \in z \diamond y\} \subseteq S$$

and thus (H1) holds in $S$. Other axioms are straightforward. □

**Theorem 3.10.** *Let $(G, \diamond)$ be a hyper $BCC$-algebra. Then the set*

$$S(G, \diamond) := \{x \in G \mid x \diamond x = \{0\}\}$$

*is a hyper $BCC$-subalgebra of $G$.*

*Proof.* From Proposition 3.3(8), we have $0 \in S(G, \diamond)$. Let $x, y \in S(G, \diamond)$ and $a \in x \diamond y$. Then $(x \diamond y) \diamond (x \diamond y) \ll x \diamond x = \{0\}$ by (H1) and so $a \diamond a \subseteq (x \diamond y) \diamond (x \diamond y) = \{0\}$ by Proposition 3.3(10). Thus $a \diamond a = \{0\}$, that is, $a \in S(G, \diamond)$. Hence $x \diamond y \subseteq S(G, \diamond)$, which completes the proof. □

**Definition 3.11.** Let $(G, \diamond)$ be a hyper $BCC$-algebra. An element $a \in G$ is said to be *left* (resp. *right*) *$BCC$-scalar* if $|a \diamond x| = 1$ (resp. $|x \diamond a| = 1$) for all $x \in G$. Denote by $R(G, \diamond)$ (resp. $L(G, \diamond)$) the set of all right (resp. left) $BCC$-scalar elements of $G$. An element $a \in R(G, \diamond) \cap L(G, \diamond)$ is called a *$BCC$-scalar element.*

**Example 3.12.** (1) In the hyper $BCC$-algebra $(G, \diamond)$ in Example 3.2(1), we have $0 \in R(G, \diamond) \cap L(G, \diamond)$.

(2) In the hyper $BCC$-algebra $(G, \diamond)$ in Example 3.2(3), we have $0 \in R(G, \diamond) \cap L(G, \diamond)$.

(3) Let $G = \{0, a, b\}$. Consider the following Cayley tables:

| $\diamond_1$ | $0$ | $a$ | $b$ |
|---|---|---|---|
| $0$ | $\{0\}$ | $\{0\}$ | $\{0\}$ |
| $a$ | $\{a\}$ | $\{0\}$ | $\{0\}$ |
| $b$ | $\{b\}$ | $\{b\}$ | $\{0, b\}$ |

| $\diamond_2$ | $0$ | $a$ | $b$ |
|---|---|---|---|
| $0$ | $\{0\}$ | $\{0\}$ | $\{0\}$ |
| $a$ | $\{a\}$ | $\{0\}$ | $\{0\}$ |
| $b$ | $\{b\}$ | $\{a, b\}$ | $\{0, a, b\}$ |

Then $(G, \diamond_1)$ and $(G, \diamond_2)$ are hyper $BCC$-algebras, and $R(G, \diamond_1) = \{0, a\} = L(G, \diamond_1)$, $R(G, \diamond_2) = \{0\}$ and $L(G, \diamond_2) = \{0, a\}$.

**Proposition 3.13.** *In any hyper $BCC$-algebra $(G, \diamond)$, we have $0 \in L(G, \diamond)$.*

*Proof.* The proof is by Proposition 3.3(3). □

**Theorem 3.14.** *Let $(G, \diamond)$ be a hyper $BCC$-algebra. Then*

(1) $R(G, \diamond) \subseteq L(G, \diamond) = S(G, \diamond)$.

(2) *$L(G, \diamond)$ is a hyper $BCC$-subalgebra of $G$.*

*Proof.* (1) Let $a \in L(G, \diamond)$. Then $|a \diamond a| = 1$. Since $0 \in a \diamond a$ by (H2), it follows that $a \diamond a = \{0\}$ so that $a \in S(G, \diamond)$. Now if $b \in S(G, \diamond)$, then $(b \diamond y) \diamond (b \diamond y) \ll b \diamond b = \{0\}$ for all $y \in G$. Using Proposition 3.3(10), we get $(b \diamond y) \diamond (b \diamond y) = \{0\}$. Let $u, v \in b \diamond y$. Then $u \diamond v \subseteq (b \diamond y) \diamond (b \diamond y) = \{0\}$ and so $u \diamond v = \{0\}$, that is, $u \ll v$. Similarly we have $v \ll u$, and thus $u = v$ by (H4). This means that $b \diamond y$ is a singleton set, that is, $|b \diamond y| = 1$ for all $y \in G$. Therefore $b \in L(G, \diamond)$, and consequently $L(G, \diamond) = S(G, \diamond)$. In order to show $R(G, \diamond) \subseteq L(G, \diamond)$, let $a \in R(G, \diamond)$. Then $|a \diamond a| = 1$ and thus $a \diamond a = \{0\}$ by (H2). Hence $a \in S(G, \diamond) = L(G, \diamond)$.

(2) is by (1) and Theorem 3.10. □

**Definition 3.15.** A subset $A$ of a hyper $BCC$-algebra $(G, \diamond)$ is called a *hyper $BCC$-ideal* of $G$ if it satisfies:

(HI1) $0 \in A$,
(HI2) $(x \diamond y) \diamond z \ll A$ and $y \in A$ imply $x \diamond z \subseteq A$.

**Example 3.16.** Let $(G, \diamond_1)$ be a hyper $BCC$-algebra in Example 3.2(4). Then $A_1 := \{0, c\}$ and $A_2 := \{0, a, c\}$ are hyper $BCC$-ideal of $G$. But the hyper $BCC$-algebra $(G, \diamond_2)$ in Example 3.2(4) have no proper hyper $BCC$-ideals.

**Theorem 3.17.** *Let $(G, \diamond)$ be a hyper $BCC$-algebra in which the equality $x \diamond 0 = \{x\}$ holds for all $x \in G$. Then every hyper $BCC$-ideal is a hyper $BCK$-ideal.*

*Proof.* Let $A$ be a hyper $BCC$-ideal of $G$. Obviously $0 \in A$. Let $x, y \in G$ be such that $x \diamond y \ll A$ and $y \in A$. Then

$$(x \diamond y) \diamond 0 = \cup\{a \diamond 0 \mid a \in x \diamond y\} = x \diamond y \ll A.$$

It follows from (HI2) and hypothesis that $\{x\} = x \diamond 0 \subseteq A$, that is, $x \in A$. Hence $A$ is a hyper $BCK$-ideal of $G$. □

The following example shows that the converse of Theorem 3.17 may not be true.

**Example 3.18.** Let $G = \{0, a, b, c\}$. Consider the following table:

| $\diamond$ | $0$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $0$ | $\{0\}$ | $\{0\}$ | $\{0\}$ | $\{0\}$ |
| $a$ | $\{a\}$ | $\{0, a\}$ | $\{0\}$ | $\{0\}$ |
| $b$ | $\{b\}$ | $\{b\}$ | $\{0\}$ | $\{0\}$ |
| $c$ | $\{c\}$ | $\{b\}$ | $\{b\}$ | $\{0\}$ |

Then $(G, \diamond)$ is a hyper $BCC$-algebra and $I := \{0, a\}$ is a hyper $BCK$-ideal of $G$ but it is not a hyper $BCC$-ideal of $G$ since $(c \diamond a) \diamond b \ll I$ and $a \in I$ but $c \diamond b \not\subseteq I$.

**Theorem 3.19.** *In a hyper $BCC$-algebra in which the equality $x \diamond 0 = \{x\}$ holds for all $x \in G$, every hyper $BCK$-ideal is a hyper $BCC$-subalgebra.*

*Proof.* Let $A$ be a hyper $BCC$-ideal and $x, y \in A$. Then $(x \diamond y) \diamond 0 = x \diamond y \ll A$ and $0 \in A$. Since $A$ is hyper $BCC$-ideal, we get $x \diamond y \subseteq A$. $\square$

**Corollary 3.20.** *In a hyper $BCC$-algebra in which the equality $x \diamond 0 = \{x\}$ holds for all $x \in G$, every hyper $BCC$-ideal is a hyper $BCC$-subalgebra.*

The following example shows that the converse of Theorem 3.19 may not be true.

**Example 3.21.** The hyper $BCC$-algebra $(G, \diamond_1)$ in Example 3.2(4), $\{0, a\}$ is a hyper $BCC$-subalgebra but not hyper $BCC$-ideal, since $(a \diamond_1 a) \diamond_1 c \ll \{0, a\}$ and $a \in \{0, a\}$ but $a \diamond_1 c \not\subseteq \{0, a\}$.

**Acknowledgements.** The authors are deeply grateful to referees for their valuable comments and suggestions for improving the paper.

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