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SIMULATIVE AND MUTANT WFI-ALGEBRAS

Young Bae Jun and Seok-Zun Song*

Abstract. The notion of simulative and mutant WFI-algebras is introduced, and several properties are investigated. Characterizations of a simulative WFI-algebra are established. A relation between an associative WFI-algebra and a simulative WFI-algebra is given. Some types for a simulative WFI-algebra to be mutant are found.

1. Introduction

In 1990, W. M. Wu [3] introduced the notion of fuzzy implication algebras (FI-algebra, for short), and investigated several properties. In [2], Z. Li and C. Zheng introduced the notion of distributive (resp. regular, commutative) FI-algebras, and investigated the relations between such FI-algberas and MV-algebras. In [1], Y. B. Jun discussed WFI-algebras (weak fuzzy implication algebras) which are weaker than FI-algebras, and gave a characterization of a WFI-algebra. He introduced the notion of associative (resp. normal, medial) WFI-algebras, and investigated several properties. He gave conditions for a WFI-algebra to be associative/medial, and provided characterizations of associative/medial WFI-algebras, and showed that every associative WFI-algebra is a group in which every element is an involution. He also verified that the class

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* Corresponding author.

of all medial WFI-algebras is a variety. In this paper, we introduce the notion of simulative and mutant WFI-algebras and investigate some properties. We establish characterizations of a simulative WFI-algebra. We give a relation between an associative WFI-algebra and a simulative WFI-algebra. We find some types for a simulative WFI-algebra to be mutant.

2. Preliminaries

We investigate an algebra $\mathscr{G} := (G; \ominus, 1)$, consisting of a set G, a binary operation $\ominus$ and a special element 1, which satisfies the four axioms:

(a1) $x \ominus (y \ominus z) = y \ominus (x \ominus z)$,
(a2) $(x \ominus y) \ominus \big((y \ominus z) \ominus (x \ominus z)\big) = 1$,
(a3) $x \ominus x = 1$,
(a4) $x \ominus y = y \ominus x = 1 \Rightarrow x = y$.

We call such algebra a *WFI-algebra*. A nonempty subset S of G is called a *subalgebra* of $\mathscr{G}$ if $x \ominus y \in S$ whenever $x, y \in S$. A nonempty subset F of G is called a *filter* of $\mathscr{G}$ if it satisfies:

- $1 \in F$,
- $x \ominus y \in F$ and $x \in F$ imply $y \in F$ for all $x, y \in G$.

In a WFI-algebra $\mathscr{G}$, the following are true (see [1]):

(b1) $x \ominus ((x \ominus y) \ominus y) = 1$,
(b2) $1 \ominus x = 1 \Rightarrow x = 1$,
(b3) $1 \ominus x = x$,
(b4) $x \ominus y = 1 \Rightarrow (y \ominus z) \ominus (x \ominus z) = 1$, $(z \ominus x) \ominus (z \ominus y) = 1$,
(b5) $(x \ominus y) \ominus 1 = (x \ominus 1) \ominus (y \ominus 1)$.

We now define a relation "$\preceq$" on $\mathscr{G}$ by $x \preceq y$ if and only if $x \ominus y = 1$. It is easy to verify that a WFI-algebra is a partially ordered set with respect to $\preceq$. A WFI-algebra $\mathscr{G}$ is said to be *associative* [1] if it satisfies

$(x \ominus y) \ominus z = x \ominus (y \ominus z)$ for all $x, y, z \in G$. A WFI-algebra $\mathscr{G}$ is said to be *medial* [1] if it satisfies

$$(x \ominus y) \ominus (a \ominus b) = (x \ominus a) \ominus (y \ominus b)$$

for all $x, y, a, b \in G$.

3. Simulative and Mutant WFI-algebras

For a WFI-algebra $\mathscr{G}$, the set

$$\mathcal{S}(\mathscr{G}) := \{x \in G \mid x \preceq 1\}$$

is called the *simulative part* of $\mathscr{G}$. Let $x, y \in \mathcal{S}(\mathscr{G})$. Then $x \ominus 1 = 1$ and $y \ominus 1 = 1$. It follows from (b5) that

$$(x \ominus y) \ominus 1 = (x \ominus 1) \ominus (y \ominus 1) = 1 \ominus 1 = 1$$

so that $x \ominus y \preceq 1$. Hence $x \ominus y \in \mathcal{S}(\mathscr{G})$, which shows that $\mathcal{S}(\mathscr{G})$ is a subalgebra of $\mathscr{G}$.

Proposition 3.1. *The simulative part of a WFI-algebra $\mathscr{G}$ is a filter of $\mathscr{G}$.*

Proof. Obviously, $1 \in \mathcal{S}(\mathscr{G})$. Let $x, y \in G$ be such that $x \in \mathcal{S}(\mathscr{G})$ and $x \ominus y \in \mathcal{S}(\mathscr{G})$. Then $x \preceq 1$ and $x \ominus y \preceq 1$. It follows from (b5) and (b3) that

$$1 = (x \ominus y) \ominus 1 = (x \ominus 1) \ominus (y \ominus 1) = 1 \ominus (y \ominus 1) = y \ominus 1,$$

that is, $y \preceq 1$. Hence $y \in \mathcal{S}(\mathscr{G})$, and the proof is complete. □

Definition 3.2. A WFI-algebra $\mathscr{G}$ is said to be *simulative* if it satisfies

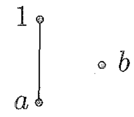
(S) $x \preceq 1 \Rightarrow x = 1$.

Note that the condition (S) is equivalent to $\mathcal{S}(\mathscr{G}) = \{1\}$.

Example 3.3. (1) Let $\mathbb{Z}$ be the set of integers. Then $\mathfrak{Z} := (\mathbb{Z}; \ominus, 0)$ is a simulative WFI-algebra, where $x \ominus y = y - x$ for all $x, y \in \mathbb{Z}$.

(2) Let $G = \{1, a, b\}$ be a set with the following Cayley table and Hasse diagram:

$\ominus$	1	a	b
1	1	a	b
a	1	1	b
b	b	b	1



Then $\mathscr{G} := (G; \ominus, 1)$ is a WFI-algebra (see [1]) which does not satisfy the condition (S).

Theorem 3.4. *Let $\mathscr{G}$ be a WFI-algebra. Then the following are equivalent.*

(i) *$\mathscr{G}$ is simulative.*
(ii) $(x \ominus 1) \ominus 1 = x$, $\forall x \in G$.
(iii) $(x \ominus 1) \ominus y = (y \ominus 1) \ominus x$, $\forall x, y \in G$.

Proof. (i) $\Rightarrow$ (ii) Suppose that $\mathscr{G}$ is simulative. Using (a1) and (a3), we have

$$x \ominus ((x \ominus 1) \ominus 1) = (x \ominus 1) \ominus (x \ominus 1) = 1,$$

that is, $x \preceq (x \ominus 1) \ominus 1$. It follows from (a3) and (b4) that

$$((x \ominus 1) \ominus 1) \ominus x \preceq x \ominus x = 1$$

so from (S) that $((x \ominus 1) \ominus 1) \ominus x = 1$, that is, $(x \ominus 1) \ominus 1 \preceq x$. Hence, by (a4), we get $(x \ominus 1) \ominus 1 = x$.

(ii) $\Rightarrow$ (iii) Assume that $\mathscr{G}$ satisfies (ii). Then

$$\begin{aligned}(x \ominus 1) \ominus y &= (x \ominus 1) \ominus ((y \ominus 1) \ominus 1) && \text{by (ii)} \\ &= (y \ominus 1) \ominus ((x \ominus 1) \ominus 1) && \text{by (a1)} \\ &= (y \ominus 1) \ominus x, && \text{by (ii)}\end{aligned}$$

which proves (iii).

(iii) $\Rightarrow$ (i) Suppose that $\mathscr{G}$ satisfies (iii) and let $x \in G$ be such that $x \preceq 1$. Then

$$x = 1 \ominus x = (1 \ominus 1) \ominus x = (x \ominus 1) \ominus 1 = 1 \ominus 1 = 1,$$

and so $\mathscr{G}$ is simulative. □

Theorem 3.5. *Let $\mathscr{G}$ be a WFI-algebra. Then the following are equivalent.*

(i) *$\mathscr{G}$ is simulative.*

(ii) $x \preceq y \Rightarrow x = y$.

(iii) $(x \ominus y) \ominus (z \ominus y) = z \ominus x$, $\forall x, y, z \in G$.

(iv) $(x \ominus y) \ominus 1 = y \ominus x$, $\forall x, y \in G$.

(v) *$\mathscr{G}$ satisfies the right cancellation law, i.e.,* $x \ominus z = y \ominus z \Rightarrow x = y$.

(vi) $(x \ominus y) \ominus (z \ominus y) = (x \ominus z) \ominus 1$, $\forall x, y, z \in G$.

Proof. (i) $\Rightarrow$ (ii) Suppose that $\mathscr{G}$ is simulative and let $x, y \in G$ be such that $x \preceq y$. Then

$$\begin{aligned} (y \ominus x) \ominus 1 &= (y \ominus 1) \ominus (x \ominus 1) && \text{by (b5)} \\ &= (y \ominus (x \ominus y)) \ominus (x \ominus 1) && \text{since } x \preceq y \\ &= (x \ominus (y \ominus y)) \ominus (x \ominus 1) && \text{by (a1)} \\ &= (x \ominus 1) \ominus (x \ominus 1) && \text{by (a3)} \\ &= 1, && \text{by (a3)} \end{aligned}$$

and so $y \ominus x \in S(\mathscr{G}) = \{1\}$. Thus $y \ominus x = 1$, i.e., $y \preceq x$. It follows from (a4) that $x = y$.

(ii) $\Rightarrow$ (iii) Assume that (ii) holds. Since $z \ominus x \preceq (x \ominus y) \ominus (z \ominus y)$, it follows from (ii) that $z \ominus x = (x \ominus y) \ominus (z \ominus y)$.

(iii) $\Rightarrow$ (iv) Suppose that (iii) is true. Then

$$(x \ominus y) \ominus 1 = (x \ominus 1) \ominus (y \ominus 1) = y \ominus x.$$

(iv) $\Rightarrow$ (v) Assume that (iv) is true and let $x, y, z \in G$ be such that $x \ominus z = y \ominus z$. Then

$$y \ominus x = (x \ominus y) \ominus 1 = (x \ominus y) \ominus ((y \ominus z) \ominus (x \ominus z)) = 1.$$

Similarly, we have $x \ominus y = 1$, and so $x = y$ by (a4).

(v) $\Rightarrow$ (i) Suppose that $\mathscr{G}$ satisfies the right cancellation law. Let $x \in \mathcal{S}(\mathscr{G})$. Then $x \ominus 1 = 1 = 1 \ominus 1$, and so $x = 1$ by using (v). Therefore $\mathcal{S}(\mathscr{G}) = \{1\}$ which shows that $\mathscr{G}$ is simulative.

(iv) $\Rightarrow$ (vi) If (iv) is true, then (iii) is valid, and so

$$(x \ominus y) \ominus (z \ominus y) = z \ominus x = (x \ominus z) \ominus 1.$$

(vi) $\Rightarrow$ (i) Suppose that (vi) is valid and let $x \in \mathcal{S}(\mathscr{G})$. Then $x \ominus 1 = 1 = x \ominus x$, which implies that

$$x = (x \ominus x) \ominus (1 \ominus x) = (x \ominus 1) \ominus 1 = 1 \ominus 1 = 1.$$

This shows that $\mathcal{S}(\mathscr{G}) = \{1\}$. Hence $\mathscr{G}$ is simulative. This completes the proof. □

Proposition 3.6. *Let $\mathscr{G}$ be a simulative WFI-algebra. Then*

$$x \ominus (y \ominus z) = ((x \ominus 1) \ominus y) \ominus z, \ \forall x, y, z \in G.$$

Proof. For any $x, y, z \in G$, we have

$$\begin{aligned}
x \ominus (y \ominus z) &= ((x \ominus 1) \ominus 1) \ominus (y \ominus z) && \text{by Theorem 3.4} \\
&= y \ominus (((x \ominus 1) \ominus 1) \ominus z) && \text{by (a1)} \\
&= y \ominus ((z \ominus 1) \ominus (x \ominus 1)) && \text{by Theorem 3.4} \\
&= (z \ominus 1) \ominus (y \ominus (x \ominus 1)) && \text{by (a1)} \\
&= ((y \ominus (x \ominus 1)) \ominus 1) \ominus z && \text{by Theorem 3.4} \\
&= ((x \ominus 1) \ominus y) \ominus z, && \text{by Theorem 3.5}
\end{aligned}$$

which completes the proof. □

Theorem 3.7. *Let $\mathscr{G}$ be a simulative WFI-algebra and define a binary operation "$\cdot$" on $\mathscr{G}$ by $x \cdot y = (y \ominus 1) \ominus x$ for all $x, y \in G$. Then $(G, \cdot, 1)$ is a commutative group.*

Proof. Using (a1) and Theorem 3.4(iii), we have

$$\begin{aligned} x \cdot (y \cdot z) &= x \cdot ((z \ominus 1) \ominus y) = (((z \ominus 1) \ominus y) \ominus 1) \ominus x \\ &= (x \ominus 1) \ominus ((z \ominus 1) \ominus y) = (z \ominus 1) \ominus ((x \ominus 1) \ominus y) \\ &= (z \ominus 1) \ominus ((y \ominus 1) \ominus x) = (x \cdot y) \cdot z, \end{aligned}$$

and $x \cdot y = (y \ominus 1) \ominus x = (x \ominus 1) \ominus y = y \cdot x$. It follows from Theorem 3.4(ii) that $x \cdot 1 = 1 \cdot x = (x \ominus 1) \ominus 1 = x$ and

$$(x \ominus 1) \cdot x = x \cdot (x \ominus 1) = ((x \ominus 1) \ominus 1) \ominus x = x \ominus x = 1.$$

Hence $(G, \cdot, 1)$ is a commutative group with $x \ominus 1$ as the inverse of x. □

Conversely, we have the following.

Theorem 3.8. *If $(G, \cdot, 1)$ is a commutative group, then $\mathscr{G} := (G; \ominus, 1)$ is a simulative WFI-algebra, where $x \ominus y = x^{-1} \cdot y$ for all $x, y \in G$.*

Proof. It is easy to verify the axioms of a *WFI*. □

Lemma 3.9. [1, Proposition 3.25] *Every medial WFI-algebra satisfies the following identities:*

(i) $(x \ominus y) \ominus 1 = y \ominus x$.
(ii) $(x \ominus 1) \ominus 1 = x$.
(iii) $(x \ominus y) \ominus y = x$.

Theorem 3.10. *Let $\mathscr{G}$ be a WFI-algebra. Then the following are equivalent:*

(i) *$\mathscr{G}$ is simulative.*
(ii) $(x \ominus y) \ominus y = x$, $\forall x, y \in G$.
(iii) *$\mathscr{G}$ is medial.*

Proof. (iii) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (i) are by Lemma 3.9 and Theorem 3.4.
(i) $\Rightarrow$ (iii) Assume that $\mathscr{G}$ is simulative. Note that, for all $x, y \in G$,

$$(x \cdot y) \ominus 1 = (x \ominus 1) \cdot (y \ominus 1) = x \ominus (y \ominus 1)$$

and $x \ominus y = ((x \ominus 1) \ominus 1) \ominus y = (x \ominus 1) \cdot y$. It follows that

$$\begin{aligned}
(x \ominus y) \ominus (a \ominus b) &= ((x \ominus 1) \cdot y) \ominus ((a \ominus 1) \cdot b) \\
&= (((x \ominus 1) \cdot y) \ominus 1) \cdot ((a \ominus 1) \cdot b) \\
&= ((x \ominus 1) \ominus (y \ominus 1)) \cdot ((a \ominus 1) \cdot b) \\
&= (((x \ominus 1) \ominus 1) \cdot (y \ominus 1)) \cdot ((a \ominus 1) \cdot b) \\
&= x \cdot (y \ominus 1) \cdot (a \ominus 1) \cdot b \\
&= x \cdot (a \ominus 1) \cdot (y \ominus 1) \cdot b \\
&= (((x \ominus 1) \ominus 1) \cdot (a \ominus 1)) \cdot ((y \ominus 1) \cdot b) \\
&= ((x \ominus 1) \ominus (a \ominus 1)) \cdot ((y \ominus 1) \cdot b) \\
&= (((x \ominus 1) \cdot a) \ominus 1) \cdot ((y \ominus 1) \cdot b) \\
&= ((x \ominus 1) \cdot a) \ominus ((y \ominus 1) \cdot b) \\
&= (x \ominus a) \ominus (y \ominus b)
\end{aligned}$$

for all $x, y, a, b \in G$. Hence $\mathscr{G}$ is medial. □

Theorem 3.11. *Every associative WFI-algebra is a simulative WFI-algebra.*

Proof. Let $\mathscr{G}$ be an associative WFI-algebra. It suffices to show that $\mathscr{G}$ satisfies the identity $(x \ominus y) \ominus y = x$ for all $x, y \in G$ (see Theorem 3.10). Obviously $x \preceq (x \ominus y) \ominus y$. Now, using (a1), (a3), and the associativity, we have

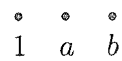
$$\begin{aligned}
((x \ominus y) \ominus y) \ominus x &= (x \ominus y) \ominus (y \ominus x) = y \ominus ((x \ominus y) \ominus x) \\
&= y \ominus (x \ominus (y \ominus x)) = (y \ominus x) \ominus (y \ominus x) = 1,
\end{aligned}$$

that is, $(x \ominus y) \ominus y \preceq x$. It follows from (a4) that $(x \ominus y) \ominus y = x$. Hence $\mathscr{G}$ is simulative. □

The converse of Theorem 3.11 may not be true as shown in the following example.

Example 3.12. Let $G = \{1, a, b\}$ be a set with the following Cayley table and Hasse diagram:

$\ominus$	1	a	b
1	1	a	b
a	b	1	a
b	a	b	1



Then $\mathscr{G} := (G; \ominus, 1)$ is a simulative WFI-algebra, but it is not an associative WFI-algebra since $b \ominus (a \ominus 1) \neq (b \ominus a) \ominus 1$.

For a WFI-algebra $\mathscr{G}$, consider the set

$$\mathcal{L}(\mathscr{G}) := \{x \in G \mid x \ominus 1 = x\}.$$

Note that $\mathcal{L}(\mathscr{G})$ is a subalgebra of $\mathscr{G}$ (see [1]).

Theorem 3.13. *Let $\mathscr{G}$ be a WFI-algebra such that $\mathcal{L}(\mathscr{G}) = G$. Then the following are equivalent:*

(i) *$\mathscr{G}$ is associative.*
(ii) $x \ominus (x \ominus y) = y$, $\forall x, y \in G$.
(iii) $x \ominus (y \ominus z) = z \ominus (y \ominus x)$, $\forall x, y, z \in G$.
(iv) $(x \ominus y) \ominus y = x$, $\forall x, y \in G$.
(v) *$\mathscr{G}$ is medial.*
(vi) *$\mathscr{G}$ is simulative.*

Proof. (i) $\Rightarrow$ (ii) is by (a3), (b3) and the associativity.

(ii) $\Rightarrow$ (iii) Note that

$$\begin{aligned}
1 &= x \ominus x && \text{by (a3)} \\
&= x \ominus (z \ominus (z \ominus x)) && \text{by (ii)} \\
&= x \ominus (z \ominus ((z \ominus x) \ominus 1)) && \text{since } \mathcal{L}(\mathscr{G}) = G \\
&= x \ominus ((z \ominus x) \ominus (z \ominus 1)) && \text{by (a1)} \\
&= x \ominus ((z \ominus x) \ominus z) && \text{since } \mathcal{L}(\mathscr{G}) = G \\
&\preceq x \ominus ((y \ominus (z \ominus x)) \ominus (y \ominus z)) && \text{by (a1), (a2) and (b4)} \\
&= x \ominus ((z \ominus (y \ominus x)) \ominus (y \ominus z)) && \text{by (a1)} \\
&= (z \ominus (y \ominus x)) \ominus (x \ominus (y \ominus z)). && \text{by (a1)}
\end{aligned}$$

Hence $z \ominus (y \ominus x) \preceq x \ominus (y \ominus z)$. Similarly, we have $x \ominus (y \ominus z) \preceq z \ominus (y \ominus x)$ by symmetry. It follows from (a4) that $x \ominus (y \ominus z) = z \ominus (y \ominus x)$.

(iii) $\Rightarrow$ (v) Suppose that (iii) holds. Then

$$\begin{aligned}(x \ominus y) \ominus (a \ominus b) &= b \ominus (a \ominus (x \ominus y)) = b \ominus (y \ominus (x \ominus a)) \\ &= (x \ominus a) \ominus (y \ominus b),\end{aligned}$$

and so $\mathscr{G}$ is medial.

(iv) $\Leftrightarrow$ (v) $\Leftrightarrow$ (vi) See Theorem 3.10.

(iv) $\Rightarrow$ (1) Assume that $\mathscr{G}$ is simulative. Then $(G, \cdot, 1)$ is a commutative group (see Theorem 3.7). Using the condition $\mathcal{L}(\mathscr{G}) = G$, we have

$$x \ominus y = (x \ominus 1) \ominus y = y \cdot x = x \cdot y = (y \ominus 1) \ominus x = y \ominus x,$$

and hence $x \ominus (y \ominus z) = x \ominus (z \ominus y) = z \ominus (x \ominus y) = (x \ominus y) \ominus z$. Therefore $\mathscr{G}$ is associative. This completes the proof. □

Let $\mathscr{G}$ be a WFI-algebra. For nonnegative integers i and j, we define a polynomial $P_{i,j}(x, y)$ of two variables x and y in G as follows:

$$P_{0,0}(x, y) = (y \ominus x) \ominus x,$$

$$P_{i+1,j}(x, y) = (y \ominus x) \ominus P_{i,j}(x, y),$$

$$P_{i,j+1}(x, y) = (x \ominus y) \ominus P_{i,j}(x, y).$$

Definition 3.14. Let i, j, m and n be nonnegative integers. A WFI-algebra $\mathscr{G}$ is said to be $(i, j; m, n)$*-mutant* if $P_{i,j}(x, y) = P_{m,n}(y, x)$ for all $x, y \in G$.

In particular, if $i = j = m = n = 0$, then we say that $\mathscr{G}$ is *mutant*, that is, a $(0, 0; 0, 0)$-mutant WFI-algebra is called a *mutant WFI.*

Example 3.15. The WFI-algebra $\mathscr{G}$ in Example 3.3(2) is a $(1, 0; 0, 0)$-mutant WFI-algebra, but the WFI-algebra $\mathfrak{Z} := (\mathbb{Z}; \ominus, 0)$ in Example 3.3(1) may not be $(1, 0; 0, 0)$-mutant because $P_{1,0}(2, 3) = 4 \neq 2 = P_{0,0}(3, 2)$.

Theorem 3.16. *Every simulative WFI-algebra $\mathscr{G}$ is $(0, n+1; n, 0)$-mutant for every nonnegative integer n.*

Proof. The proof is by induction on n. Let $x, y \in G$. If $n = 0$, then

$$\begin{aligned} P_{0,1}(x,y) &= (x \ominus y) \ominus P_{0,0}(x,y) \\ &= (x \ominus y) \ominus ((y \ominus x) \ominus x) \\ &= (x \ominus y) \ominus y \\ &= P_{0,0}(y,x). \end{aligned}$$

Assume that the result is valid for $n = k$, i.e., $P_{0,k+1}(x,y) = P_{k,0}(y,x)$. Then

$$\begin{aligned} P_{0,k+2}(x,y) &= (x \ominus y) \ominus P_{0,k+1}(x,y) \\ &= (x \ominus y) \ominus P_{k,0}(y,x) \\ &= P_{k+1,0}(y,x). \end{aligned}$$

Hence $P_{0,n+1}(x,y) = P_{n,0}(y,x)$, that is, $\mathscr{G}$ is $(0, n+1; n, 0)$-mutant. □

The second part of Example 3.15 shows that a simulative WFI-algebra $\mathscr{G}$ may not be $(1,0;0,0)$-mutant.

Theorem 3.17. *Every simulative WFI-algebra is $(\alpha, m; m, \alpha+1)$-mutant for all nonnegative integer m, where $\alpha = m + i$ for $i \geq 0$.*

Proof. Let $\mathscr{G}$ be a simulative WFI-algebra. We first show that for every nonnegative integer n,

(i) $P_{n,n}(x,y) = y$,
(ii) $P_{n,n+1}(x,y) = x$,
(iii) $P_{n+1,n}(x,y) = (y \ominus x) \ominus y$,
(iv) If we use the notation $x^k \ominus y$ instead of

$$\overbrace{x \ominus (\cdots \ominus (x \ominus (x}^{k\text{-times}} \ominus y)) \cdots),$$

then $P_{n+i,n}(x,y) = (y \ominus x)^i \ominus y = P_{n,n+i+1}(y,x)$ for $i \geq 0$.

By induction on n, we have

$$P_{0,0}(x,y) = (y \ominus x) \ominus x = y$$

by Theorem 3.10. Suppose that $P_{k,k}(x,y) = y$ for $n = k > 0$. Then

$$\begin{aligned} P_{k+1,k+1}(x,y) &= (y \ominus x) \ominus P_{k,k+1}(x,y) \\ &= (y \ominus x) \ominus ((x \ominus y) \ominus P_{k,k}(x,y)) \\ &= (y \ominus x) \ominus ((x \ominus y) \ominus y) \\ &= (y \ominus x) \ominus x = y, \end{aligned}$$

which proves (i). Note that

$$\begin{aligned} P_{0,1}(x,y) &= (x \ominus y) \ominus P_{0,0}(x,y) \\ &= (x \ominus y) \ominus ((y \ominus x) \ominus x) \\ &= (x \ominus y) \ominus y = x. \end{aligned}$$

Suppose that (ii) is valid for $n = k > 0$. Then

$$\begin{aligned} P_{k+1,k+2}(x,y) &= (x \ominus y) \ominus P_{k+1,k+1}(x,y) \\ &= (x \ominus y) \ominus ((y \ominus x) \ominus P_{k,k+1}(x,y)) \\ &= (x \ominus y) \ominus ((y \ominus x) \ominus x) \\ &= (x \ominus y) \ominus y = x. \end{aligned}$$

Hence (ii) holds. We observe that

$$P_{1,0}(x,y) = (y \ominus x) \ominus P_{0,0}(x,y) = (y \ominus x) \ominus y.$$

If (iii) is true for $n = k > 0$, then

$$P_{k+2,k+1}(x,y) = (y \ominus x) \ominus P_{k+1,k+1}(x,y) = (y \ominus x) \ominus y$$

by (i). Therefore (iii) is valid. For a fixed $n \geq 0$, we have

$$P_{n+0,n}(x,y) = P_{n,n}(x,y) = y = (y \ominus x)^0 \ominus y.$$

Assume that $P_{n+k,n}(x,y) = (y \ominus x)^k \ominus y$ for $i = k > 0$. Then

$$\begin{aligned} P_{n+k+1,n}(x,y) &= (y \ominus x) \ominus P_{n+k,n}(x,y) \\ &= (y \ominus x) \ominus ((y \ominus x)^k \ominus y) \\ &= (y \ominus x)^{k+1} \ominus y. \end{aligned}$$

From (ii), it follows that

$$P_{n,n+1}(y,x) = y = (y \ominus x)^0 \ominus y.$$

Suppose that $P_{n,n+k+1}(y,x) = (y \ominus x)^k \ominus y$ for $i = k > 0$. Then

$$\begin{aligned} P_{n,n+k+2}(y,x) &= (y \ominus x) \ominus P_{n,n+k+1}(y,x) \\ &= (y \ominus x) \ominus ((y \ominus x)^k \ominus y) \\ &= (y \ominus x)^{k+1} \ominus y. \end{aligned}$$

This proves (iv). Finally for a fixed $m \geq 0$, let $\alpha = m + i$ where $i \geq 0$. Using (iv), we conclude that $P_{\alpha,m}(x,y) = P_{m,\alpha+1}(y,x)$. Therefore $\mathscr{G}$ is $(\alpha, m; m, \alpha+1)$-mutant. This completes the proof. □

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Young Bae Jun
Department of Mathematics Education and (RINS)
Gyeongsang National University
Chinju 660-701, Korea
E-mail: skywine@gmail.com

Seok Zun Song
Department of Mathematics
and Research Institute for Basic Sciences
Cheju National University

Cheju 690-756, Korea
E-mail: szsong@cheju.ac.kr