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A NOTE ON OPERATORS ON FINSLER MODULES

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Abstract. let E be a Finsler modules over C*-algebras $\mathcal{A}$ with norm-map ρ and $L(E)$ set of all $\mathcal{A}$-linear bonded operators on E. We show that the canonical homomorphism $\phi : L(E) \rightarrow L(E_I)$ sending each operator T to its restriction $T|E_I$ is injective if and only if I is an essential ideal in the underlying C^*-algebra $\mathcal{A}$. We also show that $T \in L(E)$ is a bounded below if and only if $||x|| = ||\rho^{'}(x)||$ is complete, where $\rho^{'}(x) = \rho(Tx)$ for all $x \in E$. Also, we give a necessary and sufficient condition for the equivalence of the norms generated by the norm map.

1. Introduction

A (left) Hilbert C^*-module over a C^*-algebra A is a left A-module E equipped with A-valued inner product $\langle .,. \rangle$ which is a A-linear in the second and conjugate linear in the first variable such that E is a Banach space with the norm $\| x \| = \| \langle x, x \rangle \|^{\frac{1}{2}}$.

Finsler modules over C^*-algebras are generalization of Hilbert C^*-modules that first investigated in [6].

Definition 1.1. *Let $\mathcal{A}_+$ be the positive cone of a C^*-algebra $\mathcal{A}$ and E is a complex linear space which is a left $\mathcal{A}$-module (and $\lambda(ax) = (\lambda a)x = a(\lambda x)$ where $\lambda \in C, a \in \mathcal{A}$ and $x \in E$). An $\mathcal{A}$-valued Finsler norm (norm map)is a map $\rho : E \rightarrow \mathcal{A}_+$ such that*

1) the map $\rho_E : x \rightarrow ||\rho(x)||$ is a norm on E, and

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2) $\rho(ax)^2 = a\rho(x)^2a^*$ *for each* $a \in \mathcal{A}$ *and* $x \in E$.

E is equipped with a $\mathcal{A}$-valued Finsler norm is called a pre-Finsler $\mathcal{A}$-module. If $(E,\ \rho_E = ||.||_E)$ is complete then E is called a Finsler $\mathcal{A}$-module.

If we use the convention $|b| = (bb^*)^{1/2}$ for $b \in \mathcal{A}$, the condition (2) is equivalent to

$$\rho(ax) = |a\rho(x)|.$$

For $\mathcal{A}$ commutative this is the same as $\rho(ax) = |a|\rho(x)$, which is the usual form this sort of axiom takes in the commutative case. But this last version is not appropriate in the noncommutative case.

A Finsler module over C^*-algebra A is said to be full if the linear span $\{\rho_A(x)^2; x \in E\}$ denoted by $\mathcal{F}(E)$ is dense in A.

In [1] was defined an inner product on a arbitrary Hilbert C^*-module by a certain operator and give a necessary and sufficient condition for the equivalence of the norms generated by the inner product. The aim of this paper is to continue this work over finsler modules. Let E be a Finsler modules over C*-algebras $\mathcal{A}$ with norm-map ρ and $L(E)$ set of all $\mathcal{A}$-linear bonded operators on E. In section 3, we show that $T \in L(E)$ is a bounded below if and only if $||x|| = ||\rho'(x)||$ is complete, where $\rho'(x) = \rho(Tx)$ for all $x \in E$. Also, we give a necessary and sufficient condition for the equivalence of the norms generated by the norm map.

Ideal submodules in Hilbert C*-modules are investigated in [2] and [6]. And also, the notion of associated (essential) ideal submodule in Finsler modules over C^*-algebras is introduced in [7]. Moreover, it was shown that if essential ideal submodule E_I is a Hilbert I-module, then E is itself a Hilbert A-module.

In section 4, we show that the canonical homomorphism $\phi : L(E) \longrightarrow L(E_I)$ sending each operator T to its restriction $T|E_I$ is injection if and only if I is an essential in the underlying C^*-algebra $\mathcal{A}$.

2. Preliminaries

Definition 2.1. *let E be a Finsler modules over C^*-algebra A, and let I be an ideal in A. The associated ideal submodule E_I is defined by*

$$E_I = [EI]^- = [\{eb : e \in E, b \in I\}]^-$$

(the closed linear span of the action of I on E).

Clearly, E_I is a closed submodule of E. It can be also regarded as a Finsler module over I.

In general, there exist closed submodules which are not ideal submodule. For instance, if a C^*-algebra A is regarded as a Hilbert A-module (with the inner product $\langle a, b\rangle = a^*b$), then ideal submodules of A are precisely ideals in A, while closed submodules of A are closed right ideals in A.

We arise some properties of ideal submodules. Following results are already known of ([2],[1]). let E be a Finsler module over C*-algebra A, and I be an ideal of A. By application of Hewitt-Cohen factorization theorem ([4], Theorem 4.1,[6], proposition 2.31) it is easy to that $E_I = EI = \{eb : e \in E, b \in I\}$. If E be a full Finsler module over A, E_I will be full over I.

Remark 2.2. let E be a Finsler module and I be an ideal of A, and E_I be associated ideal submodule. Define by $q : E \to \frac{E}{E_I}$ and $\pi : A \to \frac{A}{I}$ the quotient maps. By defintion right action of $\frac{A}{I}$ on linear space $\frac{E}{E_I}$ with $q(e)\pi(a) = q(ea)$, $\frac{E}{E_I}$ will be a $\frac{A}{I}$-module and by [5 Lemma 12], $\frac{E}{E_I}$ is a Finsler $\frac{A}{I}$-module with norm Finsler $\rho_{\frac{A}{I}}(q(e)) = \pi(\rho_A(e))$. Then $\rho_{\frac{A}{I}}(q(E)) = \pi(\rho_A(E))$, so $[\rho_{\frac{A}{I}}(q(E)] = \pi([\rho_A(E)])$.

In addition, $\frac{E}{E_I}$ is a full Finsler $\frac{A}{I}$-module if and only if E is full. This follows at once from the evident equality $[\rho_{\frac{A}{I}}(q(E)] = \pi([\rho_A(E)])$.

With similar argument of [2 p. 4], if X be a closed submodule of E, J be an ideal of A such that $\rho(E) \subseteq J$, then $\frac{E}{X}$ with module action $q(x)\pi(a) = q(xa)$ is a $\frac{A}{J}$-module iff $X = E_J$. Note that smallest of such ideals is $A-$linear hull $(\rho(E)^2)$.

3. Norm maps on a Finsler module

Definition 3.1. *let E and F be Finsler modules over C^*- algebra A. We define $L(E,F)$ to be the set of all bounded operators $T : E \to F$ such that T be A-linear map in the sense $T(ax) = aT(x)$, for all $x \in E$ and $a \in A$.*

Theorem 3.2. *Let E be a Finsler modules over C^*-algebra A with map $\rho_1 : E \to A_+$ and $||.||_1$ is the corresponding norm. Let T be a linear map in $L(E)$, we define $\rho_2(x) = \rho_1(Tx)$, then we have*
i)(E, ρ_2) is pre-Finsler A-module if and only if $kerT = 0$.
In addition if $kerT = \{0\}$, then the following statements are equivalent.
ii) Norms $||.||_1$ and $||.||_2 = ||\rho_2(x)||$ are equivalent on E.
iii) T is a bounded below.
iv) $||.||_2$ is complete.

Proof. i) Let (E, ρ_2) be a pre-Finsler A-module and $x \in kerT$. Then $||x||_2 = ||\rho_2(x)|| = ||\rho_1(Tx)|| = 0$, so that $x = 0$. Hence $kerT = \{0\}$.

Conversely, suppose $kerT = \{0\}$ and $||x||_2 = 0$. Then $||Tx|| = ||\rho_1(Tx)|| = ||\rho_2(x)|| = ||x||_2 = 0$. It follows that $Tx = 0$. Thus $x = 0$. It is straightforward to show that $||.||$ is indeed a norm and also $\rho_2(ax)^2 = \rho_1(Tax)^2 = \rho_1(aTx)^2 = a\rho_1(Tx)^2a^* = a\rho_2(x)^2a^*$.

$(ii) \to (iii)$ Let T is a bounded below, the following show that $||.||_1$ and $||.||_2$ are equivalent:

$$\begin{aligned} ||x||_2 &= ||\rho_2(x)|| = ||\rho_1(Tx)|| = ||Tx||_1 \leq ||T||||x||_1. \\ ||x||_2 &= ||\rho_2(x)|| = ||\rho_1(Tx)|| = ||Tx||_1 \geq \alpha||x||_1, \end{aligned}$$

for some α.

$(iii) \to (ii)$ suppose two norms are equivalent then there exists a real number α such that $||x||_1 \leq \alpha||x||_2$, for each $x \in E$, $\alpha^{-1}||x||_1 \leq ||x||_2 = ||\rho_2(x)|| = ||\rho_1(Tx)|| = ||Tx||_1$. So that $||Tx||_1 \geq \alpha^{-1}||x||_1$.

$(iv) \to (iii)$ Since $(E, ||.||_1)$ and $(E, ||.||_2)$ are assumed to be Banach spaces and also $||x||_2 = ||\rho_2(x)|| = ||\rho_1(Tx)|| = ||Tx||_1 \leq ||T||||x||_1$, by the open mapping theorem, there exists a real number α such that $||.||_1 \leq \alpha||.||_2$, hence the normes are equivalent.

$(iii) \to (iv)$ It is obvious. □

Example. Let A be a unital C^*-algebra and E be a Finsler A-module with map ρ. Define $T_x : E \to E$ with $T_x(y) = \rho(x)y$ for some $x \in E$. It is clear that T_x is an A-linear map and by

$$||T_x(y)||_1 = ||\rho(x)y||_1 \leq ||\rho(x)||||y||_1 = ||x||||y||_1.$$

Therefore, $||T_x|| \leq ||x||_1$. That is, T_x is a bounded operator on E.

Now suppose that for some $x \in E$, $\rho(x)$ is invertible then T_x is onto operator on E and $T_x^{-1}y = \rho^{-1}(x)y$. Define $\rho^{'}(y) = \rho(T_x y)$. Since $kerT_x = 0$, E with map $\rho^{'}$ is a Finsler A-module and corresponding norms of ρ and $\rho^{'}$ are equivalent.□

4. Ideal submodule of a Finsler module

Definition 3.3. Let I be an ideal of C^*-algebra A, define $I^\perp = \{a \in A : aI = 0\}$ (that is ideal of A). I is essential if $I^\perp = \{0\}$, that is equivalent $I \bigcap J \neq \{0\}$ for all closed ideal J of A.

Theorem 3.4. let E be a Finsler module over C*-algebra A, I be an ideal of A and E_I be associated ideal submodule of it. Then linear operator $\phi : L(E) \to L(E_I)$ with $\phi(K) = K|_{E_I}$ for all $K \in L(E)$ is injective if I is a essential ideal of A.
Conversely, if ϕ is injective and E be a full Finsler A-module then I is essential ideal of A.

Proof. Suppose that I be an essential ideal of A and $\gamma(K) = 0$ for $K \in L(E)$. Then $K(bx) = 0$ for all $b \in I$ and $x \in E$. Now by [2, Lemma 1.10], we have

$$\begin{aligned} \| Kx \|^2 &= \| \rho(Kx) \|^2 = sup_{b \in I, \|b\| \leq 1} \| b(\rho(Kx)^2)b^* \| \\ &= sup_{b \in I, \|b\| \leq 1} \| (bKx)^2 \| = sup_{b \in I, \|b\| \leq 1} \| K(bx) \|^2 \\ &= 0 \end{aligned}$$

Then ϕ is injective.
Conversely, let E be full and γ is injective, but I is not essential means $I^\perp \neq 0$. Then there exists $0 \neq c \in I^\perp$ and either fullness condition of E by [1 proof Theorem 3.2(iii)], show that $\exists x_0 \in E$ such that $cx_0 \neq 0$. First suppose that A a commutative C^*-algebra. Hence A-linear operator $K_c : E \to E$ with $K_c(x) = cx$ is non zero in value x_0 and $K_c|_{E_I} = 0$, that is contradiction with injectivity of ϕ. Then I is essential.

To prove the general case, Recall that if A, B, and D are C^*-algebra , and if homomorphisms $\varphi : A \to D$ and $\psi : B \to D$ are given , then the

C^*-algebra $A \oplus_D B$ is defined as

$$A \oplus_D B = \{(a, b) \in A \oplus B \ : \varphi(a) = \psi(b)\}.$$

We use the same notation for modules, Banach spaces, etc.

Let A be a C^*-algebra. By [5 lemmas 10 and 11] A has a unique maximal commutative ideal I_0 and a closed ideal J such that $I_0 \bigcap J = \{0\}$ and $\frac{A}{J}$ is commutative, moreover, $A \cong \frac{A}{J} \oplus_{\frac{A}{I_0+J}} \frac{A}{I_0}$ by *-isomorphism $\varphi : A \to \frac{A}{J} \oplus_{\frac{A}{I_0+J}} \frac{A}{I_0}$ such that $\varphi(a) = (a + J, a + I_0)$.

By theorem 17 of [5], we can write $E \cong E_1 \oplus_{E_0} E_2$, where E_2 and E_0 are Hilbert $\frac{A}{I_0}$ and $\frac{A}{I_0+J}$ modules resp., and E_1 is a Finsler module over commutative C*-algebra $\frac{A}{I}$. Clearly, E_1 and E_2 are full if and only if E is full.

If I be an essential ideal in C*-algebra A, then by [7, Lemma 3.5] $\frac{I}{J}$ and $\frac{I}{I_0}$ are essential ideal in commutative C^*-algebras $\frac{A}{J}$ and $\frac{A}{I_0}$ respectively. Let $E_{11} = E_{1\frac{I}{I_0}}$ and $E_{22\frac{I}{J}}$ are associated ideal submodule E_1 and E_2 respectively. Then $E_I = E_{11} \oplus E_{22}$.

$I^{\perp} \neq 0$ implies that $I_1^{\perp} \neq 0$ or $I_2^{\perp} \neq 0$. Suppose that $I_1^{\perp} \neq 0$ since $\frac{A}{J}$ is a commutative C^*-algebra there exists a non Zero $\mathcal{A}$-linear map $K_1 : E_1 \longrightarrow E$ such that $K|_{E_{11}} = 0$. We can extend K on E by $K|_{E_2} = 0$. This is impossible because ϕ is injective. If $I_2^{\perp} \neq 0$ by assertion follows of [2, Theorem 1.12] because E_2 is a Hilbert $\frac{A}{J}$-module.□

Let E be a Finsler module over C*-algebra A, We say that norm Finsler map ρ' is induced from norm Finsler map ρ, if there exists $K \in L(E)$ such that $\rho'(x) = \rho(K(x))$ for all $x \in E$.

Corollary 3.5. let E be a Finsler module over C*-algebra A, I be an ideal of A and E_I be associated ideal submodule of it. Induced norm

Finsler maps of a norm Finsler map ρ are equal iff are equal over essential ideal submodule E_I of E. □

Proof. It is enough to show that $K_1 = K_2$ iff $K_1|_{E_I} = K_2|_{E_I}$. Since I is a essential ideal of A, previous assertion is straightforward of Lemma 2.6 .

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