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INTEGER MATRICES WITH PRESCRIBED PERMANENT AND ITS APPLICATIONS

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Abstract. In this paper, we construct a procedure of Maple programming for $(0,1)$-matrix with a prescribed permanent, $1, 2, \ldots, 2^{n-1}$. An application of such construction is given, and we obtain the some results of $(0,1)$-matrices with the permanent less than or equal to $n!$ by replacing elements 0's by 1's.

1. Introduction and preliminaries

Let $A = [a_{ij}]$ be an m-by-n matrix with $m \leq n$. The *permanent* of A is defined by

$$\text{per}A = \sum a_{1i_1} a_{2i_2} \cdots a_{mi_m},$$

where the summation extends over all the m-permutations $(i_1, i_2, \cdots, i_m)$ of the integers $1, 2, \cdots, n$. Thus per$A$ is the sum of all possible products of m elements of the m-by-n matrix A with the property that the elements in each of the products lie on different lines of A. This scalar valued function of the matrix A occurs throughout the combinatorial literature in connection with various enumeration and extremal problems.

If A is a $(0,1)$-matrix of order n, then clearly $0 \leq \text{per}A \leq n!$. In [1], it has known that there is a $(0,1)$-matrix A such that per$A = k$ for each $k = 0, 1, \ldots, 2^{n-1}$, and there is no $(0,1)$-matrix of order n such that per$A = k$ for some k in $2^{n-1} < k \leq n!$. In [3], Kim et. al

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studied a construction method for a $(0,1)$-matrix with per$A = k$ for each $k = 0, 1, \ldots, 2^{n-1}$ by using the binary number system of k. In this note, we obtain a Maple procedure of this construction. By the same construction method we explore the $(0,1)$-matrix A with perA $= k$ such that $2^{n-1} < k \leq n!$.

First note that the n-by-n $(0,1)$-matrix

$$(1)\qquad B_n = [b_{ij}] = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdot & \cdot & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & \cdot & \cdot & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & \cdot & \cdot & 0 & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & 1 & \cdot & \cdot & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & \cdot & \cdot & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & \cdot & \cdot & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

is row equivalent to the lower Hessenberg matrix H_n. In Maple program, the command

"Array(Hessenberg[lower],1..n,1..n,(i,j) $\rightarrow$ 1);"

produces the following lower Hessenberg matrix H_n.

$$(2)\qquad H_n = [b_{ij}] = \begin{bmatrix} 1 & 1 & 0 & 0 & \cdot & \cdot & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & \cdot & \cdot & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & \cdot & \cdot & 0 & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & 1 & \cdot & \cdot & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & \cdot & \cdot & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & \cdot & \cdot & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

It is well known (for example see [3]) that

$$\text{per}H_n = 2^{n-1}.$$

Throughout the present paper, we define

$$H_n(n,k) = H_n - E_{n,k}, \tag{3}$$

where $E_{n,k}$ is a cell which is a $(0,1)$-matrix with only one entry equal to 1 at the (n,k) entry. And let $H_n(n|k)$ be the submatrix of order $n-1$ obtained from H_n by deleting the n-th row and k-th column. It immediately follows from [3] that $\mathrm{per}H_n(n,k) = 2^{n-1} - 2^{k-2}$, $(k = 2,3,\cdots,n)$ and $\mathrm{per}H_n(n,1) = 2^{n-1} - 1$.

Let ℓ be a positive integer such that $0 < \ell \leq 2^{n-1}$. Then there exists the least number m such that $2^{m-2} < \ell \leq 2^{m-1}$ where $2 \leq m \leq n$. Now let us introduce the construction method of the $(0,1)$-matrix A for which $\mathrm{per}A = \ell$ where $2^{m-2} < \ell \leq 2^{m-1}$. First let t be a positive integer with $2^{n-2} \leq t \leq 2^{n-1}$ and $t_{(2)} = x_{n-2}x_{n-3}\cdots x_1x_0$ be the binary representation of t, where the x_i is 0 or 1. Define $x_k^c = 1$ if $x_k = 0$ and $x_k^c = 0$ if $x_k = 1$. Let $H_m(t)$ be obtained from H_m by replacing $b_{n,k}$ of H_m to x_{k-2}^c, for all $k = 2,3,\cdots,n$. The following theorem shows that for a given positive integer ℓ, we can construct an $n \times n$ $(0,1)$-matrix with permanent ℓ $(\ell = 1,2,\ldots,2^{n-1})$.

Theorem 1.1. [3] *Let ℓ be a positive integer with $0 < \ell < 2^{n-1}$. Then $\mathrm{per}H_m(t) = \ell$ where $t = 2^{m-1} - l$ such that m is the least number with $2^{m-2} < \ell \leq 2^{m-1}$.*

2. A Maple procedure for the construction of the $(0,1)$-matrix A with $\mathrm{per}A = \ell$

The Maple program gives useful helps for us solving many computational problems. In this section, first we give the algorithm for the construction of the $(0,1)$-matrix A with $\mathrm{per}A = \ell$.

ALGORITHM

Step 1. Find the least number n with $\ell \leq 2^{n-1}$.

Step 2. Take $t = 2^{n-1} - \ell$.

Step 3. Convert t to binary representation $b_{n-2}b_{n-3}\ldots b_0$.

Step 4. Compute the complement $b_{n-2}^c b_{n-3}^c \cdots b_0^c$.

Step 5. Replacing $b_{n,k}$ to $b_{k-2}^c, k = 2, 3, \ldots, n$ of H_n.

Now we consider a Maple procedure for the construction of the $(0,1)$-matrix A with per$A = \ell$, as following:

MAPLE PROCEDURE

```
restart:with(LinearAlgebra):L:=n:
L2:=convert(L,binary):
N:=length(L2):
T:=2^N-L:
TB:=convert(T,binary):
k:=length(TB):
T2:=convert(T,base,2):
T2V:=Vector[row]([T2,0$(N-k)]):
T2VC:=Vector[row]([1$N]):
T2V1:=VectorScalarMultiply(T2V,-1):
T2VA1:=VectorAdd(T2VC,T2V1):
ADDT:=Vector[row]([1,T2VA1]):
B_N:=Array(Hessenberg[lower],1..N+1,1..N+1,(a,b)->1):
C:=convert(B_N, Matrix):
Permanent(C):
delB:=DeleteRow(C,N+1..N+1):
B(L):=Matrix(N+1,N+1,[[delB],[ADDT]]);Permanent(B(L));
```

For example, let $n = 407$. By the Maple procedure we obtain the $(0,1)$-matrix with permanent 407 as following:

$$(4) \qquad H_n(407) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

By the row exchange, $H_n(407)$ is permutation equivalent to $B_n(407)$ in [3].

$$(5) \qquad B_n(407) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

3. Behavior of perB_n

In this section, we obtain the permanent of the $(0,1)$-matrices obtained from B_n of matrices by replacing 0 entries in B_n by 1. We recall that there is no $(0,1)$-matrix A such that per$A = k$ for some k in

$2^{n-1} < k \leq n!$. Let $B_n^{(k)}$ be the set of all n-by-n $(0,1)$-matrix from B_n by replacing k 0's by 1's where $1 \leq k \leq \frac{(n-1)(n-2)}{2}$. Then clearly

$$2^{n-1} = \text{per}A_n < \text{per}{A_n}^{(1)} < \text{per}{A_n}^{(2)} < \cdots < \text{per}{A_n}^{(\frac{(n-1)(n-2)}{2})} = n!$$

where $A_n^{(k)}$ is a matrix in $B_n^{(k)}$. We note that for a fixed k, per$A_n^{(k)}$ depends on the positions of k 1's. Hence per$A_n^{(k)}$ can be have different values.

Lemma 3.1. [3]*Let B_n be the same matrix as in* [3] *and $E_{i,j}$ a cell with appropriate size.*

(1) $\text{per}(B_n + E_{2,k}) = 2^{n-k}\text{per}(B_k + E_{2,k}), (k = 3, 4, ..., n-1)$
(2) $\text{per}(B_n + E_{2,n}) = 2\text{per}B_{n-1} + \text{per}(B_{n-1} + E_{2,3} + E_{3,4} + ... + E_{n-2,n-1})$
(3) $\text{per}(B_n + E_{3,k}) = 2^{n-k+1}\text{per}(B_{k-1} + E_{2,k-1}),\ (k = 4, 5, ..., n-1)$
(4) $\text{per}(B_n + E_{k,\ell}) = 2^{k-2}\text{per}(B_{n-k+2} + E_{2,\ell-k+2}), (k = 4, 5, \ldots, n - 1, \ell > k)$
(5) $\text{per}(B_n + E_{2,3} + E_{3,4} + ... + E_{n-1,n}) = 2 \times 3^{n-2}$

Now we obtain the different permanent values for the matrices in $B_n^{(k)}$.

Theorem 3.2. *The permanent value of a matrix in $B_n^{(1)}$ exactly has:*

$$\text{per}(B_n + E_{k,\ell}) = 2^{n-1}(1 + \frac{1}{3}(\frac{3}{2})^{\ell-k}), 2 \leq k < \ell.$$

Proof. (Case 1) $B_n + E_{2,k}$, for $k = 3, 4, \ldots, n$.
By applying (2),(5) we obtain

$$\begin{aligned}
\text{per}(B_n + E_{2,k}) &= 2^{n-k}\text{per}(B_k + E_{2,k}) \\
&= 2^{n-k}(2 \times 2^{k-2} + 2 \times 3^{k-3}) \\
&= 2^{n-1}\{(1 + \frac{1}{3}(\frac{3}{2})^{k-2}\}, (k = 3, 4, \ldots, n-1).
\end{aligned}$$

If $k=n$, by (2) in lemma 3.1 we obtain

$$\mathrm{per}(B_n+E_{2,n})=2^{n-1}\{(1+\frac{1}{3}(\frac{3}{2})^{n-2}\}.$$

Consequently, we have

$$\mathrm{per}(B_n+E_{2,k})=2^{n-1}\{(1+\frac{1}{3}(\frac{3}{2})^{k-2}\}$$

for all $k=3,4,\ldots,n$.

(Case 2) $B_n+E_{3,k}$, for $k=4,5,\ldots,n$.

$$\begin{aligned}\mathrm{per}(B_n+E_{3,k}) &= 2^{n-k+1}\mathrm{per}(B_{k-1}+E_{2,k-1})\\ &= 2^{n-k+1}\times(2^{k-2}+2\times 3^{k-4})\\ &= 2^{n-1}+2^{n-k+2}\times 3^{k-4}\\ &= 2^{n-1}\{(1+\frac{1}{3}(\frac{3}{2})^{k-3}\}.\end{aligned}$$

(Case 3) $B_n+E_{k,\ell}$, for $k=4,5,\ldots,n-1$. From (1) in Lemma 3.1, $\mathrm{per}(B_{n-k+2}+E_{2,\ell-k+2})=2^{n-\ell}\mathrm{per}(B_{\ell-k+2}+E_{2,\ell-k+2})$. So from (4) in Lemma 3.1

$$\begin{aligned}\mathrm{per}(B_n+E_{k,\ell}) &= 2^{k-2}\mathrm{per}(B_{n-k+2}+E_{2,\ell-k+2})\\ &= 2^{n-1}+2^{n-\ell+k-1}\times 3^{\ell-k-1}.\end{aligned}$$

So we obtain

$$\mathrm{per}(B_n+E_{k,\ell})=2^{n-1}(1+\frac{1}{3}(\frac{3}{2})^{\ell-k}),2\le k<\ell.$$

So we have the possible values. The proof is complete. □

Theorem 3.3. *Let* $A_n^{(2)} \in B_n^{(2)}$*. Then the possible values of* $\mathrm{per}A_n^{(2)}$ *are*

$$\begin{cases} \mathrm{per}(B_n + E_{2,3} + E_{2,k}) = 3 \times 2^{n-k-2} + 2^{n-k+1} \times 3^{k-3}, \\ \hfill (k = 3, 4, \ldots, n-1) \\ \mathrm{per}(B_n + E_{2,3} + E_{k,\ell}) = 3 \times [2^{n-k} + 2^{n-k-1} \times 3^{\ell-k-3}], \\ \hfill (k = 3, 4, \ldots, n-1, k < \ell) \\ \mathrm{per}(B_n + E_{2,4} + E_{k,\ell}) = 7 \times [2^{n-k} + 2^{n-k-1} \times 3^{\ell-k-5}], \\ \hfill (k = 3, 4, \ldots, n-1, k < \ell) \\ \mathrm{per}(B_n + E_{t,m} + E_{\ell,k}) = 2^{n-2} + 9 \times 2^{n-k} + 3^{\ell-k-1} \times \\ \hfill (2^{n-k-1} + 2^{n-\ell+k-2}), (t < \ell, m < k). \end{cases}$$

Proof. (1) By the Laplace expansion of $B_n + E_{2,3} + E_{2,k}$ on row 3, we obtain the

$$\begin{aligned} \mathrm{per}(B_n + E_{2,3} + E_{2,k}) &= 3\mathrm{per}(B_{n-1} + E_{2,k-1}) \\ &= 3[2^{n-k}\mathrm{per}(B_{k-1} + E_{2,k-1})] \\ &= 3 \times 2^{n-k}[2\mathrm{per}B_{k-2}\mathrm{per}(B_{k-2} + E_{2,3} \\ &+ E_{3,4} + \ldots + E_{k-3,k-2})] \\ &= 3 \times 2^{n-k-2} + 2^{n-k+1} \times 3^{k-3}. \end{aligned}$$

(2) By the Laplace expansion on the second row of $B_n + E_{2,3} + E_{\ell,k}$,

$$\mathrm{per}(B_n + E_{2,3} + E_{k,\ell}) = 3 \times \mathrm{per}(B_{n-1} + E_{k-1,\ell-1}).$$

Applying (4) in 3.1, we obtain the

$$\mathrm{per}(B_{n-1} + E_{k-1,\ell-1}) = 2^{k-3}\mathrm{per}(B_{n-k+1} + E_{2,l-k+1}).$$

So

$$\mathrm{per}(B_n + E_{2,3} + E_{k,\ell}) = 3 \times [2^{n-k} + 2^{n-k-1} \times 3^{\ell-k-3}].$$

(3) By the Laplace expansion on the second row of $B_n + E_{2,4} + E_{k,\ell}$, we obtain

$$\begin{aligned} \mathrm{per}(B_n + E_{2,4} + E_{k,\ell}) &= 2 \times \mathrm{per}(B_{n-1} + E_{k-1,\ell-1}) \\ &+ \mathrm{per}(B_{n-1} + E_{2,3} + E_{k-1,\ell-1}) \\ &= 2 \times (2^{n-k+1} + 2^{n-k} \times 3^{\ell-k-1}) \\ &+ 3 \times (2^{n-k} + 2^{n-k-1} \times 3^{\ell-k-3}) \\ &= 7 \times (2^{n-k} + 2^{n-k-1} \times 3^{\ell-k-2}). \end{aligned}$$

(4) By the Laplace expansion on the second row of $B_n + E_{t,m} + E_{k,\ell}$, we obtain

$$\begin{aligned} \mathrm{per}(B_n + E_{t,m} + E_{k,\ell}) &= 2^{t-2} \times \mathrm{per}(B_{n-t+2} + E_{2,m-t+2} + E_{k-t+2,\ell-t+2}) \\ &= 2^{t-2} \times [2^{k-t-1} \times (2^{n-k+1} + 2^{n-\ell+1} \times 3^{\ell-k-1}) \\ &+ 3^2 \times (2^{n-k} + 2^{n-k-1} \times 3^{\ell-k-3})] \\ &= 2^{n-2} + 9 \times 2^{n-k} \\ &+ 3^{\ell-k-1} \times (2^{n-k-1} + 2^{n-\ell+k-2}). \end{aligned}$$

4. Application

Let $B = \{0, 1\}$ be the Boolean algebra and $B^n = \{(b_1, b_2, \ldots, b_n) | b_i \in B, i = 1, 2, \ldots, n\}$ the product algebra of B. Then B^n has useful expression for the many situations of computer science. Sometimes the elements of B^n are called by *n-bit strings*. If we have a security number with n-digits, then we can have the corresponding number encoding by n square matrices of order 5.

Example 4.1. Let $\ell = 4679$ be a security number with 4-digits. Since every number in each digits is less than or equal to 9. So we consider the matrix B_5 of order 5:

$$B_5 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \mathrm{per} B_5 = 16.$$

Take $t_1 = 4$. Then $16 - t_1 = 12_{(2)} = 1100$. Similarly take $t_2 = 6$ with $10_{(2)} = 1010$, $t_3 = 7$ with $9_{(2)} = 1001$ and $t_4 = 9$ with $7_{(2)} = 0111$. Then we obtain

$$B_5(12) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}, \mathrm{per} B_5(12) = 16 - 12 = 4.$$

$$B_5(10) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}, \mathrm{per} B_5(10) = 16 - 10 = 6.$$

$$B_5(9) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}, \mathrm{per} B_5(2) = 16 - 9 = 7.$$

$$B_5(7) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 0 & 0 \\ & 1 & 1 & 0 & 0 \\ & 1 & 1 & 1 & 0 \\ & 0 & 0 & 0 & 1 \end{bmatrix}, \mathrm{per} B_5(7) = 16 - 7 = 9.$$

So we can get the matrix code corresponding to the security number $\ell = 4679$ as followings:

$$4679 \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

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