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ON THE HIGHER ORDER KOBAYASHI METRICS[‡]

Jong Jin Kim[†], Jeong Kyun Kim and Jeong Seung Lee

Abstract. The purpose of this note is to prove some properties related to the higher order Kobayashi metrics(resp. pseudodistances) as the counterpart for the usual Kobayashi metrics(resp. pseudodistances).

1. Introduction

Kobayashi([5]) initiated studying his pseudodistance and Royden published the infinitesimal form in [7] as a modification of the Carathéodory metric which has a number of advantages. The infinitesimal form that is called as the Kobayashi metric has been developed by many mathematicans. The higher order Kobayashi metric is introduced in [9] by Yu as the generalization of the Kobayashi metric. Nikolov([6]) also investigated the higher order Kobayashi metric.

We first introduce some notations which are used in the sequel. By $\mathbb{N}$ and $\mathbb{C}$ we denote the set of natural numbers and the set of complex numbers, respectively. Also, by F_Ω^c and K_Ω we denote the Carathéodory metric and the usual Kobayashi metric for some domain Ω, respectively. Moreover, by a domain we mean the open and connected set. We will also use the notations $<,>$ and $||\cdot||$ for the usual inner product and norm on complex Euclidean spaces, respectively.

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2. The higher order Kobayashi metrics

Let $D \subset \mathbb{C}^n$ be a domain and denote by $\mathcal{O}(\Delta, D)$ the space of all holomorphic mappings from the unit disk $\Delta \subset \mathbb{C}$ into D. For $t \in D$, we mean by $\mathcal{O}_t(\Delta, D)$ the set $\{\varphi \in \mathcal{O}(\Delta, D) \mid \varphi(0) = t\}$.

For each $m \in \mathbb{N}$ and $(z, X) \in D \times \mathbb{C}^n$, the *m-th order Kobayashi metric* is defined by

$$K_D^m(z,X) := \inf\{|\alpha|^{-1} \mid \exists \psi \in \mathcal{O}_z(\Delta, D) \text{ s.t. } \nu(\psi) \geq m, \psi^{(m)}(0) = m!\alpha X\}$$

where $\nu(\psi)$ stands for the order of vanishing of $\psi - \psi(0)$ at 0. Clearly $K_D^1(z, X)$ is the usual Kobayashi metric.

Proposition 2.1. ([4][9]) *Let $D \subset \mathbb{C}^n$ be a domain. Then for each $m \geq 1$, the following hold;*

(1) *K_D^m has the length decreasing property. In particular, K_D^m is biholomorphically invariant.*

(2) *$K_\Delta^m \equiv K_\Delta^1$, the usual Kobayashi metric for the unit disc Δ.*

(3) *$F_D^c(z,X) \leq K_D^m(z,X) \leq K_D^1(z,X)$ for all $(z,X) \in D \times \mathbb{C}^n$.*

(4) *$K_D^m(z, \mu X) = |\mu| K_D^m(z,X)$ for all $(z,X) \in D \times \mathbb{C}^n$ and $\mu \in \mathbb{C}$.*

A set $A \subset \mathbb{C}^k$ is called a *balanced set* if $\lambda z \in A$ for arbitrary $\lambda \in \bar{\Delta}$ and $z \in A$.

Theorem 2.2. *Let $G \subset \mathbb{C}^n$ be a balanced pseudoconvex domain given by $G := \{z \in \mathbb{C}^n \mid h(z) < 1\}$ with Minkowski function h, i.e., $h : \mathbb{C}^n \longrightarrow [0, \infty)$ is a plurisubharmonic function*[1] *for which $h(\lambda z) = |\lambda| h(z)$ for all $\lambda \in \mathbb{C}$ and $z \in \mathbb{C}^n$. Then we have $K_G^m(0, X) = h(X)$ for all $X \in \mathbb{C}^n$.*

Proof To show that $K_G^m(0,X) \leq h(X)$, let us assume that $h(X) \neq 0$. If we define a map $\phi : \Delta \longrightarrow G$ by $\phi(\lambda) = \lambda^m X / h(X)$, then we have

$$\phi \in \mathcal{O}_0(\Delta, G), \nu(\phi) \geq m \text{ and } \phi^{(m)}(0) = m! \frac{X}{h(X)}.$$

[1]Refer [2][3] for plurisubharmonic functions and more their informations

Now let us consider the case $h(X) = 0$. For any $t > 1$, if we define a map $\phi_t : \Delta \longrightarrow G$ by $\phi_t(\lambda) = t\lambda^m X$, then we know that

$$\phi_t \in \mathcal{O}_0(\Delta, G), \nu(\phi_t) \geq m \text{ and } \phi_t^{(m)}(0) = m!tX.$$

It follows from this fact that

$$K_G^m(0, X) \leq \frac{1}{t} \longrightarrow 0 \text{ as } t \longrightarrow \infty.$$

Thus in either cases, the inequality $K_G^m(0, X) \leq h(X)$ holds.

Conversely, let $\phi \in \mathcal{O}_0(\Delta, G)$ for which

$$\nu(\phi) \geq m \text{ and } \phi^{(m)}(0)\alpha = m!X \ (\alpha > 0).$$

If we define a map $\tilde{\phi} : \Delta \longrightarrow \mathbb{C}^n$ by

$$\tilde{\phi}(\lambda) := \begin{cases} \frac{\phi(\lambda)}{\lambda^m} & \text{if } \lambda \neq 0 \\ \frac{\phi^{(m)}(0)}{m!} & \text{if } \lambda = 0 \end{cases}$$

then we have $\tilde{\phi} \in \mathcal{O}(\Delta, \mathbb{C}^n)$ and $\phi(\lambda) = \lambda^m \tilde{\phi}(\lambda)$ for all $\lambda \in \Delta$. On the other hand, since $1 > h(\phi(\lambda)) = |\lambda|^m h(\tilde{\phi}(\lambda))$ for all $\lambda \in \Delta$ and $h \circ \tilde{\phi}$ is a subharmonic function on Δ, it follows from the maximum principle for subharmonic function that $h \circ \tilde{\phi} \leq 1$ on Δ. Hence

$$m!\frac{1}{\alpha}h(X) = h(\phi^{(m)}(0)) = h(m!\tilde{\phi}(0)) = m!(h \circ \tilde{\phi})(0) \leq m!$$

and so $h(X) \leq \alpha$. By the assumption for ϕ and α, we obtain $h(X) \leq K_G^m(0, X)$. □

Let $B \subset \mathbb{C}^n$ be an open unit ball with center 0 in $\mathbb{C}^n$. Then the Minkowski function for B is the usual Euclidean norm. Recall that B is a balanced pseudoconvex domain and K_B^m is biholomorphic invariant. Thus we have the following(cf [4]);

Corollary 2.3. *Let $B \subset \mathbb{C}^n$ be an open unit ball in $\mathbb{C}^n$ with center 0. Then we have*

$$K_B^m(z, X) = \left[\frac{||X||^2}{1 - ||z||^2} + \frac{|\langle z, X\rangle|^2}{(1 - ||z||^2)^2}\right]^{\frac{1}{2}}.$$

for all $(z, X) \in B \times \mathbb{C}^n$.

Let G and D be domains in $\mathbb{C}^n$. A holomorphic map $\pi : G \longrightarrow D$ is called a *holomorphic covering* if for any point $z \in D$ there exists an open neighborhood U of z with the property that each connected components of $\pi^{-1}(U)$ is mapped biholomorphically onto U by π.

Theorem 2.4. *Let $\tilde{G}$ and G be domains in $\mathbb{C}^n$ and let $\pi : \tilde{G} \longrightarrow G$ be a holomorphic covering map*[2]*. Then for each $(\tilde{p}, X) \in \tilde{G} \times \mathbb{C}^n$ we have the following*

$$K^m_{\tilde{G}}(\tilde{p}, X) = K^m_G(\pi(\tilde{p}), d\pi(\tilde{p})X).$$

Proof By the holomorphic contraction property (Proposition 2.1), we have

$$K^m_{\tilde{G}}(\tilde{p}, X) \geq K^m_G(\pi(\tilde{p}), d\pi(\tilde{p})X).$$

Let us now show the reverse inequality. To do this, let $\epsilon > 0$ be arbitrary and let $\phi \in \mathcal{O}_{\pi(\tilde{p})}(\Delta, G)$ for which $\nu(\phi) \geq m, \phi^{(m)}(0)\eta = m!d\pi(\tilde{p})X$ and $0 < \eta < K^m_G(\pi(\tilde{p}), d\pi(\tilde{p})X) + \epsilon$. Then there is a lifting $\tilde{\phi} \in \mathcal{O}(\Delta, \tilde{G})$ such that $\pi \circ \tilde{\phi} = \phi$ and $\tilde{\phi}(0) = \tilde{p}$. It hence suffices to show that $\nu(\tilde{\phi}) \geq m$ and $\tilde{\phi}^{(m)}(0)\eta = m!X$. If so, then by the definition of m-th order Kobayashi metric, we have $K^{(m)}_{\tilde{G}}(\tilde{p}, X) \leq \eta$. Since $0 < \eta < K^m_G(\pi(\tilde{p}), d\pi(\tilde{p})X) + \epsilon$ and ϵ was arbitrary, the following inequality holds;

$$K^m_{\tilde{G}}(\tilde{p}, X) \leq K^m_G(\pi(\tilde{p}), d\pi(\tilde{p})X),$$

which is our claim.

It follows from $\nu(\phi) \geq m$ and the differential of $\phi = \pi \circ \tilde{\phi}$ that $\nu(\tilde{\phi}) \geq m$ and $\phi^{(m)}(0) = d\pi(\tilde{p})\tilde{\phi}^{(m)}(0)$. Hence we obtain

$$m!d\pi(\tilde{p})X = \phi^{(m)}(0)\eta = d\pi(\tilde{p})\tilde{\phi}^{(m)}(0)\eta.$$

But since π is locally biholomorphic, $\tilde{\phi}^{(m)}(0)\eta = m!X$. So we have the required assertion. □

[2]Refer [1][8] for a covering map and more informations

3. The higher order Kobayashi distances

The higher order Kobayashi metric is uppersemicontinuous([4]). So it can be used to define the length of a piecewise C^1-curve and then the minimal length of all such curves connecting two fixed points will yield a new pseudodistance.

For a domain $D \subset \mathbb{C}^n$, let us define the K_D^m*-length* of a piecewise C^1-curve $\alpha : [0,1] \longrightarrow D$ by

$$L_m(\alpha) := \int_0^1 K_D^m(\alpha(t), \alpha'(t))dt.$$

Then $L_m(\alpha) \in [0, \infty)$ and so we may define a map $k_D^m : D \times D \longrightarrow \mathbb{R}$, which is called the *integrated form* of K_D^m, by

$$k_D^m(z,w) := \inf_\alpha L_m(\alpha)$$

where the infimum is taken over all piecewise C^1-curves α joining z and w.

Proposition 3.1. ([4]) *Let $D \subset \mathbb{C}^n$ be a domain. Then k_D^m is a pseudodistance on D.*

We call k_D^m the *m-th order Kobayashi pseudodistance* on D.

Let $B \subset \mathbb{C}^n$ be the open unit ball with center 0 and let $z, w \in B$. Then by Corollary 2.3,

$$\begin{aligned} k_B^m(z,w) &= \inf_\alpha \int_0^1 K_B^m(\alpha(t), \alpha'(t))dt \\ &= \inf_\alpha \int_0^1 \left[\frac{||\alpha'(t)||^2}{1 - ||\alpha(t)||^2} + \frac{|< \alpha(t), \alpha'(t) >|^2}{(1 - ||\alpha(t)||^2)^2} \right]^{\frac{1}{2}} dt, \end{aligned}$$

where the infimum is taken over all piecewise C^1-curves α joining z and w. Hence, as expected from Lempert's Theorem([3]), the following holds;

Corollary 3.2. *Let $B \subset \mathbb{C}^n$ be the open unit ball with center 0. Then we have $k_B^m(z,w) = k_B(z,w)$ for all $z, w \in B$. Here k_B stands for the usual Kobayashi distance for B.*

Proposition 2.1 and the definition of k_D^m induce the following

Proposition 3.3. ([4]) *Let $\Omega \subset \mathbb{C}^l$ and $D \subset \mathbb{C}^n$ be two domains. If $f : \Omega \longrightarrow D$ is a holomorphic map, then $k_\Omega^m(z,w) \geq k_D^m(f(z), f(w))$ for any $z, w \in \Omega$. That is, k_D^m has the distance decreasing property under holomorphic mappings.*

Theorem 3.4. *Let $\pi : \tilde{G} \longrightarrow G$ be a holomorphic covering map, and let $p, q \in G$ and $\tilde{p} \in \tilde{G}$ such that $\pi(\tilde{p}) = p$. Then the following holds;*

$$k_G^m(p,q) = \inf_{\tilde{q} \in \pi^{-1}(q)} k_{\tilde{G}}^m(\tilde{p}, \tilde{q}).$$

Proof By the holomorphic contraction property(Proposition 3.3), we have

$$k_G^m(p,q) \leq \inf_{\tilde{q} \in \pi^{-1}(q)} k_{\tilde{G}}^m(\tilde{p}, \tilde{q}).$$

Hence to show the reverse inequality, suppose that there exists an $\epsilon > 0$ such that the inequality

$$k_G^m(p,q) + 2\epsilon \leq \inf_{\tilde{q} \in \pi^{-1}(q)} k_{\tilde{G}}^m(\tilde{p}, \tilde{q})$$

holds. Then by the definition of $k_G^m(p,q)$, there is a piecewise C^1-curve $\alpha : [0,1] \longrightarrow G$ connecting p and q such that

$$\int_0^1 K_G^m(\alpha(t), \alpha'(t))dt < k_G^m(p,q) + \epsilon.$$

Since $\pi : \tilde{G} \longrightarrow G$ is a holomorphic covering, there are a $\tilde{q} \in \pi^{-1}(q)$ and a piecewise C^1-curve $\tilde{\alpha} : [0,1] \longrightarrow \tilde{G}$ connecting $\tilde{p}$ and $\tilde{q}$ such that $\pi \circ \tilde{\alpha} = \alpha$.

On the other hand, by Theorem 2.4 for m-th order Kobayashi metric, we have

$$\begin{aligned}\int_0^1 K_G^m(\alpha(t), \alpha'(t))dt &= \int_0^1 K_G^m((\pi \circ \tilde{\alpha})(t), (\pi \circ \tilde{\alpha})'(t))dt \\ &= \int_0^1 K_{\tilde{G}}^m(\tilde{\alpha}(t), \tilde{\alpha}'(t))dt.\end{aligned}$$

Hence we have

$$\begin{aligned} k_{\tilde{G}}^m(\tilde{p},\tilde{q}) &\leq \int_0^1 K_{\tilde{G}}^m(\tilde{\alpha}(t),\tilde{\alpha}'(t))dt \\ &= \int_0^1 K_G^m(\alpha(t),\alpha'(t))dt \\ &< k_G^m(p,q)+\epsilon, \end{aligned}$$

which is a contradiction to our assumption. □

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Jong Jin Kim
Department of Mathematics, and
Institute of Pure and Applied Mathematics
Chonbuk National University
Chonju, 561-756, Korea
e-mail : jjkim@chonbuk.ac.kr

Jeong Kyun Kim
Department of Mathematics, and
Institute of Pure and Applied Mathematics
Chonbuk National University
Chonju, 561-756, Korea
e-mail : jgkim911@dreamwiz.com

Jeong Seung Lee
Department of Mathematics
Chonbuk National University
Chonju, 561-756, Korea
e-mail : leejs0724@chonbuk.ac.kr