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CHARACTERIZATIONS OF FILTERS AND IDEALS ON WFI-ALGEBRAS

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Abstract. The notion of ideals in WFI-algebras is introduced, and several properties are investigated. Relations between a filter and an ideal are given, and characterizations of an ideal are provided. An extension property for an ideal is established.

1. Introduction

In 1990, W. M. Wu [4] introduced the notion of fuzzy implication algebras (FI-algebra, for short), and investigated several properties. In [3], Z. Li and C. Zheng introduced the notion of distributive (resp. regular, commutative) FI-algebras, and investigated the relations between such FI-algberas and MV-algebras. In [1], Y. B. Jun discussed several aspects of WFI-algebras, and gave a characterization of a WFI-algebra. He introduced the notion of associative (resp. normal, medial) WFI-algebras, and investigated several properties. He gave conditions for a WFI-algebra to be associative/medial, and provided characterizations of associative/medial WFI-algebras, and showed that every associative WFI-algebra is a group in which every element is an involution. He also verified that the class of all medial WFI-algebras is a variety. Y. B. Jun and S. Z. Song [2] introduced the notions of simulative and/or mutant

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WFI-algebras and investigated some properties. They established characterizations of a simulative WFI-algebra, and gave a relation between an associative WFI-algebra and a simulative WFI-algebra. They also found some types for a simulative WFI-algebra to be mutant. In this paper we introduced the concept of ideals of WFI-algebras. We give relations between a filter and an ideal, and provide characterizations of an ideal. We establish an extension property for an ideal.

2. Preliminaries

Let $K(\tau)$ be the class of all algebras of type $\tau = (2, 0)$. By a *WFI-algebra* we mean a system $\mathfrak{X} = (X, \ominus, 1) \in K(\tau)$ such that for all $x, y, z \in X$:

(a1) $x \ominus (y \ominus z) = y \ominus (x \ominus z)$,
(a2) $(x \ominus y) \ominus \big((y \ominus z) \ominus (x \ominus z)\big) = 1$,
(a3) $x \ominus x = 1$,
(a4) $x \ominus y = y \ominus x = 1 \Rightarrow x = y$.

For the convenience of notation, we shall write $[x, y_1, y_2, \cdots, y_n]$ for

$$(\cdots((x \ominus y_1) \ominus y_2) \ominus \cdots) \ominus y_n.$$

We define $[x, y]^0 = x$, and for $n > 0$, $[x, y]^n = [x, y, y, \cdots, y]$, where y occurs n-times.

Proposition 2.1. [1] *In a WFI-algebra $\mathfrak{X}$, the following are true:*

(b1) $x \ominus [x, y]^2 = 1$,
(b2) $1 \ominus x = 1 \Rightarrow x = 1$,
(b3) $1 \ominus x = x$,
(b4) $x \ominus y = 1 \Rightarrow (y \ominus z) \ominus (x \ominus z) = 1$, $(z \ominus x) \ominus (z \ominus y) = 1$,
(b5) $(x \ominus y) \ominus 1 = (x \ominus 1) \ominus (y \ominus 1)$,
(b6) $[x, y]^3 = x \ominus y$.

A nonempty subset S of a WFI-algebra $\mathfrak{X}$ is called a *subalgebra* of $\mathfrak{X}$ if $x \ominus y \in S$ whenever $x, y \in S$. A nonempty subset F of a WFI-algebra $\mathfrak{X}$ is called a *filter* of $\mathfrak{X}$ if it satisfies:

(c1) $1 \in F$,

(c2) $x \ominus y \in F$ and $x \in F$ imply $y \in F$ for all $x, y \in X$.

A filter F of a WFI-algebra $\mathfrak{X}$ is said to be *closed* [1] if F is also a subalgebra of $\mathfrak{X}$.

Proposition 2.2. [1] *Let F be a filter of a WFI-algebra $\mathfrak{X}$. Then F is closed if and only if $x \ominus 1 \in F$ for all $x \in F$.*

Proposition 2.3. [1] *In a finite WFI-algebra, every filter is closed.*

We now define a relation "$\preceq$" on $\mathfrak{X}$ by $x \preceq y$ if and only if $x \ominus y = 1$. It is easy to verify that a WFI-algebra is a partially ordered set with respect to $\preceq$. A WFI-algebra $\mathfrak{X}$ is said to be *associative* [1] if it satisfies $(x \ominus y) \ominus z = x \ominus (y \ominus z)$ for all $x, y, z \in X$. A WFI-algebra $\mathfrak{X}$ is said to be *medial* [1] if it satisfies

$$(x \ominus y) \ominus (a \ominus b) = (x \ominus a) \ominus (y \ominus b)$$

for all $x, y, a, b \in X$. For a WFI-algebra $\mathfrak{X}$, the set

$$\mathcal{S}(\mathfrak{X}) := \{x \in X \mid x \preceq 1\}$$

is called the *simulative part* of $\mathfrak{X}$. A WFI-algebra $\mathfrak{X}$ is said to be *simulative* [2] if it satisfies

(S) $x \preceq 1 \Rightarrow x = 1$.

Note that the condition (S) is equivalent to $\mathcal{S}(\mathfrak{X}) = \{1\}$.

3. Simulatively closed filters

In what follows let $\mathfrak{X}$ denote a WFI-algebra $(X; \ominus, 1)$ unless otherwise specified.

Definition 3.1. A nonempty subset F of $\mathfrak{X}$ is called a *simulatively closed filter* of $\mathfrak{X}$ if it satisfies:

(f1) F is a closed filter of $\mathfrak{X}$.

(f2) $(\forall x \in X)\ (x \in F \Rightarrow x = [x,1]^2)$.

Example 3.2. (1) Let $X = \{a, b, c, 1\}$ be a set with the following Cayley table:

$\ominus$	1	a	b	c
1	1	a	b	c
a	c	1	c	a
b	1	a	1	c
c	a	c	a	1

Then $\mathfrak{X} = (X; \ominus, 1)$ is a WFI-algebra. We know that $F := \{1\}$ is the only simulatively closed filter of $\mathfrak{X}$.

(2) Let $X = \{a, b, c, 1\}$ be a set with the following Cayley table:

$\ominus$	1	a	b	c
1	1	a	b	c
a	1	1	c	c
b	c	c	1	1
c	c	b	a	1

Then $\mathfrak{X} = (X; \ominus, 1)$ is a WFI-algebra. We have two simulatively closed filters $F_1 := \{1\}$ and $F_2 := \{1, c\}$ of $\mathfrak{X}$.

Lemma 3.3. [2] *Let $\mathfrak{X}$ be a WFI-algebra. Then the following are equivalent.*

(i) *$\mathfrak{X}$ is simulative.*

(ii) $(\forall x \in X)\ ([x,1]^2 = x)$.

(iii) $(\forall x, y \in X)\ ((x \ominus 1) \ominus y = (y \ominus 1) \ominus x)$.

Lemma 3.4. [2] *Let $\mathfrak{X}$ be a simulative WFI-algebra and define a binary operation "$\cdot$" on $\mathfrak{X}$ by $x \cdot y = (y \ominus 1) \ominus x$ for all $x, y \in X$. Then $(X, \cdot, 1)$ is a commutative group.*

Theorem 3.5. *In a simulative WFI-algebra, every subalgebra is a filter.*

Proof. Let F be a subalgebra of a simulative WFI-algebra $\mathfrak{X}$. Then $1 \in F$. Now $x, y \in F$ implies $x \cdot y^{-1} = (y^{-1} \ominus 1) \ominus x = [y, 1]^2 \ominus x = y \ominus x \in F$. Hence F is a subgroup of $(X, \cdot, 1)$. Let $x, y \in X$ be such that $x \ominus y \in F$ and $x \in F$. Then $y = (y \cdot x^{-1}) \cdot x \in F$, and so F is a filter of $\mathfrak{X}$. □

Combining Proposition 2.3 and Theorem 3.5, we have the following corollary.

Corollary 3.6. *In a finitely simulative WFI-algebra, every subalgebra is a simulatively closed filter.*

The *doubly simulative part* of $\mathfrak{X}$ is defined to be the set

$$\mathcal{DS}(\mathfrak{X}) := \{x \in X \mid [x, 1]^2 = x\}.$$

Obviously, $1 \in \mathcal{DS}(\mathfrak{X})$ and $\mathcal{DS}(\mathfrak{X}) \cap \mathcal{S}(\mathfrak{X}) = \{1\}$. Let $x, y \in \mathcal{DS}(\mathfrak{X})$. Then $[x, 1]^2 = x$ and $[y, 1]^2 = y$, which imply from (b5) that

$$[x \ominus y, 1]^2 = ((x \ominus 1) \ominus (y \ominus 1)) \ominus 1 = [x, 1]^2 \ominus [y, 1]^2 = x \ominus y,$$

that is, $x \ominus y \in \mathcal{DS}(\mathfrak{X})$. Hence $\mathcal{DS}(\mathfrak{X})$ is a simulative subalgebra of $\mathfrak{X}$.

Theorem 3.7. *Let F be a simulatively closed filter of $\mathfrak{X}$. Then we have*

(1) $$(\forall x, y \in X)(\forall a \in F)(a \ominus x = a \ominus y \Rightarrow x = y).$$

Proof. Let $x, y \in X$ and $a \in F$ be such that

(2) $$a \ominus x = a \ominus y.$$

Then

$$\begin{aligned} a \ominus (x \ominus y) &= x \ominus (a \ominus y) && \text{by (a1)} \\ &= x \ominus (a \ominus x) && \text{by (2)} \\ &= a \ominus (x \ominus x) && \text{by (a1)} \\ &= a \ominus 1 \in F, && \text{by (a3) and Proposition 2.2} \end{aligned}$$

and so $x \ominus y \in F$ by (c2). It follows from (f2) that $x \ominus y = [x \ominus y, 1]^2$, that is, $x \ominus y \in \mathcal{DS}(\mathfrak{X})$. Using (2), (a1), (a2) and (a3), we get $x \ominus y \preceq (a \ominus x) \ominus (a \ominus y) = 1$, and so $x \ominus y \in \mathcal{S}(\mathfrak{X})$. Therefore $x \ominus y \in \mathcal{DS}(\mathfrak{X}) \cap \mathcal{S}(\mathfrak{X}) = \{1\}$, i.e., $x \ominus y = 1$. Similarly, we have $y \ominus x = 1$. It follows from (a4) that $x = y$. This completes the proof. □

We know that the doubly simulative part of $\mathfrak{X}$ is not a filter of $\mathfrak{X}$ in general as seen in the following example.

Example 3.8. Let $X = \{a, b, c, d, 1\}$ be a set with the following Cayley table:

$\ominus$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	b	d
b	d	d	1	a	b
c	d	d	1	1	b
d	b	b	d	d	1

Then $\mathfrak{X} = (X; \ominus, 1)$ is a WFI-algebra and $\mathcal{DS}(\mathfrak{X}) = \{1, b, d\}$ is not a filter of $\mathfrak{X}$ since $b \ominus a = d \in \mathcal{DS}(\mathfrak{X})$ and $b \in \mathcal{DS}(\mathfrak{X})$, but $a \notin \mathcal{DS}(\mathfrak{X})$.

We now give conditions for the doubly simulative part of $\mathfrak{X}$ to be a filter of $\mathfrak{X}$.

Theorem 3.9. *The doubly simulative part $\mathcal{DS}(\mathfrak{X})$ of $\mathfrak{X}$ is a filter of $\mathfrak{X}$ if and only if it satisfies:*

(3) $(\forall x, y \in \mathcal{S}(\mathfrak{X}))(\forall a, b \in \mathcal{DS}(\mathfrak{X}))(a \ominus x = b \ominus y \Rightarrow x = y, a = b)$.

Proof. Assume that $\mathcal{DS}(\mathfrak{X})$ is a filter of $\mathfrak{X}$. Let $x, y \in \mathcal{S}(\mathfrak{X})$ and $a, b \in \mathcal{DS}(\mathfrak{X})$ be such that

(4) $$a \ominus x = b \ominus y.$$

Then

$$\begin{aligned} a &= [a,1]^2 \quad \text{since } a \in \mathcal{DS}(\mathfrak{X}) \\ &= (a \ominus 1) \ominus (x \ominus 1) \quad \text{since } x \in \mathcal{S}(\mathfrak{X}) \\ &= (a \ominus x) \ominus 1 \quad \text{by (b5)} \\ &= (b \ominus y) \ominus 1 \quad \text{by (4)} \\ &= (b \ominus 1) \ominus (y \ominus 1) \quad \text{by (b5)} \\ &= [b,1]^2 \quad \text{since } y \in \mathcal{S}(\mathfrak{X}) \\ &= b. \quad \text{since } b \in \mathcal{DS}(\mathfrak{X}) \end{aligned}$$

Using (4), we get $a \ominus x = a \ominus y$, and so

$$a \ominus (x \ominus y) = x \ominus (a \ominus y) = x \ominus (a \ominus x) = a \ominus (x \ominus x) = a \ominus 1 \in \mathcal{DS}(\mathfrak{X})$$

by (a1) and (a3). Since $\mathcal{DS}(\mathfrak{X})$ is a filter of $\mathfrak{X}$ and $a \in \mathcal{DS}(\mathfrak{X})$, it follows from (c2) that $x \ominus y \in \mathcal{DS}(\mathfrak{X})$. Since $\mathcal{S}(\mathfrak{X})$ is a subalgebra, we have $x \ominus y \in \mathcal{S}(\mathfrak{X})$. Thus $x \ominus y \in \mathcal{DS}(\mathfrak{X}) \cap \mathcal{S}(\mathfrak{X})$, and so $x \ominus y = 1$. Similarly, we get $y \ominus x = 1$. Hence $x = y$. Conversely, the condition (3) is valid and let $x, y \in X$ be such that $x \ominus y \in \mathcal{DS}(\mathfrak{X})$ and $x \in \mathcal{DS}(\mathfrak{X})$. Then

$$\begin{aligned} x \ominus y &= [x \ominus y, 1]^2 \quad \text{since } x \ominus y \in \mathcal{DS}(\mathfrak{X}) \\ &= ((x \ominus 1) \ominus (y \ominus 1)) \ominus 1 \quad \text{by (b5)} \\ &= [x,1]^2 \ominus [y,1]^2 \quad \text{by (b5)} \\ &= x \ominus [y,1]^2, \quad \text{since } x \in \mathcal{DS}(\mathfrak{X}) \end{aligned}$$

and therefore

$$\begin{aligned} x \ominus ([y,1]^2 \ominus y) &= [y,1]^2 \ominus (x \ominus y) \\ &= [y,1]^2 \ominus (x \ominus [y,1]^2) \\ &= x \ominus ([y,1]^2 \ominus [y,1]^2) = x \ominus 1. \end{aligned}$$

Note that $([y,1]^2 \ominus y) \ominus 1 = [y,1]^3 \ominus (y \ominus 1) = (y \ominus 1) \ominus (y \ominus 1) = 1$ by (b5), (b6) and (a3), that is, $[y,1]^2 \ominus y \in \mathcal{S}(\mathfrak{X})$. It follows from (3) that $[y,1]^2 \ominus y = 1$, that is, $[y,1]^2 \preceq y$ so that $[y,1]^2 = y$ by using (b1) and (a4). Hence $y \in \mathcal{DS}(\mathfrak{X})$, and $\mathcal{DS}(\mathfrak{X})$ is a filter of $\mathfrak{X}$. □

Lemma 3.10. *For a WFI-algebra $\mathfrak{X}$, we have*

$$(\forall x \in X)(((x \ominus 1) \ominus ([x,1]^2 \ominus x)) \ominus x \in \mathcal{S}(\mathfrak{X})).$$

Proof. It is straightforward by using (a3), (b5) and (b6). □

Theorem 3.11. *If the doubly simulative part of $\mathfrak{X}$ is a filter of $\mathfrak{X}$, then*

$$(\forall x \in X)(\exists a \in \mathcal{DS}(\mathfrak{X}), \exists b \in \mathcal{S}(\mathfrak{X}))(x = a \ominus b),$$

and in this case a and b are unique.

Proof. Note that $[x,1]^2 \ominus x \in \mathcal{S}(\mathfrak{X})$ and $x \ominus 1 \in \mathcal{DS}(\mathfrak{X})$ for all $x \in X$. Using (a3), (a1) and (b6), we have

$$\begin{aligned}&(x \ominus 1) \ominus ([x,1]^2 \ominus (((x \ominus 1) \ominus ([x,1]^2 \ominus x)) \ominus x))\\ &\quad = (x \ominus 1) \ominus (((x \ominus 1) \ominus ([x,1]^2 \ominus x)) \ominus ([x,1]^2 \ominus x))\\ &\quad = ((x \ominus 1) \ominus ([x,1]^2 \ominus x)) \ominus ((x \ominus 1) \ominus ([x,1]^2 \ominus x))\\ &\quad = 1 \in \mathcal{DS}(\mathfrak{X}).\end{aligned}$$

Since $x \ominus 1$, $[x,1]^2 \in \mathcal{DS}(\mathfrak{X})$ and $\mathcal{DS}(\mathfrak{X})$ is a filter of $\mathfrak{X}$, it follows from (c2) that

$$((x \ominus 1) \ominus ([x,1]^2 \ominus x)) \ominus x \in \mathcal{DS}(\mathfrak{X})$$

so from Lemma 3.10 that

$$((x \ominus 1) \ominus ([x,1]^2 \ominus x)) \ominus x \in \mathcal{DS}(\mathfrak{X}) \cap \mathcal{S}(\mathfrak{X}) = \{1\}.$$

Hence $((x \ominus 1) \ominus ([x,1]^2 \ominus x)) \ominus x = 1$. On the other hand, we get

$$\begin{aligned}&x \ominus ((x \ominus 1) \ominus ([x,1]^2 \ominus x))\\ &\quad = (x \ominus 1) \ominus ([x,1]^2 \ominus (x \ominus x))\\ &\quad = (x \ominus 1) \ominus [x,1]^3 = 1\end{aligned}$$

by (a3), (a1) and (b6). Therefore we have $x = (x \ominus 1) \ominus ([x,1]^2 \ominus x)$. Taking $a = x \ominus 1$ and $b = [x,1]^2 \ominus x$, and using Theorem 3.9, we obtain the desired result. □

Theorem 3.12. *The doubly simulative part of $\mathfrak{X}$ is a filter of $\mathfrak{X}$ if and only if for each $a \in \mathcal{DS}(\mathfrak{X})$, the self map f_a of $\mathfrak{X}$ is bijective, where f_a is defined by $f_a(x) = a \ominus x$ for all $x \in X$.*

Proof. Assume that $\mathcal{DS}(\mathfrak{X})$ is a filter of $\mathfrak{X}$ and let $a \in \mathcal{DS}(\mathfrak{X})$. Let $x, y \in X$ be such that $f_a(x) = f_a(y)$. Then $a \ominus x = a \ominus y$, and so $a \ominus (x \ominus y) = a \ominus 1 \in \mathcal{DS}(\mathfrak{X})$. It follows from (c2) that $x \ominus y \in \mathcal{DS}(\mathfrak{X})$. On the other hand, note that

$$\begin{aligned} a &= [a,1]^2 \quad \text{since } a \in \mathcal{DS}(\mathfrak{X}) \\ &= (a \ominus (x \ominus y)) \ominus 1 \quad \text{since } a \ominus (x \ominus y) = a \ominus 1 \\ &= (a \ominus 1) \ominus ((x \ominus y) \ominus 1) \quad \text{by (b5)} \\ &= (x \ominus y) \ominus [a,1]^2 \quad \text{by (a1)} \\ &= (x \ominus y) \ominus a. \quad \text{since } a \in \mathcal{DS}(\mathfrak{X}) \end{aligned}$$

Thus, in particular, $(x \ominus y) \ominus 1 = 1$ since $1 \in \mathcal{DS}(\mathfrak{X})$, that is, $x \ominus y \in \mathcal{S}(\mathfrak{X})$. Therefore $x \ominus y \in \mathcal{DS}(\mathfrak{X}) \cap \mathcal{S}(\mathfrak{X}) = \{1\}$ and thus $x \ominus y = 1$. Similarly we have $y \ominus x = 1$. Consequently, $x = y$, which shows that f_a is injective. Since $a \ominus 1 \in \mathcal{DS}(\mathfrak{X})$, we know also that $f_{a \ominus 1}$ is injective. Next, we get

$$\begin{aligned} &x \ominus ((a \ominus 1) \ominus (a \ominus x)) = (a \ominus 1) \ominus (x \ominus (a \ominus x)) \\ &\quad = (a \ominus 1) \ominus (a \ominus (x \ominus x)) = (a \ominus 1) \ominus (a \ominus 1) = 1, \end{aligned}$$

that is, $x \preceq (a \ominus 1) \ominus (a \ominus x)$ for all $x \in X$. Now we obtain

$$\begin{aligned} &f_a(f_{a \ominus 1}(((a \ominus 1) \ominus (a \ominus x)) \ominus x) \\ &\quad = f_a((a \ominus 1) \ominus (((a \ominus 1) \ominus (a \ominus x)) \ominus x)) \\ &\quad = a \ominus ((a \ominus 1) \ominus (((a \ominus 1) \ominus (a \ominus x)) \ominus x)) \\ &\quad = ((a \ominus 1) \ominus (a \ominus x)) \ominus ((a \ominus 1) \ominus (a \ominus x)) \\ &\quad = 1 = (a \ominus 1) \ominus (a \ominus 1) = a \ominus [a,1]^2 \\ &\quad = f_a(f_{a \ominus 1}(1)), \end{aligned}$$

and so $((a \ominus 1) \ominus (a \ominus x)) \ominus x = 1$, i.e., $(a \ominus 1) \ominus (a \ominus x) \preceq x$ for all $x \in X$. Hence $x = (a \ominus 1) \ominus (a \ominus x) = a \ominus ((a \ominus 1) \ominus x) = f_a((a \ominus 1) \ominus x)$. Consequently, f_a is surjective. Conversely, suppose that f_a is bijective for every $a \in \mathcal{DS}(\mathfrak{X})$. Let $x, y \in \mathcal{S}(\mathfrak{X})$ and $a, b \in \mathcal{DS}(\mathfrak{X})$ be such that $a \ominus x = b \ominus y$. Then $f_a(x) = f_b(y)$, which implies that $a = b$ and $x = y$. Using Theorem 3.9, we know that $\mathcal{DS}(\mathfrak{X})$ is a filter of $\mathfrak{X}$. □

Proposition 3.13. *If the doubly simulative part of $\mathfrak{X}$ is a filter of $\mathfrak{X}$, then $f_a \circ f_b = f_{(b\ominus 1)\ominus a}$ for all $a, b \in \mathcal{DS}(\mathfrak{X})$.*

Proof. Let $a, b \in \mathcal{DS}(\mathfrak{X})$. Then

$$\begin{aligned}
&f_a \circ f_b((a \ominus (b \ominus x)) \ominus (((b \ominus 1) \ominus a) \ominus x)) \\
&\quad = a \ominus (b \ominus ((a \ominus (b \ominus x)) \ominus (((b \ominus 1) \ominus a) \ominus x))) \\
&\quad = ((b \ominus 1) \ominus a) \ominus 1 = [b, 1]^2 \ominus (a \ominus 1) \\
&\quad = b \ominus (a \ominus 1) = a \ominus (b \ominus 1) = f_a \circ f_b(1),
\end{aligned}$$

and so

$$(a \ominus (b \ominus x)) \ominus (((b \ominus 1) \ominus a) \ominus x) = 1 \tag{5}$$

because f_a and f_b are injective and so $f_a \circ f_b$ is also injective. Now we get

$$\begin{aligned}
&f_{(b\ominus 1)\ominus a}((((b \ominus 1) \ominus a) \ominus x) \ominus (a \ominus (b \ominus x))) \\
&\quad = ((b \ominus 1) \ominus a) \ominus ((((b \ominus 1) \ominus a) \ominus x) \ominus (a \ominus (b \ominus x))) \\
&\quad = (((b \ominus 1) \ominus a) \ominus x) \ominus (((b \ominus 1) \ominus a) \ominus (a \ominus (b \ominus x))) \\
&\quad = (((b \ominus 1) \ominus a) \ominus x) \ominus (a \ominus (b \ominus (((b \ominus 1) \ominus a) \ominus x))) \\
&\quad = a \ominus (b \ominus 1) = ((b \ominus 1) \ominus a) \ominus 1 = f_{(b\ominus 1)\ominus a}(1),
\end{aligned}$$

which implies that

$$(((b \ominus 1) \ominus a) \ominus x) \ominus (a \ominus (b \ominus x)) = 1 \tag{6}$$

since $(b \ominus 1) \ominus a \in \mathcal{DS}(\mathfrak{X})$ and thus $f_{(b\ominus 1)\ominus a}$ is injective. Combining (a4), (5) and (6), we obtain $f_a \circ f_b(x) = a \ominus (b \ominus x) = ((b \ominus 1) \ominus a) \ominus x = f_{(b\ominus 1)\ominus a}(x)$ for all $x \in X$, and consequently the equality $f_a \circ f_b = f_{(b\ominus 1)\ominus a}$ is valid. □

4. Ideals of WFI-algebras

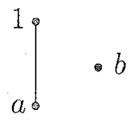
Definition 4.1. A nonempty subset G of $\mathfrak{X}$ is called an *ideal* of $\mathfrak{X}$ if it satisfies:

(c1) $1 \in G$,

(c3) $(\forall x, y \in X)\ (\forall z \in G)\ ((x \ominus z) \ominus y \in G \Rightarrow x \ominus y \in G)$.

Example 4.2. (1) Let $X = \{1, a, b\}$ be a set with the following Cayley table and Hasse diagram:

$\ominus$	1	a	b
1	1	a	b
a	1	1	b
b	b	b	1



Then $\mathfrak{X} = (X; \ominus, 1)$ is a WFI-algebra (see [1]). We know that $G_1 := \{1\}$ and $G_2 := \{1, a\}$ are ideals of $\mathfrak{X}$, but $G_3 := \{1, b\}$ is not an ideal of $\mathfrak{X}$ since $(1 \ominus b) \ominus a = b \ominus a = b \in G_3$, but $1 \ominus a = a \notin G_3$.

(2) Let $X = \{1, a, b, c\}$ be a set with the following Cayley table:

$\ominus$	1	a	b	c
1	1	a	b	c
a	a	1	c	b
b	b	c	1	a
c	c	b	a	1

Then $\mathfrak{X} = (X; \ominus, 1)$ is a WFI. It is easy to verify that a set $G := \{1, a\}$ is an ideal of $\mathfrak{X}$.

Theorem 4.3. *Every ideal of a WFI-algebra is a closed filter.*

Proof. Let G be an ideal of $\mathfrak{X}$. Taking $x = 1$ in (c3) and using (b3), we have the condition (c2). Putting $x = z$ in (c3) and using (a3) and (b3), we know that $z \ominus y \in G$ whenever $y, z \in G$. Hence G is a closed filter of $\mathfrak{X}$. □

The converse of Theorem 4.3 is not true in general as seen in the following example.

Example 4.4. Let $X = \{a, b, c, 1\}$ be a set with the following Cayley table:

$\ominus$	1	a	b	c
1	1	a	b	c
a	c	1	a	b
b	b	c	1	a
c	a	b	c	1

Then $\mathfrak{X} = (X; \ominus, 1)$ is a WFI-algebra. We know that $G := \{1\}$ is a closed filter of $\mathfrak{X}$. But it is not an ideal of $\mathfrak{X}$ since $(a \ominus 1) \ominus c = c \ominus c = 1 \in G$, but $a \ominus c = b \notin G$.

We give conditions for a filter to be an ideal.

Theorem 4.5. *Let G be a filter of $\mathfrak{X}$ such that*

$$(7) \qquad (\forall x, y \in X)\,(y \in G \;\Rightarrow\; x \ominus y \in G).$$

Then G is an ideal of $\mathfrak{X}$.

Proof. Let $x, y \in X$ and $z \in G$ be such that $(x \ominus z) \ominus y \in G$. Using (a1) and (7), we have $(x \ominus z) \ominus (x \ominus y) = x \ominus ((x \ominus z) \ominus y) \in G$. Since $x \ominus z \in G$ by (7) and since G is a filter, it follows from (c2) that $x \ominus y \in G$. Hence G is an ideal of $\mathfrak{X}$. □

Theorem 4.6. *Let G be a filter of $\mathfrak{X}$. Then the following are equivalent:*

(i) *G is an ideal of $\mathfrak{X}$.*
(ii) $(\forall x, y \in X)$ $((x \ominus 1) \ominus y \in G \Rightarrow x \ominus y \in G)$.
(iii) $(\forall x, y, z \in X)$ $((x \ominus y) \ominus z \in G \Rightarrow x \ominus (y \ominus z) \in G)$.

Proof. Assume that G is an ideal of $\mathfrak{X}$. Let $x, y \in X$ be such that $(x \ominus 1) \ominus y \in G$. Then $x \ominus y \in G$ by (c3), and so (ii) is valid. Suppose that (ii) holds and let $x, y, z \in X$ be such that $(x \ominus y) \ominus z \in G$. Note that

$$\begin{aligned} 1 &= (x \ominus 1) \ominus (x \ominus 1) = (x \ominus 1) \ominus (x \ominus (y \ominus y)) && \text{by (a3)} \\ &= (x \ominus 1) \ominus (y \ominus (x \ominus y)) && \text{by (a1)} \\ &\preceq (x \ominus 1) \ominus (((x \ominus y) \ominus z) \ominus (y \ominus z)) && \text{by (a2) and (b4)} \\ &= ((x \ominus y) \ominus z) \ominus ((x \ominus 1) \ominus (y \ominus z)). && \text{by (a1)} \end{aligned}$$

Since $1 \in G$, it follows that $((x \ominus y) \ominus z) \ominus ((x \ominus 1) \ominus (y \ominus z)) \in G$ so that $(x \ominus 1) \ominus (y \ominus z) \in G$. Using (ii), we get $x \ominus (y \ominus z) \in G$. Finally suppose that (iii) is valid and let $x, y \in X$ and $z \in G$ be such that $(x \ominus z) \ominus y \in G$. It follows from (a1) and (iii) that $z \ominus (x \ominus y) = x \ominus (z \ominus y) \in G$ so from (c2) that $x \ominus y \in G$. Hence G is an ideal of $\mathfrak{X}$. This completes the proof. □

Theorem 4.7. (Extension property for an ideal) *Let G and H be filters of $\mathfrak{X}$ such that $G \subseteq H$. If G is an ideal of $\mathfrak{X}$, then so is H.*

Proof. Let G be a filter of $\mathfrak{X}$ and let $x, y \in X$ be such that $(x \ominus 1) \ominus y \in H$. Using (a1) and (a3), we have

$$(x \ominus 1) \ominus (((x \ominus 1) \ominus y) \ominus y) = ((x \ominus 1) \ominus y) \ominus ((x \ominus 1) \ominus y) = 1 \in G.$$

It follows from (a1) and Theorem 4.6(ii) that

$$((x \ominus 1) \ominus y) \ominus (x \ominus y) = x \ominus (((x \ominus 1) \ominus y) \ominus y) \in G \subseteq H$$

so from (c2) that $x \ominus y \in H$. Hence, by Theorem 4.6, we conclude that H is an ideal of $\mathfrak{X}$. □

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References

[1] Y. B. Jun, *Weak fuzzy implication algebras,* Adv. Stud. Contemp. Math. **7** (2003), no. 1, 41–52.

[2] Y. B. Jun and S. Z. Song, *Simulative and mutant WFI-algebras,* Honam Math. J. (to appear).

[3] Z. Li and C. Zheng, *Relations between fuzzy implication algebra and MV-algebra,* J. Fuzzy Math. 9 (2001), no. 1, 201–205.

[4] W. M. Wu, *Fuzzy implication algebras,* Fuzzy Systems Math. 4 (1990), no. 1, 56–63.

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