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INTRINSIC PRODUCT OF INTUITIONISTIC FUZZY SUBRINGS/IDEALS IN RINGS

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Abstract. Intrinsic product of intuitionistic fuzzy sets are considered. Using this, characterizations of intuitionistic fuzzy subrings/ideals are discussed. The notions of intuitionistic fuzzy quasi ideals and intuitionistic fuzzy bi-ideals are introduced. Characterizations of regular rings are provided.

1. Introduction

After the introduction of the concept of fuzzy sets by Zadeh [12], several researchers were conducted on the generalizations of the notion of fuzzy sets. The idea of *intuitionistic fuzzy set* was first published by Atanassov [2, 3] as a generalization of the notion of fuzzy sets. In [5], Banerjee and Basnet applied the concept of intuitionistic fuzzy sets to the theory of rings, and introduced the notions of intuitionistic fuzzy subrings and intuitionistic fuzzy ideals of a ring. Hur et al. [9] introduced the notions of intuitionistic fuzzy (completely) prime ideals and intuitionistic fuzzy weak completely prime ideals in a ring. In this paper we consider the intrinsic product of intuitionistic fuzzy sets, and investigate their basic properties. Using such notion, we discuss characterizations of intuitionistic fuzzy subrings and intuitionistic fuzzy ideals.

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We introduce the concepts of intuitionistic fuzzy bi-ideals and intuitionistic fuzzy quasi ideals in a ring. We deal with the related properties on regular rings.

2. Preliminaries

An ideal A of a ring R is said to be *semiprime* if, whenever $a^n \in A$ for some $a \in A$ and some positive integer n, then $a \in A$. Note that a commutative ring R is regular if and only if every ideal of R is semiprime.

An additive subgroup Q of a ring R is called a *quasi-ideal* of R if $QR \cap RQ \subseteq Q$, and an additive subgroup B of a ring R is called a *bi-ideal* of R if $BB \subseteq B$ and $BRB \subseteq B$, where

$$AB = \Big\{ \sum_{\text{finite}} a_i b_i \mid a_i \in A,\ b_i \in B \Big\}$$

for any non-empty subsets A and B of R.

A mapping $\mu : M \to [0,1]$, where M is an arbitrary non-empty set, is called a *fuzzy set* in M. For any fuzzy set μ in M and any $\alpha \in [0,1]$ we define two sets

$$U(\mu;\alpha) = \{x \in M \mid \mu(x) \geq \alpha\} \quad \text{and} \quad L(\mu;\alpha) = \{x \in M \mid \mu(x) \leq \alpha\},$$

which are called an *upper* and *lower level set* of μ.

As an important generalization of the notion of fuzzy sets in M, Atanassov [2, 3] introduced the concept of an *intuitionistic fuzzy set* (IFS for short) defined on a non-empty set M as objects having the form

$$A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle \mid x \in M\},$$

where the functions $\mu_A : M \to [0,1]$ and $\gamma_A : M \to [0,1]$ denote the *degree of membership* (namely $\mu_A(x)$) and the *degree of nonmembership* (namely $\gamma_A(x)$) of each element $x \in M$ to A respectively, and $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for all $x \in M$ [8]).

Such defined objects are studied by many authors (see for example two journals: 1. *Fuzzy Sets and Systems* and 2. *Notes on Intuitionistic Fuzzy Sets*) and have many interesting applications not only in mathematics (see Chapter 5 in the book [4]).

For the sake of simplicity, we shall use the symbol $A = (\mu_A, \gamma_A)$ for the IFS $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle \mid x \in M\}$.

Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be IFSs in a set M. We define

- $A \subseteq B \Leftrightarrow (\forall x \in M)\,(\mu_A(x) \leq \mu_B(x),\ \gamma_A(x) \geq \gamma_B(x))$.
- $A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$.
- $A \cap B = (\mu_A \wedge \mu_B, \gamma_A \vee \gamma_B)$.
- $A \cup B = (\mu_A \vee \mu_B, \gamma_A \wedge \gamma_B)$.
- $0_\sim = (0, 1)$ and $1_\sim = (1, 0)$.

Let f be a mapping from a set X to a set Y. If $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ are IFSs in X and Y, respectively, then the *preimage* of $B = (\mu_B, \gamma_B)$ under f is defined to be an IFS $f^{-1}(B) = (\mu_{f^{-1}(B)}, \gamma_{f^{-1}(B)})$ where $\mu_{f^{-1}(B)}(x) = \mu_B(f(x))$ and $\gamma_{f^{-1}(B)}(x) = \gamma_B(f(x))$ for all $x \in X$, and the *image* of $A = (\mu_A, \gamma_A)$ under f is defined to be an IFS $f(A) = (\mu_{f(A)}, \gamma_{f(A)})$ where

$$\mu_{f(A)}(y) := \begin{cases} \bigvee\limits_{x \in f^{-1}(y)} \mu_A(x) & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

$$\gamma_{f(A)}(y) := \begin{cases} \bigwedge\limits_{x \in f^{-1}(y)} \gamma_A(x) & \text{if } f^{-1}(y) \neq \emptyset, \\ 1 & \text{otherwise} \end{cases}$$

for all $y \in Y$.

Definition 2.1. [5, 10] An IFS $A = (\mu_A, \gamma_A)$ in a ring R is called an *intuitionistic fuzzy subring* of R if it satisfies the following conditions:

(i) $(\forall x, y \in R)\ (\mu_A(x - y) \geq \min\{\mu_A(x), \mu_A(y)\})$,

(ii) $(\forall x, y \in R)\ (\mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y)\})$,

(iii) $(\forall x, y \in R)\ (\gamma_A(x - y) \leq \max\{\gamma_A(x), \gamma_A(y)\})$,

(iv) $(\forall x, y \in R)$ $(\gamma_A(xy) \leq \max\{\gamma_A(x), \gamma_A(y)\})$,

Definition 2.2. [5, 10] An IFS set $A = (\mu_A, \gamma_A)$ in a ring R is called an *intuitionistic fuzzy left* (resp. *right*) *ideal* of R if it satisfies the following conditions:

(i) $(\forall x, y \in R)$ $(\mu_A(x-y) \geq \min\{\mu_A(x), \mu_A(y)\})$,
(ii) $(\forall x, y \in R)$ $(\gamma_A(x-y) \leq \max\{\gamma_A(x), \gamma_A(y)\})$,
(iii) $(\forall a, x \in R)$ $(\mu_A(ax) \geq \mu_A(x))$ (resp. $\gamma_A(xa) \leq \gamma_A(x)$).

If $A = (\mu_A, \gamma_A)$ is both an intuitionistic fuzzy left and intuitionistic fuzzy right ideal of a ring R, then $A = (\mu_A, \gamma_A)$ is called an *intuitionistic fuzzy ideal* of R.

3. Intrinsic products

Definition 3.1. Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be IFSs in a ring R. The *intrinsic product* of $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ is defined to be the IFS $A * B = (\mu_{A*B}, \gamma_{A*B})$ in R given by

$$\mu_{A*B}(x) := \bigvee_{\substack{x=\sum a_i b_i \\ \text{finite}}} \min\left\{ \begin{array}{l} \mu_A(a_1), \mu_A(a_2), \cdots, \mu_A(a_m), \\ \mu_B(b_1), \mu_B(b_2), \cdots, \mu_B(b_m) \end{array} \right\}$$

$$\gamma_{A*B}(x) := \bigwedge_{\substack{x=\sum a_i b_i \\ \text{finite}}} \max\left\{ \begin{array}{l} \gamma_A(a_1), \gamma_A(a_2), \cdots, \gamma_A(a_m), \\ \gamma_B(b_1), \gamma_B(b_2), \cdots, \gamma_B(b_m) \end{array} \right\}$$

if we can express $x = a_1b_1 + a_2b_2 + \cdots + a_mb_m$ for some $a_i, b_i \in R$ and for some positive integer m where each $a_ib_i \neq 0$. Otherwise, we define $A * B = 0_\sim$, i.e., $\mu_{A*B}(x) = 0$ and $\gamma_{A*B}(x) = 1$.

Proposition 3.2. *Let $A = (\mu_A, \gamma_A)$, $B = (\mu_B, \gamma_B)$ and $C = (\mu_C, \gamma_C)$ be IFSs in a ring R. If $A \subseteq B$, then $A * C \subseteq B * C$ and $C * A \subseteq C * B$. Also we have $A \circ B \subseteq A * B$ where $A \circ B = (\mu_{A\circ B}, \gamma_{A\circ B})$ is an IFS in R*

given by

$$\mu_{A\circ B}(x) := \begin{cases} \bigvee\limits_{x=ab} \min\{\mu_A(a), \mu_B(b)\} & \text{if } x \text{ is factorizable in } R \\ 0 & \text{otherwise,} \end{cases}$$

$$\gamma_{A\circ B}(x) := \begin{cases} \bigwedge\limits_{x=ab} \min\{\gamma_A(a), \gamma_B(b)\} & \text{if } x \text{ is factorizable in } R \\ 1 & \text{otherwise.} \end{cases}$$

Proof. Straightforward. □

Proposition 3.3. *For any IFSs $A = (\mu_A, \gamma_A)$, $B = (\mu_B, \gamma_B)$ and $C = (\mu_C, \gamma_C)$ in a ring R, we have $A * (B * C) = (A * B) * C$.*

Proof. Let $A = (\mu_A, \gamma_A)$, $B = (\mu_B, \gamma_B)$ and $C = (\mu_C, \gamma_C)$ be IFSs in a ring R and $x \in R$. We assume that x is expressible as $x = a_1b_1c_1 + a_2b_2c_2 + \cdots + a_mb_mc_m$ where $a_i, b_i, c_i \in R$ and $a_ib_ic_i \neq 0$. Then

$$\begin{aligned} &\min\{\mu_A(a_1), \mu_A(a_2), \cdots, \mu_A(a_m), \mu_B(b_1), \mu_B(b_2), \cdots, \mu_B(b_m), \\ &\qquad\qquad \mu_C(c_1), \mu_C(c_2), \cdots, \mu_C(c_m)\} \\ (1) \quad &\leq \min\{\mu_A(a_1), \mu_A(a_2), \cdots, \mu_A(a_m), \mu_{B*C}(b_1c_1), \\ &\qquad\qquad \mu_{B*C}(b_2c_2), \cdots, \mu_{B*C}(b_mc_m), \} \\ &\leq \mu_{A*(B*C)}(x), \end{aligned}$$

and

$$\begin{aligned} &\max\{\gamma_A(a_1), \gamma_A(a_2), \cdots, \gamma_A(a_m), \gamma_B(b_1), \gamma_B(b_2), \cdots, \gamma_B(b_m), \\ &\qquad\qquad \gamma_C(c_1), \gamma_C(c_2), \cdots, \gamma_C(c_m)\} \\ (2) \quad &\geq \max\{\gamma_A(a_1), \gamma_A(a_2), \cdots, \gamma_A(a_m), \gamma_{B*C}(b_1c_1), \\ &\qquad\qquad \gamma_{B*C}(b_2c_2), \cdots, \gamma_{B*C}(b_mc_m), \} \\ &\geq \gamma_{A*(B*C)}(x). \end{aligned}$$

Now

$$
\begin{aligned}
&\mu_{(A*B)*C}(x)\\
&= \bigvee_{x=\sum_{\text{finite}} a_i b_i} \min\left\{ \begin{array}{l} \mu_{A*B}(a_1), \mu_{A*B}(a_2), \cdots, \mu_{A*B}(a_m), \\ \mu_C(b_1), \mu_C(b_2), \cdots, \mu_C(b_m) \end{array} \right\}\\
&= \bigvee_{x=\sum_{\text{finite}} a_i b_i} \Bigg(\bigvee_{a_1=\sum_{\text{finite}} a_{1i} b_{i1}} \Bigg(\bigvee_{a_2=\sum_{\text{finite}} a_{2i} b_{i2}}\\
&\qquad \Bigg(\cdots \Bigg(\bigvee_{a_m=\sum_{\text{finite}} a_{mi} b_{im}} \min\{\mu_A(a_{11}), \mu_A(a_{12}), \cdots, \mu_A(a_{1m_1}),\\
&\qquad \mu_A(a_{21}), \mu_A(a_{22}), \cdots, \mu_A(a_{2m_2}), \cdots, \mu_A(a_{m1}),\\
&\qquad \mu_A(a_{m2}), \cdots, \mu_A(a_{mm_m}), \mu_B(b_{11}), \mu_B(b_{21}),\\
&\qquad \cdots, \mu_B(b_{m_m m}), \mu_C(b_1), \mu_C(b_2), \cdots, \mu_C(b_m)\} \Bigg) \cdots \Bigg)\Bigg)\Bigg)\\
&\leq \mu_{A*(B*C)}(x), \qquad \text{by (1)}
\end{aligned}
$$

and

$$
\begin{aligned}
&\gamma_{(A*B)*C}(x)\\
&= \bigwedge_{x=\sum_{\text{finite}} a_i b_i} \max\left\{ \begin{array}{l} \gamma_{A*B}(a_1), \gamma_{A*B}(a_2), \cdots, \gamma_{A*B}(a_m), \\ \gamma_C(b_1), \gamma_C(b_2), \cdots, \gamma_C(b_m) \end{array} \right\}\\
&= \bigwedge_{x=\sum_{\text{finite}} a_i b_i} \Bigg(\bigwedge_{a_1=\sum_{\text{finite}} a_{1i} b_{i1}} \Bigg(\bigwedge_{a_2=\sum_{\text{finite}} a_{2i} b_{i2}}\\
&\qquad \Bigg(\cdots \Bigg(\bigwedge_{a_m=\sum_{\text{finite}} a_{mi} b_{im}} \max\{\gamma_A(a_{11}), \gamma_A(a_{12}), \cdots, \gamma_A(a_{1m_1}),\\
&\qquad \gamma_A(a_{21}), \gamma_A(a_{22}), \cdots, \gamma_A(a_{2m_2}), \cdots, \gamma_A(a_{m1}),\\
&\qquad \gamma_A(a_{m2}), \cdots, \gamma_A(a_{mm_m}), \gamma_B(b_{11}), \gamma_B(b_{21}),\\
&\qquad \cdots, \gamma_B(b_{m_m m}), \gamma_C(b_1), \gamma_C(b_2), \cdots, \gamma_C(b_m)\} \Bigg) \cdots \Bigg)\Bigg)\Bigg)\\
&\geq \gamma_{A*(B*C)}(x). \qquad \text{by (2)}
\end{aligned}
$$

Hence $A*(B*C) \subseteq (A*B)*C$. Similarly $(A*B)*C \subseteq A*(B*C)$. If x is not expressible as $x = a_1b_1c_1 + a_2b_2c_2 + \cdots + a_mb_mc_m$, then $\mu_{(A*B)*C}(x) = 0 = \mu_{A*(B*C)}(x)$ and $\gamma_{(A*B)*C}(x) = 1 = \gamma_{A*(B*C)}(x)$. Therefore $A*(B*C) = (A*B)*C$. □

Proposition 3.4. *Let $A = (\mu_A, \gamma_A)$, $B = (\mu_B, \gamma_B)$ and $C = (\mu_C, \gamma_C)$ be IFSs in a ring R. Then*

(i) $A * (B + C) \subseteq (A * B) + (A * C)$, *and the equality is valid if* $\mu_C(0) = \mu_B(0) \geq \max\{\mu_B(x), \mu_C(x)\}$ *and* $\gamma_C(0) = \gamma_B(0) \leq \min\{\gamma_B(x), \gamma_C(x)\}$ *for all* $x \in R$.

(ii) $(A + B) * C \subseteq (A * C) + (B * C)$, *and the equality is valid if* $\mu_B(0) = \mu_A(0) \geq \max\{\mu_A(x), \mu_B(x)\}$ *and* $\gamma_B(0) = \gamma_A(0) \leq \min\{\gamma_A(x), \gamma_B(x)\}$ *for all* $x \in R$.

Proof. (i) Let $x \in R$. If x is expressible in the form

$$x = a_1(u_1 + v_1) + a_2(u_2 + v_2) + \cdots + a_m(u_m + v_m) \tag{3}$$

where no term is zero, then

$$\begin{aligned}
&(\mu_{A*B} + \mu_{A*C})(x)\\
&\geq \min\{\mu_{A*B}(\textstyle\sum_{i=1}^m a_iu_i), \mu_{A*C}(\sum_{i=1}^m a_iv_i)\}\\
&\geq \min\{\min\{\mu_A(a_1), \mu_A(a_2), \cdots, \mu_A(a_m), \mu_B(u_1), \mu_B(u_2), \cdots, \mu_B(u_m)\},\\
&\qquad \min\{\mu_A(a_1), \mu_A(a_2), \cdots, \mu_A(a_m), \mu_C(v_1), \mu_C(v_2), \cdots, \mu_C(v_m)\}\}\\
&= \min\{\mu_A(a_1), \mu_A(a_2), \cdots, \mu_A(a_m), \min\{\mu_B(u_1), \mu_C(v_1)\},\\
&\qquad\qquad \min\{\mu_B(u_2), \mu_C(v_2)\}, \cdots, \min\{\mu_B(u_m), \mu_C(v_m)\}\},
\end{aligned}$$

and

$$\begin{aligned}
&(\gamma_{A*B} + \gamma_{A*C})(x)\\
&\leq \max\{\gamma_{A*B}(\textstyle\sum_{i=1}^m a_iu_i), \gamma_{A*C}(\sum_{i=1}^m a_iv_i)\}\\
&\leq \max\{\max\{\gamma_A(a_1), \gamma_A(a_2), \cdots, \gamma_A(a_m), \gamma_B(u_1), \gamma_B(u_2), \cdots, \gamma_B(u_m)\},\\
&\qquad \max\{\gamma_A(a_1), \gamma_A(a_2), \cdots, \gamma_A(a_m), \gamma_C(v_1), \gamma_C(v_2), \cdots, \gamma_C(v_m)\}\}\\
&= \max\{\gamma_A(a_1), \gamma_A(a_2), \cdots, \gamma_A(a_m), \max\{\gamma_B(u_1), \gamma_C(v_1)\},\\
&\qquad\qquad \max\{\gamma_B(u_2), \gamma_C(v_2)\}, \cdots, \max\{\gamma_B(u_m), \gamma_C(v_m)\}\}.
\end{aligned}$$

Using the above results, we have

$$\begin{aligned}
&\mu_{A*(B+C)}(x)\\
&= \bigvee_{x=\sum_{\text{finite}} a_i b_i} \min\left\{\begin{array}{l} \mu_A(a_1), \mu_A(a_2), \cdots, \mu_A(a_m), \\ \mu_{B+C}(b_1), \mu_{B+C}(b_2), \cdots, \mu_{B+C}(b_m) \end{array}\right\}\\
&= \bigvee_{x=\sum_{\text{finite}} a_i b_i} \Bigg(\bigvee_{b_1=u_1+v_1} \Bigg(\bigvee_{b_2=u_2+v_2} \Bigg(\cdots\\
&\qquad \Bigg(\bigvee_{b_m=u_m+v_m} \min\{\mu_A(a_1), \mu_A(a_2), \cdots, \mu_A(a_m),\\
&\qquad\qquad \min\{\mu_B(u_1), \mu_C(v_1)\}, \cdots, \min\{\mu_B(u_m), \mu_C(v_m)\}\}\Bigg) \cdots \Bigg)\Bigg)\Bigg)\\
&\le \mu_{(A*B)+(A*C)}(x),
\end{aligned}$$

and

$$\begin{aligned}
&\gamma_{A*(B+C)}(x)\\
&= \bigwedge_{x=\sum_{\text{finite}} a_i b_i} \min\left\{\begin{array}{l} \mu_A(a_1), \mu_A(a_2), \cdots, \mu_A(a_m), \\ \mu_{B+C}(b_1), \mu_{B+C}(b_2), \cdots, \mu_{B+C}(b_m) \end{array}\right\}\\
&= \bigwedge_{x=\sum_{\text{finite}} a_i b_i} \Bigg(\bigwedge_{b_1=u_1+v_1} \Bigg(\bigwedge_{b_2=u_2+v_2} \Bigg(\cdots\\
&\qquad \Bigg(\bigwedge_{b_m=u_m+v_m} \max\{\gamma_A(a_1), \gamma_A(a_2), \cdots, \gamma_A(a_m),\\
&\qquad\qquad \max\{\gamma_B(u_1), \gamma_C(v_1)\}, \cdots, \max\{\gamma_B(u_m), \gamma_C(v_m)\}\}\Bigg) \cdots \Bigg)\Bigg)\Bigg)\\
&\ge \gamma_{(A*B)+(A*C)}(x).
\end{aligned}$$

If x is not expressible as in (3), then $\mu_{A*(B+C)}(x) = 0 = \mu_{(A*B)+(A*C)}(x)$ and $\gamma_{A*(B+C)}(x) = 1 = \gamma_{(A*B)+(A*C)}(x)$. Hence we have

$$A * (B + C) \subseteq (A * B) + (A * C).$$

Now suppose that $\mu_C(0) = \mu_B(0) \ge \max\{\mu_B(x), \mu_C(x)\}$ and $\gamma_C(0) = \gamma_B(0) \le \min\{\gamma_B(x), \gamma_C(x)\}$ for all $x \in R$. Let $x \in R$ and suppose that x is expressible in the form

$$x = \sum_{i=1}^{m} a_i b_i + \sum_{j=1}^{n} c_j d_j \tag{4}$$

where $a_ib_i \neq 0$ and $c_jd_j \neq 0$ for $i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, n$. Then

$$x = \sum_{i=1}^{m} a_i(b_i + 0) + \sum_{j=1}^{n} c_j(d_j + 0),$$

and so

$$\begin{aligned}
&\mu_{A*(B+C)}(x) \\
&= \mu_{A*(B+C)}\left(\textstyle\sum_{i=1}^{m} a_ib_i + \sum_{j=1}^{n} c_jd_j\right) \\
&\geq \min\{\mu_A(a_1), \mu_A(a_2), \cdots, \mu_A(a_m), \mu_A(c_1), \mu_A(c_2), \cdots, \mu_A(c_n), \\
&\qquad \mu_{B+C}(b_1), \mu_{B+C}(b_2), \cdots, \mu_{B+C}(b_m), \\
&\qquad \mu_{B+C}(d_1), \mu_{B+C}(d_2), \cdots, \mu_{B+C}(d_n)\} \\
&\geq \min\{\mu_A(a_1), \mu_A(a_2), \cdots, \mu_A(a_m), \mu_A(c_1), \mu_A(c_2), \cdots, \mu_A(c_n), \\
&\qquad \mu_B(b_1), \mu_C(0), \mu_B(b_2), \mu_C(0), \cdots, \mu_B(b_m), \mu_C(0), \\
&\qquad \mu_B(0), \mu_C(d_1), \mu_B(0), \mu_C(d_2), \cdots, \mu_B(0), \mu_C(d_n)\} \\
&\geq \min\{\mu_A(a_1), \mu_A(a_2), \cdots, \mu_A(a_m), \mu_A(c_1), \mu_A(c_2), \cdots, \mu_A(c_n), \\
&\qquad \mu_B(b_1), \mu_B(b_2), \cdots, \mu_B(b_m), \mu_C(d_1), \mu_C(d_2), \cdots, \mu_C(d_n)\}
\end{aligned}$$

and

$$\begin{aligned}
&\gamma_{A*(B+C)}(x) \\
&= \gamma_{A*(B+C)}\left(\textstyle\sum_{i=1}^{m} a_ib_i + \sum_{j=1}^{n} c_jd_j\right) \\
&\leq \max\{\gamma_A(a_1), \gamma_A(a_2), \cdots, \gamma_A(a_m), \gamma_A(c_1), \gamma_A(c_2), \cdots, \gamma_A(c_n), \\
&\qquad \gamma_{B+C}(b_1), \gamma_{B+C}(b_2), \cdots, \gamma_{B+C}(b_m), \\
&\qquad \gamma_{B+C}(d_1), \gamma_{B+C}(d_2), \cdots, \gamma_{B+C}(d_n)\} \\
&\leq \max\{\gamma_A(a_1), \gamma_A(a_2), \cdots, \gamma_A(a_m), \gamma_A(c_1), \gamma_A(c_2), \cdots, \gamma_A(c_n), \\
&\qquad \gamma_B(b_1), \gamma_C(0), \gamma_B(b_2), \gamma_C(0), \cdots, \gamma_B(b_m), \gamma_C(0), \\
&\qquad \gamma_B(0), \gamma_C(d_1), \gamma_B(0), \gamma_C(d_2), \cdots, \gamma_B(0), \gamma_C(d_n)\} \\
&\leq \max\{\gamma_A(a_1), \gamma_A(a_2), \cdots, \gamma_A(a_m), \gamma_A(c_1), \gamma_A(c_2), \cdots, \gamma_A(c_n), \\
&\qquad \gamma_B(b_1), \gamma_B(b_2), \cdots, \gamma_B(b_m), \gamma_C(d_1), \gamma_C(d_2), \cdots, \gamma_C(d_n)\}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\mu_{(A*B)+(A*C)}(x)\\
&= \bigvee_{x=u+v} \min\{\mu_{A*B}(u), \mu_{A*C}(v)\}\\
&= \bigvee_{x=u+v}\Big(\bigvee_{u=\sum_{i=1}^{m} a_i b_i}\Big(\bigvee_{v=\sum_{j=1}^{n} c_j d_j} \min\{\mu_A(a_1), \mu_A(a_2), \cdots, \mu_A(a_m),\\
&\qquad \mu_A(c_1), \mu_A(c_2), \cdots, \mu_A(c_n), \mu_B(b_1), \mu_B(b_2), \cdots, \mu_B(b_m),\\
&\qquad\qquad \mu_C(d_1), \mu_C(d_2), \cdots, \mu_C(d_n)\}\Big)\Big)\\
&\leq \mu_{A*(B+C)}(x)
\end{aligned}$$

and

$$\begin{aligned}
&\gamma_{(A*B)+(A*C)}(x)\\
&= \bigwedge_{x=u+v} \max\{\gamma_{A*B}(u), \gamma_{A*C}(v)\}\\
&= \bigwedge_{x=u+v}\Big(\bigwedge_{u=\sum_{i=1}^{m} a_i b_i}\Big(\bigwedge_{v=\sum_{j=1}^{n} c_j d_j} \max\{\gamma_A(a_1), \gamma_A(a_2), \cdots, \gamma_A(a_m),\\
&\qquad \gamma_A(c_1), \gamma_A(c_2), \cdots, \gamma_A(c_n), \gamma_B(b_1), \gamma_B(b_2), \cdots, \gamma_B(b_m),\\
&\qquad\qquad \gamma_C(d_1), \gamma_C(d_2), \cdots, \gamma_C(d_n)\}\Big)\Big)\\
&\geq \gamma_{A*(B+C)}(x)
\end{aligned}$$

so that $A*(B+C) = (A*B)+(A*C)$.

(ii) Similar the the proof of (i). □

Lemma 3.5. *If $A = (\mu_A, \gamma_A)$ is an IFS in a ring R, then $A*A \subseteq A$ if and only if it satisfies the following conditions:*

$$\begin{aligned}
&\min\{\mu_A(a_1), \mu_A(a_2), \cdots, \mu_A(a_m), \mu_A(b_1), \mu_A(b_2), \cdots, \mu_A(b_m)\}\\
&\leq \mu_A(a_1b_1 + a_2b_2 + \cdots a_mb_m),
\end{aligned}$$

$$\begin{aligned}
&\max\{\gamma_A(a_1), \gamma_A(a_2), \cdots, \gamma_A(a_m), \gamma_A(b_1), \gamma_A(b_2), \cdots, \gamma_A(b_m)\}\\
&\geq \gamma_A(a_1b_1 + a_2b_2 + \cdots a_mb_m),
\end{aligned}$$

where $a_i, b_i \in R$.

Proof. Straightforward. □

Theorem 3.6. *For an IFS $A = (\mu_A, \gamma_A)$ in a ring R, the following assertions are equivalent:*

(i) $A = (\mu_A, \gamma_A)$ *is an intuitionistic fuzzy subring of* R.

(ii) $A = (\mu_A, \gamma_A)$ *is an intuitionistic fuzzy subgroup of the additive group* $(R, +)$ *and* $A * A \subseteq A$.

Proof. (i) $\Rightarrow$ (ii). The condition

$$\mu_A(x-y) \geq \min\{\mu_A(x), \mu_A(y)\} \text{ and } \gamma_A(x-y) \leq \max\{\mu_A(x), \mu_A(y)\}$$

is equivalent to saying that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy subgroup of $(R, +)$. Let $a_i, b_i \in R$ where $i = 1, 2, \cdots, m$. Then

$$\begin{aligned} &\min\{\mu_A(a_1), \mu_A(a_2), \cdots, \mu_A(a_m), \mu_A(b_1), \mu_A(b_2), \cdots, \mu_A(b_m)\} \\ \leq\ &\min\{\mu_A(a_1b_1), \mu_A(a_2b_2), \cdots, \mu_A(a_mb_m)\} \\ \leq\ &\mu_A(a_1b_1 + a_2b_2 + \cdots, a_mb_m), \end{aligned}$$

and

$$\begin{aligned} &\max\{\gamma_A(a_1), \gamma_A(a_2), \cdots, \gamma_A(a_m), \gamma_A(b_1), \gamma_A(b_2), \cdots, \gamma_A(b_m)\} \\ \geq\ &\max\{\gamma_A(a_1b_1), \gamma_A(a_2b_2), \cdots, \gamma_A(a_mb_m)\} \\ \geq\ &\gamma_A(a_1b_1 + a_2b_2 + \cdots, a_mb_m), \end{aligned}$$

which shows from Lemma 3.5 that $A * A \subseteq A$.

(ii) $\Rightarrow$ (i). Applying Lemma 3.5 and the definition of intuitionistic fuzzy subring, we have the desired result. □

Lemma 3.7. *For any IFS* $A = (\mu_A, \gamma_A)$ *in* R, *the following are equivalent.*

(i) $A \circ A \subseteq A$.

(ii) $\mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y)\}$ *and* $\gamma_A(xy) \leq \max\{\gamma_A(x), \gamma_A(y)\}$ *for all* $x, y \in R$.

Proof. Straightforward. □

Theorem 3.8. *If* $A = (\mu_A, \gamma_A)$ *and* $B = (\mu_B, \gamma_B)$ *are intuitionistic fuzzy subrings of a ring* R *such that* $A * B = B * A$, *then* $A * B$ *is an intuitionistic fuzzy subring of* R.

Proof. For any $x, y \in R$, let $x = \sum_{i=1}^{m} a_i b_i$ and $y = \sum_{j=1}^{n} c_j d_j$. Then

$$x - y = \sum_{i=1}^{m} a_i b_i + \sum_{j=1}^{n} (-c_j) d_j,$$

and so

$$\begin{aligned}
&\mu_{A*B}(x-y) = \mu_{A*B}(\textstyle\sum_{i=1}^{m} a_i b_i + \sum_{j=1}^{n}(-c_j)d_j) \\
&\geq \min\{\mu_A(a_1), \mu_A(a_2), \cdots, \mu_A(a_m), \mu_A(-c_1), \\
&\qquad \mu_A(-c_2), \cdots, \mu_A(-c_n), \mu_B(b_1), \mu_B(b_2), \cdots, \mu_B(b_m), \\
&\qquad\qquad \mu_B(d_1), \mu_B(d_2), \cdots, \mu_B(d_n)\}
\end{aligned}$$

$$\begin{aligned}
&\gamma_{A*B}(x-y) = \gamma_{A*B}(\textstyle\sum_{i=1}^{m} a_i b_i + \sum_{j=1}^{n}(-c_j)d_j) \\
&\leq \max\{\gamma_A(a_1), \gamma_A(a_2), \cdots, \gamma_A(a_m), \gamma_A(-c_1), \\
&\qquad \gamma_A(-c_2), \cdots, \gamma_A(-c_n), \gamma_B(b_1), \gamma_B(b_2), \cdots, \gamma_B(b_m), \\
&\qquad\qquad \gamma_B(d_1), \gamma_B(d_2), \cdots, \gamma_B(d_n)\}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\mu_{A*B}(x-y) \\
&\geq \bigvee_{x=\sum_{\text{finite}} a_i b_i} \Big(\bigvee_{y=\sum_{\text{finite}} c_j d_j} \min\{\mu_A(a_1), \mu_A(a_2), \cdots, \mu_A(a_r), \\
&\qquad \mu_A(-c_1), \mu_A(-c2), \cdots, \mu_A(-c_t), \mu_B(b_1), \\
&\qquad \mu_B(b_2), \cdots, \mu_B(b_r), \mu_B(d_1), \mu_B(d_2), \cdots, \mu_B(d_t)\}\Big) \\
&= \min\{\mu_{A*B}(x), \mu_{A*B}(y)\}
\end{aligned}$$

and

$$\begin{aligned}
&\gamma_{A*B}(x-y) \\
&\leq \bigwedge_{x=\sum_{\text{finite}} a_i b_i} \Big(\bigvee_{y=\sum_{\text{finite}} c_j d_j} \max\{\gamma_A(a_1), \gamma_A(a_2), \cdots, \gamma_A(a_r), \\
&\qquad \gamma_A(-c_1), \gamma_A(-c2), \cdots, \gamma_A(-c_t), \gamma_B(b_1), \\
&\qquad \gamma_B(b_2), \cdots, \gamma_B(b_r), \gamma_B(d_1), \gamma_B(d_2), \cdots, \gamma_B(d_t)\}\Big) \\
&= \max\{\gamma_{A*B}(x), \gamma_{A*B}(y)\}
\end{aligned}$$

so that $A * B$ is an intuitionistic fuzzy subgroup of $(R, +)$. Since $A * B = B * A$ by assumption, it follows from Propositions 3.2 and 3.3 and

Theorem 3.6(ii) that

$$\begin{aligned}(A*B)\circ(A*B) &\subseteq (A*B)*(A*B)=A*(B*A)*B\\ &= A*(A*B)*B=(A*A)*(B*B)\\ &\subseteq A*B\end{aligned}$$

so from Lemma 3.7 that

$$\mu_{A*B}(ab)\geq \min\{\mu_{A*B}(a),\mu_{A*B}(b)\},$$

$$\gamma_{A*B}(ab)\leq \max\{\gamma_{A*B}(a),\gamma_{A*B}(b)\}$$

for all $a,b\in R$. Hence $A*B$ is a intuitionistic fuzzy subring of R. □

Lemma 3.9. *An IFS* $A=(\mu_A,\gamma_A)$ *in a ring* R *is an intuitionistic fuzzy left* (resp. *left*) *ideal of a ring* R *if and only if*

(i) $\mu_A(x-y)\geq \min\{\mu_A(x),\mu_A(y)\}$ *and* $\gamma_A(x-y)\leq \max\{\gamma_A(x),\gamma_A(y)\}$,
(ii) $1_\sim * A\subseteq A$ (resp. $A*1_\sim\subseteq A$).

Proof. Suppose that $A=(\mu_A,\gamma_A)$ is an intuitionistic fuzzy left ideal of R. It is clear that the condition (i) holds. Let x be any element of R. We assume that x is expressible as $x=a_1b_1+a_2b_2+\cdots+a_mb_m$, where $a_i,b_i\in R$ and $a_ib_i\neq 0$. Then

$$\begin{aligned}\mu_{1_\sim * A}(x) &= \bigvee_{x=\sum\limits_{\text{finite}} a_ib_i}\min\left\{\begin{array}{l}1(a_1),1(a_2),\cdots,1(a_m),\\ \mu_A(b_1),\mu_A(b_2),\cdots,\mu_A(b_m)\end{array}\right\}\\ &\leq \bigvee_{x=\sum\limits_{\text{finite}} a_ib_i}\min\{1,\mu_A(a_1b_1),\mu_A(a_2b_2),\cdots,\mu_A(a_mb_m)\}\\ &= \bigvee_{x=\sum\limits_{\text{finite}} a_ib_i}\min\{\mu_A(a_1b_1),\mu_A(a_2b_2),\cdots,\mu_A(a_mb_m)\}\\ &\leq \mu_A(a_1b_1+a_2b_2+\cdots+a_mb_m)\\ &= \mu_A(x),\end{aligned}$$

$$\begin{aligned}
\gamma_{1_\sim * A}(x) &= \bigwedge_{x=\sum_{\text{finite}} a_i b_i} \max\left\{ \begin{array}{l} 0(a_1), 0(a_2), \cdots, 0(a_m), \\ \gamma_A(b_1), \gamma_A(b_2), \cdots, \gamma_A(b_m) \end{array} \right\} \\
&\geq \bigwedge_{x=\sum_{\text{finite}} a_i b_i} \min\{1, \gamma_A(a_1b_1), \gamma_A(a_2b_2), \cdots, \gamma_A(a_mb_m)\} \\
&= \bigwedge_{x=\sum_{\text{finite}} a_i b_i} \min\{\gamma_A(a_1b_1), \gamma_A(a_2b_2), \cdots, \gamma_A(a_mb_m)\} \\
&\geq \gamma_A(a_1b_1 + a_2b_2 + \cdots + a_mb_m) \\
&= \gamma_A(x),
\end{aligned}$$

which shows that $1_\sim * A \subseteq A$. We remark that is x is not expressible as $x = a_1b_1 + a_2b_2 + \cdots + a_mb_m$, then $\mu_{1_\sim * A}(x) = 0 \leq \mu_A(x)$ and $\gamma_{1_\sim * A}(x) = 1 \geq \gamma_A(x)$. Therefore the condition (ii) is valid.

Conversely, assume that (i) and (ii) hold. Let $x, y \in R$. Using the condition (ii), we have

$$\begin{aligned}
\mu_A(xy) &\geq \mu_{1_\sim * A}(xy) \\
&= \bigvee_{xy=\sum_{\text{finite}} a_i b_i} \min\left\{ \begin{array}{l} 1(a_1), 1(a_2), \cdots, 1(a_m), \\ \mu_A(b_1), \mu_A(b_2), \cdots, \mu_A(b_m) \end{array} \right\} \\
&\geq \min\{1(x), \mu_A(y)\} \\
&= \mu_A(y),
\end{aligned}$$

$$\begin{aligned}
\gamma_A(xy) &\leq \gamma_{1_\sim * A}(xy) \\
&= \bigwedge_{xy=\sum_{\text{finite}} a_i b_i} \max\left\{ \begin{array}{l} 0(a_1), 0(a_2), \cdots, 0(a_m), \\ \gamma_A(b_1), \gamma_A(b_2), \cdots, \gamma_A(b_m) \end{array} \right\} \\
&\leq \max\{0(x), \gamma_A(y)\} \\
&= \gamma_A(y).
\end{aligned}$$

This means that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy left ideal of R. In a similar way, we can prove that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy right ideal of R. □

Theorem 3.10. *If $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy right ideal of a ring R and $B = (\mu_B, \gamma_B)$ is an intuitionistic fuzzy left ideal of R, then $A * B \subseteq A \cap B$.*

Proof. Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be IFSs in a ring R and $x \in R$. We assume that x is expressible as $x = a_1b_1 + a_2b_2 + \cdots + a_mb_m$ where $a_i, b_i \in R$ and $a_ib_i \neq 0$. Then

$$\begin{aligned}
&\min\{\mu_A(a_1), \mu_A(a_2), \cdots, \mu_A(a_m), \mu_B(b_1), \mu_B(b_2), \cdots, \mu_B(b_m)\} \\
&\leq \min\{\mu_A(a_1), \mu_A(a_2), \cdots, \mu_A(a_m)\} \\
&\leq \min\{\mu_A(a_1b_1), \mu_A(a_2b_2), \cdots, \mu_A(a_mb_m)\} \\
&\leq \mu_A(a_1b_1 + a_2b_2, \cdots, a_mb_m)
\end{aligned} \tag{5}$$

and

$$\begin{aligned}
&\max\{\gamma_A(a_1), \gamma_A(a_2), \cdots, \gamma_A(a_m), \gamma_B(b_1), \gamma_B(b_2), \cdots, \gamma_B(b_m)\} \\
&\geq \max\{\gamma_A(a_1), \gamma_A(a_2), \cdots, \gamma_A(a_m)\} \\
&\geq \max\{\gamma_A(a_1b_1), \gamma_A(a_2b_2), \cdots, \gamma_A(a_mb_m)\} \\
&\geq \gamma_A(a_1b_1 + a_2b_2, \cdots, a_mb_m)
\end{aligned} \tag{6}$$

It follows from (5) and (6) that

$$\begin{aligned}
&\mu_{(A*B)}(x) \\
&= \bigvee_{\substack{x=\sum a_ib_i \\ \text{finite}}} \min\left\{ \begin{array}{l} \mu_A(a_1), \mu_A(a_2), \cdots, \mu_A(a_m), \\ \mu_B(b_1), \mu_B(b_2), \cdots, \mu_B(b_m) \end{array} \right\} \\
&\leq \min\{\mu_A(a_1), \mu_A(a_2), \cdots, \mu_A(a_m), \mu_B(b_1), \mu_B(b_2), \cdots, \mu_B(b_m)\} \\
&\leq \mu_A(a_1b_1 + a_2b_2, \cdots, a_mb_m) \\
&= \mu_A(x)
\end{aligned}$$

and

$$\begin{aligned}
&\gamma_{(A*B)}(x) \\
&= \bigwedge_{\substack{x=\sum a_ib_i \\ \text{finite}}} \max\left\{ \begin{array}{l} \gamma_A(a_1), \gamma_A(a_2), \cdots, \gamma_A(a_m), \\ \gamma_B(b_1), \gamma_B(b_2), \cdots, \mu_B(b_m) \end{array} \right\} \\
&\geq \max\{\gamma_A(a_1), \gamma_A(a_2), \cdots, \gamma_A(a_m), \gamma_B(b_1), \gamma_B(b_2), \cdots, \gamma_B(b_m)\} \\
&\geq \gamma_A(a_1b_1 + a_2b_2, \cdots, a_mb_m) \\
&= \gamma_A(x)
\end{aligned}$$

Hence $\mu_{A*B}(x) \leq \mu_A(x)$ and $\gamma_{A*B}(x) \geq \gamma_A(x)$. Similarly $\mu_{A*B}(x) \leq \mu_A(x)$ and $\gamma_{A*B}(x) \geq \gamma_A(x)$. Thus $\mu_{A*B}(x) \leq \min\{\mu_A(x), \mu_B(x)\} =$

$(\mu_A \wedge \mu_B)(x)$ and $\gamma_{A*B}(x) \geq \max\{\gamma_A(x), \gamma_B(x)\} = (\gamma_A \vee \gamma_B)(x)$. Therefore $A * B \subseteq A \cap B$. □

Theorem 3.11. *If $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy left ideal of a ring R and $B = (\mu_B, \gamma_B)$ is an intuitionistic fuzzy right ideal of R, then $A * B$ is an intuitionistic fuzzy ideal of R.*

Proof. Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be IFSs in a ring R. From the first part of the proof of Theorem 3.8, we see that $A * B$ is an intuitionistic fuzzy subgroup of $(R, +)$. For any $x, y \in R$,let $x = a_1b_1 + a_2b_2 + \cdots + a_mb_m$ where $a_i, b_i \in R$ and $a_ib_i \neq 0$. Then $xy = a_1(b_1y) + a_2(b_2y) + \cdots + a_m(b_my)$. and so

$$\begin{aligned}
&\min\{\mu_A(a_1), \mu_A(a_2), \cdots, \mu_A(a_m), \mu_B(b_1), \mu_B(b_2), \cdots, \mu_B(b_m)\} \\
&\leq \min\{\mu_A(a_1), \mu_A(a_2), \cdots, \mu_A(a_m), \mu_B(b_1y), \mu_B(b_2y), \cdots, \mu_B(b_my)\} \\
&\leq \mu_{A*B}(xy)
\end{aligned}$$

and

$$\begin{aligned}
&\max\{\gamma_A(a_1), \gamma_A(a_2), \cdots, \gamma_A(a_m), \gamma_B(b_1), \gamma_B(b_2), \cdots, \gamma_B(b_m)\} \\
&\geq \max\{\gamma_A(a_1), \gamma_A(a_2), \cdots, \gamma_A(a_m), \gamma_B(b_1y), \gamma_B(b_2y), \cdots, \gamma_B(b_my)\} \\
&\geq \gamma_{A*B}(xy)
\end{aligned}$$

It follows that

$$\begin{aligned}
&\mu_{(A*B)}(x) \\
&= \bigvee_{\substack{x=\sum a_ib_i \\ \text{finite}}} \min\left\{\begin{array}{l} \mu_A(a_1), \mu_A(a_2), \cdots, \mu_A(a_m), \\ \mu_B(b_1), \mu_B(b_2), \cdots, \mu_B(b_m) \end{array}\right\} \\
&\min\{\mu_A(a_1), \mu_A(a_2), \cdots, \mu_A(a_m), \mu_B(b_1), \mu_B(b_2), \cdots, \mu_B(b_m)\} \\
&\leq \min\{\mu_A(a_1), \mu_A(a_2), \cdots, \mu_A(a_m), \mu_B(b_1y), \mu_B(b_2y), \cdots, \mu_B(b_my)\} \\
&\leq \mu_{A*B}(xy)
\end{aligned}$$

and

$$\gamma_{(A*B)}(x)$$
$$= \bigvee_{x=\sum\limits_{\text{finite}} a_ib_i} \max\left\{ \begin{array}{l} \gamma_A(a_1), \gamma_A(a_2), \cdots, \gamma_A(a_m), \\ \gamma_B(b_1), \gamma_B(b_2), \cdots, \gamma_B(b_m) \end{array} \right\}$$
$$\max\{\gamma_A(a_1), \gamma_A(a_2), \cdots, \gamma_A(a_m), \gamma_B(b_1), \gamma_B(b_2), \cdots, \gamma_B(b_m)\}$$
$$\geq \max\{\gamma_A(a_1), \gamma_A(a_2), \cdots, \gamma_A(a_m), \gamma_B(b_1y), \gamma_B(b_2y), \cdots, \gamma_B(b_my)\}$$
$$\geq \gamma_{A*B}(xy).$$

Hence $\mu_{A*B}(x) \leq \mu_{A*B}(xy)$ and $\gamma_{A*B}(x) \geq \gamma_{A*B}(xy)$. Simlary ,we get $\mu_{A*B}(y) \leq \mu_{A*B}(xy)$ and $\gamma_{A*B}(y) \geq \gamma_{A*B}(xy)$. There $A * B$ is a intuitionistic fuzzy ideal of R □

Definition 3.12. An IFS $A = (\mu_A, \gamma_A)$ in a ring R is called an *intuitionistic fuzzy quasi-ideal* of R if

(i) $(\forall x, y \in R)$ $(\mu_A(x - y) \geq \min\{\mu_A(x), \mu_A(y)\})$,
(ii) $(\forall x, y \in R)$ $(\gamma_A(x - y) \leq \max\{\gamma_A(x), \gamma_A(y)\})$,
(iii) $(A * 1_\sim) \cap (1_\sim * A) \subseteq A$.

Example 3.13. Let

$$R = \left\{0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, b = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, c = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\right\}.$$

Then $R = \{0, a, b, c\}$ is a ring of matrices under matrix addition and multiplication modulo 2.

$+$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	0	0	0
c	0	a	b	c

Define an intuitionistic fuzzy set $Q = (\mu_Q, \gamma_Q)$ in R as follows:

$$\mu_Q(0) = 1, \mu_Q(a) = 0.6, \mu_Q(b) = \mu_Q(c) = 0,$$

$$\gamma_Q(0) = 0, \gamma_Q(a) = 0.1, \gamma_Q(b) = \gamma_Q(c) = 1.$$

It is easy to verify that $Q = (\mu_Q, \gamma_Q)$ is an intuitionistic fuzzy quasi-ideal of R which is not an intuitionistic fuzzy right ideal of R.

Definition 3.14. An IFS $A = (\mu_A, \gamma_A)$ in a ring R is called an *intuitionistic fuzzy bi-ideal* of R if

(i) $(\forall x, y \in R)$ $(\mu_A(x - y) \geq \min\{\mu_A(x), \mu_A(y)\})$,
(ii) $(\forall x, y \in R)$ $(\gamma_A(x - y) \leq \max\{\gamma_A(x), \gamma_A(y)\})$,
(iii) $A * A \subseteq A$ and $A * 1_\sim * A \subseteq A$.

Example 3.15. Let $R = \{0, a, b, c\}$ be a ring with the following tables:

$+$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	0	0	b
b	0	0	0	0
c	0	b	0	a

Let $B = (\mu_B, \gamma_B)$ be an IFS in R defined by

$$\mu_B(0) = 1, \mu_B(a) = 0.6, \mu_B(b) = \mu_B(c) = 0,$$

$$\gamma_B(0) = 0, \gamma_B(a) = 0.2, \gamma_B(b) = \gamma_B(c) = 1.$$

It can be easily seen that $B = (\mu_B, \gamma_B)$ is an intuitionistic fuzzy bi-ideal of R which is not an intuitionistic fuzzy quasi ideal of R.

Lemma 3.16. *Every intuitionistic fuzzy left* (resp. *right, two-sided*) *ideal of a ring R is an intuitionistic fuzzy quasi-ideal of R.*

Proof. Straightforward. □

Lemma 3.17. *Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be any intuitionistic fuzzy right ideal and any intuitionistic fuzzy left ideal of a ring R, respectively. Then $A \cap B$ is an intuitionistic fuzzy quasi-ideal of R.*

Proof. Straightforward. □

Theorem 3.18. *Any intuitionistic fuzzy quasi-ideal of a ring R is an intuitionistic fuzzy bi-ideal of R.*

Proof. Let A be any intuitionistic fuzzy quasi-ideal of R. Then

$$A * A = (A * A) \cap (A * A) \subseteq (A * 1_\sim) \cap (1_\sim * A) \subseteq A.$$

Since $A*1_\sim*A \subseteq A*(1_\sim*1_\sim) \subseteq A*1_\sim$ and $A*1_\sim*A \subseteq (1_\sim*1_\sim)*A \subseteq 1_\sim * A$, we have $1_\sim * A * 1_\sim \subseteq (A * 1_\sim) \cap (1_\sim * A) \subseteq A$. Therefore $A * B$ is an intuitionistic fuzzy bi-ideal of R. □

The converse of Theorem 3.18 is not true in general. For example, the intuitionistic fuzzy bi-ideal $B = (\mu_B, \gamma_B)$ described in Example 3.15 is not an intuitionistic fuzzy quasi ideal of R.

Theorem 3.19. *The intrinsic product of two intuitionistic fuzzy quasi-ideals of a ring R is an intuitionistic fuzzy bi-ideal of R.*

Proof. Let A and B be any two intuitionistic fuzzy quasi-ideals of R. From Theorem 3.18, it follows that B is an intuitionistic fuzzy bi-ideal of R. Then we have $A * 1_\sim * A \subseteq A$ and so

$$(A * B) * (A * B) = (A * B * A) * B \subseteq (A * 1_\sim * A) * B \subseteq A * B$$

and

$$\begin{aligned}(A * B) * 1_\sim * (A * B) &= A * \{B * (A * 1_\sim) * B\} \\ &\subseteq A * \{B * (1_\sim * 1_\sim) * B\} \\ &\subseteq A * (B * 1_\sim * B) \\ &\subseteq A * B.\end{aligned}$$

Therefore $A * B$ is an intuitionistic fuzzy bi-ideal of R. □

4. Regular rings

A ring R is said to be *regular* if for each element x of R, there exists an element $a \in R$ such that $x = xax$. A ring R is regular if and only if $AB = A \cap B$ whenever A is a right ideal of R and B is a left ideal of R (see [6]). The corresponding intuitionistic fuzzy ideals nicely simulate this basic result.

For a subset X of ring R,we denote $\widetilde{X} = \{\langle x, \mu_{\widetilde{X}}(x), \gamma_{\widetilde{X}}(x)\rangle \mid x \in R\}$ defined by

$$\mu_{\widetilde{X}}(x) := \begin{cases} 1 & \text{if } x \in X \\ 0 & \text{otherwise} \end{cases} \quad \text{and } \gamma_{\widetilde{X}}(x) := \begin{cases} 0 & \text{if } x \in X \\ 1 & \text{otherwise} \end{cases}$$

for all $x \in R$. For the sake of simplicity, we shall use the symbol $\widetilde{X} = (\mu_{\widetilde{X}}, \gamma_{\widetilde{X}})$ for the $\widetilde{X} = \{\langle x, \mu_{\widetilde{X}}(x), \gamma_{\widetilde{X}}(x)\rangle \mid x \in X\}$.

Lemma 4.1. *Let A be a nonempty subset of R, Then*

(i) *A is a subring of R if and only if $\widetilde{A}$ is intuitionistic fuzzy subring of R*

(ii) *A is a left(right) ideal of R if and only if $\widetilde{A}$ is intuitionistic fuzzy left(right) ideal of R*

(iii) *A is a quas ideal of R if and only if $\widetilde{A}$ is intuitionistic fuzzy quasi ideal of R*

Proof. Straightforward. □

Theorem 4.2. *A ring R is regular if and only if $A * B = A \cap B$ for every intuitionistic fuzzy right ideal $A = (\mu_A, \gamma_A)$ of R and every intuitionistic fuzzy left ideal $B = (\mu_B, \gamma_B)$ of R.*

Proof. Suppose that R is a regular ring. By Theorem 3.10, we have $A * B \subseteq A \cap B$. To prove the opposite inclusion, let $x \in R$. Since R is a regular ring, there exists $y \in R$ such that $x = xyx$. Hence

$$\mu_{(A*B)}(x) = \bigvee_{x=\sum_{\text{finite}} a_i b_i} \min\{\mu_A(a_1), \mu_A(a_2), \cdots, \mu_A(a_m), \mu_B(b_1), \mu_B(b_2), \cdots, \mu_B(b_m)\}$$

$$\geq \min\{\mu_A(xy), \mu_B(x)\} \geq \min\{\mu_A(x), \mu_B(x)\} = (\mu_A \wedge \mu_B)(x)$$

and

$$\gamma_{(A*B)}(x) = \bigwedge_{x=\sum_{\text{finite}} a_i b_i} \max\{\gamma_A(a_1), \gamma_A(a_2), \cdots, \gamma_A(a_m), \gamma_B(b_1), \gamma_B(b_2), \cdots, \gamma_B(b_m)\}$$

$$\leq \max\{\gamma_A(x), \gamma_B(yx)\} \leq \max\{\gamma_A(x), \gamma_B(x)\} = (\gamma_A \vee \gamma_B)(x).$$

Hence $\mu_{A*B}(x) \geq (\mu_A \wedge \mu_B)(x)$ and $\gamma_{A*B}(x) \leq (\gamma_A \vee \gamma_B)(x)$ for all $x \in R$. This show that $A * B \supseteq A \cap B$. Therefore $A * B = A \cap B$. Conversely, suppose that $A * B = A \cap B$ whenever A is an intuitionistic fuzzy right ideal and B is an intuitionistic fuzzy left ideal of R. Let U be a right ideal and V be a left ideal of R. By Lemma 4.1(ii), $\widetilde{U} = (\mu_{\widetilde{U}}, \gamma_{\widetilde{U}})$ is an intuitionistic fuzzy right ideal and $\widetilde{V} = (\mu_{\widetilde{V}}, \gamma_{\widetilde{V}})$ is an intuitionistic fuzzy left ideal of R. Therefore we get $\widetilde{U} \cap \widetilde{V} = \widetilde{U} * \widetilde{V}$. We always have $UV \subset U \cap V$. We show that $U \cap V \subseteq UV$. Let $x \in U \cap V$. Since

$$\mu_{\widetilde{U}*\widetilde{V}}(x) = \bigvee_{x=\sum_{\text{finite}} a_i b_i} \min\{\mu_{\widetilde{U}}(a_1), \mu_{\widetilde{U}}(a_2), \cdots, \mu_{\widetilde{U}}(a_m), \mu_{\widetilde{V}}(b_1), \mu_{\widetilde{V}}(b_2), \cdots, \mu_{\widetilde{V}}(b_m)\}$$

$$= (\mu_{\widetilde{U}} \wedge \mu_{\widetilde{V}})(x) = \min\{\mu_{\widetilde{U}}(x), \mu_{\widetilde{V}}(x)\} = \min\{1, 1\} = 1$$

and

$$\gamma_{\widetilde{U}*\widetilde{V}}(x) = \bigwedge_{x=\sum\limits_{\text{finite}} a_ib_i} \max\{\gamma_{\widetilde{U}}(a_1), \gamma_{\widetilde{U}}(a_2), \cdots, \gamma_{\widetilde{U}}(a_m), \gamma_{\widetilde{V}}(b_1), \gamma_{\widetilde{V}}(b_2), \cdots, \gamma_{\widetilde{V}}(b_m)\}$$

$$= (\gamma_{\widetilde{U}} \vee \gamma_{\widetilde{V}})(x) = \max\{\gamma_{\widetilde{U}}(x), \gamma_{\widetilde{V}}(x)\} = \max\{0, 0\} = 0,$$

there must exist $a_i, b_i \in R$, for which $\mu_{\widetilde{U}}(a_i) = \mu_{\widetilde{V}}(b_i) = 1$ and $\gamma_{\widetilde{U}}(a_i) = \gamma_{\widetilde{V}}(b_i) = 0$. This implies that $x = \sum\limits_{\text{finite}} a_ib_i \in UV$. Accordingly, $U \cap V \subseteq UV$. Therefore R is a regular ring. □

Corollary 4.3. *For a ring R, the following assertions are valid.*

(i) *If R is a commutative ring such that $A * B = A \cap B$ for all intuitionistic fuzzy ideals A, B of R, then R is a regular ring.*
(ii) *If R is a regular ring, then every intuitionistic fuzzy two-sided ideal of R is idempotent.*
(iii) *If R is a commutative ring such that $A^2 = A$ for all intuitionistic fuzzy ideals of R then R is a regular ring.*

Proof. Straightforward. □

Theorem 4.4. *For a ring R, the following conditions are equivalent:*

(i) *R is a regular rings,*
(ii) *$A = A * 1_\sim * A$ for every intuitionistic fuzzy bi-ideal A of R,*
(iii) *$A = A * 1_\sim * A$ for every intuitionistic fuzzy quasi-ideal A of R.*

Proof. (i) $\Rightarrow$ (ii). Let $A = (\mu_A, \gamma_A)$ be any intuitionistic fuzzy bi-ideal of R. Since A is intuitionistic fuzzy bi-ideal of R, we have $A * 1_\sim * A \subseteq A$ To prove the opposite inclusion, let a any element of R. Then since R is regular, there exists an element $x \in R$ such that $a = axa$. Then we have

$$\begin{aligned}
\mu_{A*1_\sim*A}(a) &= \bigvee_{x=\sum\limits_{\text{finite}} a_i b_i} \min\left\{\begin{array}{l}\mu_A(a_1), \mu_A(a_2), \cdots, \mu_A(a_m), \\ \mu_{1_\sim*A}(b_1), \mu_{1_\sim*A}(b_2), \cdots, \mu_{1_\sim*A}(b_m)\end{array}\right\} \\
&\geq \min\{\mu_A(a), \mu_{1_\sim*A}(xa)\} \\
&= \min\{\mu_A(a), \bigvee_{xa=\sum\limits_{\text{finite}} p_i q_i} \min\{1(p_1), 1(p_2), \cdots, 1(p_m), \\
&\qquad \mu_A(q_1), \mu_A(q_2), \cdots, \mu_A(q_m)\}\} \\
&\geq \min\{\mu_A(a), \min\{1(x), \mu_A(a)\}\} \\
&= \min\{\mu_A(a), \min\{1, \mu_A(a)\}\} \\
&= \mu_A(a),
\end{aligned}$$

and

$$\begin{aligned}
\gamma_{A*1_\sim*A}(a) &= \bigwedge_{x=\sum\limits_{\text{finite}} a_i b_i} \max\left\{\begin{array}{l}\gamma_A(a_1), \gamma_A(a_2), \cdots, \gamma_A(a_m), \\ \gamma_{1_\sim*A}(b_1), \gamma_{1_\sim*A}(b_2), \cdots, \gamma_{1_\sim*A}(b_m)\end{array}\right\} \\
&\leq \max\{\gamma_A(a), \gamma_{1_\sim*A}(xa)\} \\
&= \max\{\gamma_A(a), \bigwedge_{xa=\sum\limits_{\text{finite}} p_i q_i} \max\{0(p_1), 0(p_2), \cdots, 0(p_m), \\
&\qquad \mu_A(q_1), \gamma_A(q_2), \cdots, \gamma_A(q_m)\}\} \\
&\leq \max\{\gamma_A(a), \max\{0(x), \gamma_A(a)\}\} \\
&= \max\{\gamma_A(a), \max\{0, \gamma_A(a)\}\} \\
&= \gamma_A(a),
\end{aligned}$$

This is shows that $A * 1_\sim * A \subset A$. Therefore $A = A * 1_\sim * A$.

(ii)$\Rightarrow$ (iii) Since any intuitionistic fuzzy quasi-ideal of R is an intuitionistic fuzzy bi-ideal of R by Theorem 3.18, the implication (ii)$\Rightarrow$ (iii) is valid.

(iii)$\Rightarrow$ (i). Let Q be any quasi-ideal of R, and a any element of Q. By Lemma 4.1(iii), $\widetilde{Q}$ is an intuitionistic fuzzy quasi-ideal of R. Then we have

$$\begin{aligned}\mu_{\tilde{Q}*1_{\sim}*\tilde{Q}}(a) &= \bigvee_{x=\sum_{\text{finite}} a_ib_i} \min\left\{\begin{array}{l}\mu_{\tilde{Q}}(a_1), \mu_{\tilde{Q}}(a_2), \cdots, \mu_{\tilde{Q}}(a_m), \\ \mu_{1_{\sim}*\tilde{Q}}(b_1), \mu_{1_{\sim}*\tilde{Q}}(b_2), \cdots, \mu_{1_{\sim}*\tilde{Q}}(b_m)\end{array}\right\} \\ &= \mu_{\tilde{Q}}(a) = 1\end{aligned}$$

and

$$\begin{aligned}\gamma_{\tilde{Q}*1_{\sim}*\tilde{Q}}(a) &= \bigwedge_{x=\sum_{\text{finite}} a_ib_i} \min\left\{\begin{array}{l}\gamma_{\tilde{Q}}(a_1), \gamma_{\tilde{Q}}(a_2), \cdots, \gamma_{\tilde{Q}}(a_m), \\ \gamma_{1_{\sim}*\tilde{Q}}(b_1), \gamma_{1_{\sim}*\tilde{Q}}(b_2), \cdots, \gamma_{1_{\sim}*\tilde{Q}}(b_m)\end{array}\right\} \\ &= \gamma_{\tilde{Q}}(a) = 0.\end{aligned}$$

This implies that there exist $a_i, b_i \in R$ such that $\mu_{\tilde{Q}}(a_i) = \mu_{1_{\sim}*\tilde{Q}}(b_i) = 1$ and $\gamma_{\tilde{Q}}(a_i) = \gamma_{1_{\sim}*\tilde{Q}}(b_i) = 0$ with $a = \sum_{\text{finite}} a_ib_i$. Since

$$1 = \mu_{1_{\sim}*\tilde{Q}}(b_i) = \bigvee_{b_i=\sum_{\text{finite}} p_iq_i} \min\left\{\begin{array}{l}1(p_1), 1(p_2), \cdots, 1(p_m), \\ \mu_{\tilde{Q}}(q_1), \mu_{\tilde{Q}}(q_2), \cdots, \mu_{\tilde{Q}}(q_m)\end{array}\right\}$$

and

$$0 = \mu_{1_{\sim}*\tilde{Q}}(b_i) = \bigwedge_{b_i=\sum_{\text{finite}} p_iq_i} \max\left\{\begin{array}{l}1(p_1), 1(p_2), \cdots, 1(p_m), \\ \gamma_{\tilde{Q}}(q_1), \gamma_{\tilde{Q}}(q_2), \cdots, \gamma_{\tilde{Q}}(q_m)\end{array}\right\},$$

there exist $p_1, q_i \in R$ such that $p_i \in R$ and $q_i \in Q$ with $b_i = \sum_{\text{finite}} p_iq_i$. Hence $a_i, q_i \in Q$ and $p_i \in R$. Then we have

$$a = \sum_{\text{finite}} a_ib_i = \sum_{\text{finite}} a_i(\sum_{\text{finite}} p_iq_i) = \sum_{\text{finite}} a_i(p_iq_i) \in QRQ$$

and so , $a \in QRQ$. Thus $Q \subseteq QRQ$. On the other hand, Q is a quasi-ideal of R,

$$QRQ \subseteq QR \cap RQ \subseteq Q,$$

and so we have $Q = QRQ$. Then it follows from [11] that R is regular.

□

Theorem 4.5. *For a ring R, the following conditions are equivalent:*

(i) *R is regular.*

(ii) $A \cap B = B * A * B$ *for every intuitionistic fuzzy ideal* A *of* R *and every intuitionistic fuzzy bi-ideal* B *of* R.

(iii) $A \cap B = B * A * B$ *for every intuitionistic fuzzy ideal* A *of* R *and every intuitionistic fuzzy quasi-ideal* B *of* R.

Proof. (i)$\Rightarrow$ (ii). Let $A = (\mu_A, \gamma_A)$ and $B = (\gamma_B, \gamma_B)$ be any intuitionistic fuzzy ideal and any intuitionistic fuzzy bi-ideal of R, respectively. Then

$$B * A * B \subseteq B * 1_\sim * B \subseteq B,$$

and

$$B * A * B \subseteq 1_\sim * A * 1_\sim \subseteq A,$$

Thus we have $B * A * B \subseteq A \cap B$. In order to see that the converse inclusion holds, let a be any element of R. Then, since R is regular, there exists an element x in R such that $a = axa (= axaxa)$. Since A is an intuitionistic fuzzy ideal of R, we get $\mu_A(xax) \leq \mu_A(xa) \leq \mu_A(a)$ and $\gamma_A(xax) \geq \gamma_A(xa) \geq \gamma_A(a)$, so

$$\begin{aligned}
\mu_{B*A*B}(a) &= \bigvee_{a=\sum\limits_{\text{finite}} a_i b_i} \min\left\{ \begin{array}{l} \mu_B(a_1), \mu_B(a_2), \cdots, \mu_B(a_m), \\ \mu_{A*B}(b_1), \mu_{A*B}(b_2), \cdots, \mu_{A*B}(b_m) \end{array} \right\} \\
&\geq \min\{\mu_B(a), \mu_{A*B}(xaxa)\} \\
&= \min\{\mu_B(a), \bigvee_{xaxa=\sum\limits_{\text{finite}} p_i q_i} \min\{\mu_A(p_1), \mu_A(p_2), \cdots, \\
&\qquad \mu_A(p_m), \mu_B(q_1), \mu_B(q_2), \cdots, \mu_B(q_m)\}\} \\
&\geq \min\{\mu_B(a), \min\{\mu_A(xax), \mu_B(a)\}\} \\
&\geq \min\{\mu_B(a), \min\{\mu_A(a), \mu_B(a)\}\} \\
&= \min\{\mu_B(a), \mu_A(a)\} \\
&= (\mu_B \wedge \mu_A)(a),
\end{aligned}$$

and

$$\begin{aligned}
\gamma_{B*A*B}(a) &= \bigwedge_{a=\sum\limits_{\text{finite}} a_i b_i} \max\left\{ \begin{array}{l} \gamma_B(a_1), \gamma_B(a_2), \cdots, \gamma_B(a_m), \\ \gamma_{A*B}(b_1), \gamma_{A*B}(b_2), \cdots, \gamma_{A*B}(b_m) \end{array} \right\} \\
&\leq \max\{\gamma_B(a), \gamma_{A*B}(xaxa)\} \\
&= \max\{\gamma_B(a), \bigwedge_{xaxa=\sum\limits_{\text{finite}} p_i q_i} \max\{\gamma_A(p_1), \gamma_A(p_2), \cdots, \\
&\qquad \gamma_A(p_m), \gamma_B(q_1), \gamma_B(q_2), \cdots, \gamma_B(q_m)\}\} \\
&\leq \max\{\gamma_B(a), \max\{\gamma_A(xax), \gamma_B(a)\}\} \\
&\geq \min\{\gamma_B(a), \min\{\gamma_A(a), \gamma_B(a)\}\} \\
&= \min\{\gamma_B(a), \gamma_A(a)\} \\
&= (\gamma_B \vee \gamma_A)(a),
\end{aligned}$$

Hence $A \cap B \supseteq B * A * B$. Therefore $A \cap B = B * A * B$.

(ii) $\Rightarrow$ (iii) Since any intuitionistic fuzzy quasi-ideal of R is an intuitionistic fuzzy bi-ideal of R by Theorem 3.18, it follows that (ii) implies (iii).

(iii) $\Rightarrow$ (i) Let U and V be any ideal and any quasi-ideal of R. By Lemma 4.1(iii), $\widetilde{U}$ is an intuitionistic fuzzy ideal and $\widetilde{V}$ is an intuitionistic fuzzy quasi ideal of R. Let a be any element of $U \cap V$.

$$\begin{aligned}
\mu_{\widetilde{V}*\widetilde{U}*\widetilde{V}}(a) &= \bigvee_{a=\sum\limits_{\text{finite}} a_i b_i} \min\left\{ \begin{array}{l} \mu_{\widetilde{V}}(a_1), \mu_{\widetilde{V}}(a_2), \cdots, \mu_{\widetilde{V}}(a_m), \\ \mu_{\widetilde{U}*\widetilde{V}}(b_1), \mu_{\widetilde{U}*\widetilde{V}}(b_2), \cdots, \mu_{\widetilde{U}*\widetilde{V}}(b_m) \end{array} \right\} \\
&= (\mu_{\widetilde{U}} \wedge \mu_{\widetilde{V}})(a) \\
&= \min\{\mu_{\widetilde{U}}(a), \mu_{\widetilde{V}}(a)\} \\
&= \min\{1, 1\} \\
&= 1
\end{aligned}$$

and

$$\begin{aligned}\gamma_{\widetilde{V}*\widetilde{U}*\widetilde{V}}(a) &= \bigwedge_{a=\sum\limits_{\text{finite}} a_i b_i} \max\left\{\begin{array}{l} \mu_{\widetilde{V}}(a_1), \mu_{\widetilde{V}}(a_2), \cdots, \mu_{\widetilde{V}}(a_m), \\ \gamma_{\widetilde{U}*\widetilde{V}}(b_1), \gamma_{\widetilde{U}*\widetilde{V}}(b_2), \cdots, \gamma_{\widetilde{U}*\widetilde{V}}(b_m) \end{array}\right\} \\ &= (\gamma_{\widetilde{U}} \vee \gamma_{\widetilde{V}})(a) \\ &= \max\{\gamma_{\widetilde{U}}(a), \gamma_{\widetilde{V}}(a)\} \\ &= \max\{0, 0\} \\ &= 0\end{aligned}$$

This implies that there exists $a_i, b_i \in R$ such that $\mu_{\widetilde{V}}(a_i) = \mu_{\widetilde{U}*\widetilde{V}}(b_i) = 1$ and $\gamma_{\widetilde{V}}(a_i) = \gamma_{\widetilde{U}*\widetilde{V}}(b_i) = 0$ with $a = \sum\limits_{\text{finite}} a_i b_i$. Since

$$1 = \mu_{\widetilde{U}*\widetilde{V}}(b_i) = \bigvee_{b_i=\sum\limits_{\text{finite}} p_i q_i} \min\left\{\begin{array}{l} \mu_{\widetilde{U}}(p_1), \mu_{\widetilde{U}}(p_2), \cdots, \mu_{\widetilde{U}}(p_m), \\ \mu_{\widetilde{V}}(q_1), \mu_{\widetilde{V}}(q_2), \cdots, \mu_{\widetilde{V}}(q_m) \end{array}\right\}$$

and

$$1 = \gamma_{\widetilde{U}*\widetilde{V}}(b_i) = \bigwedge_{b_i=\sum\limits_{\text{finite}} p_i q_i} \max\left\{\begin{array}{l} \gamma_{\widetilde{U}}(p_1), \gamma_{\widetilde{U}}(p_2), \cdots, \gamma_{\widetilde{U}}(p_m), \\ \gamma_{\widetilde{V}}(q_1), \gamma_{\widetilde{V}}(q_2), \cdots, \gamma_{\widetilde{V}}(q_m) \end{array}\right\}$$

there exist $p_1, q_i \in R$ such that $p_i \in U$ and $q_i \in V$ with $b_i = \sum\limits_{\text{finite}} p_i q_i$. Hence $a_i, q_i \in V$ and $p_i \in U$. Then we have

$$a = \sum_{\text{finite}} a_i b_i = \sum_{\text{finite}} a_i (\sum_{\text{finite}} p_i q_i) = \sum_{\text{finite}} a_i (p_i q_i) \in VUV.$$

Thus we have $a \in VUV$, and so $U \cap V = VUV$. Then it follows from [11] that R is regular. □

Theorem 4.6. *For a ring R, the following conditions are equivalent:*

(i) *R is regular ring*

(ii) *$A \cap B \subseteq A * B$ for every intuitionistic fuzzy right ideal A and every intuitionistic fuzzy bi-ideal B of R*

(iii) *$A \cap B \subseteq A * B$ for every intuitionistic fuzzy right ideal A and every intuitionistic fuzzy quasi-ideal B of R*

(iv) $B \cap C \subseteq B * C$ *for every intuitionistic fuzzy left ideal* C *and every intuitionistic fuzzy bi-ideal* B *of* R

(v) $B \cap C \subseteq B * C$ *for every intuitionistic fuzzy left ideal* C *and every intuitionistic fuzzy quasi-ideal* B *of* R

Proof. (i) $\Rightarrow$ (ii). First assume that (i) holds. Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be any intuitionistic fuzzy right ideal and any intuitionistic fuzzy bi-ideal of R, respectively. Let a be any element of R. Then, since R is regular, there exists an element x in R such that $a = axa$. Then we have

$$\begin{aligned} \mu_{A*B}(a) &= \bigvee_{a=\sum\limits_{\text{finite}} a_i b_i} \min\left\{ \begin{array}{l} \mu_A(a_1), \mu_A(a_2), \cdots, \mu_A(a_m), \\ \mu_B(b_1), \mu_B(b_2), \cdots, \mu_B(b_m) \end{array} \right\} \\ &\geq \min\{\mu_B(ax), \mu_B(a)\} \\ &\geq \min\{\mu_A(a), \mu_B(a)\} \\ &= (\mu_A \wedge \mu_B)(a) \end{aligned}$$

and

$$\begin{aligned} \mu_{A*B}(a) &= \bigwedge_{a=\sum\limits_{\text{finite}} a_i b_i} \max\left\{ \begin{array}{l} \mu_A(a_1), \mu_A(a_2), \cdots, \mu_A(a_m), \\ \mu_B(b_1), \mu_B(b_2), \cdots, \mu_B(b_m) \end{array} \right\} \\ &\leq \max\{\mu_B(ax), \mu_B(a)\} \\ &\leq \max\{\mu_A(a), \mu_B(a)\} \\ &= (\mu_A \vee \mu_B)(a) \end{aligned}$$

and so we have $A \cap B \subseteq A * B$

(ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v). Straightforward

(v) $\Rightarrow$ (i). Since any intuitionistic fuzzy left ideal of R is an intuitionistic fuzzy quasi-ideal, it follows from Theorem 4.2, that R is regular. □

Theorem 4.7. *For a ring* R*, the following conditions are equivalent:*

(i) R *is regular ring*

(ii) $A \cap B \cap C \subseteq A * B * C$ *for every intuitionistic fuzzy right ideal* A*,every intuitionistic fuzzy left ideal* C *and every intuitionistic fuzzy bi-ideal* B *of* R

(iii) $A \cap B \cap C \subseteq A * B * C$ *for every intuitionistic fuzzy right ideal* A, *every intuitionistic fuzzy left ideal* C *and every intuitionistic fuzzy quasi-ideal* B *of* R

Proof. (i) $\Rightarrow$ (ii) Let $A = (\mu_A, \gamma_A), B = (\mu_B, \gamma_B)$ and $C = (\mu_C, \gamma_C)$ be any intuitionistic fuzzy right ideal, any intuitionistic fuzzy left ideal, and any intuitionistic fuzzy bi-ideal of R, respectively. Let a be any element of R. Since R is regular, there exists an element x in R such that $a = axa(= axaxa)$. Then we have

$$\begin{aligned}
\mu_{A*B*C}(a) &= \bigvee_{\substack{x=\sum a_ib_i \\ \text{finite}}} \min\left\{ \begin{array}{l} \mu_A(a_1), \mu_A(a_2), \cdots, \mu_A(a_m), \\ \mu_{B*C}(b_1), \mu_{B*C}(b_2), \cdots, \mu_{B*C}(b_m) \end{array} \right\} \\
&\geq \min\{\mu_A(ax), \mu_{B*C}(axa)\} \\
&= \min\{\mu_A(ax), \bigvee_{\substack{axa=\sum p_iq_i \\ \text{finite}}} \min\{\mu_B(p_1), \mu_B(p_2), \cdots, \\
&\qquad \mu_B(p_m), \mu_C(q_1), \mu_C(q_2), \cdots, \mu_C(q_m)\}\} \\
&\geq \min\{\mu_A(ax), \min\{\mu_B(a), \mu_C(xa)\}\} \\
&= \min\{\mu_A(a), \min\{\mu_B(a), \mu_C(a)\}\} \\
&= (\mu_A) \wedge \mu_B) \wedge \mu_C)(a),
\end{aligned}$$

and

$$\begin{aligned}
\gamma_{A*B*C}(a) &= \bigwedge_{\substack{x=\sum a_ib_i \\ \text{finite}}} \max\left\{ \begin{array}{l} \gamma_A(a_1), \gamma_A(a_2), \cdots, \gamma_A(a_m), \\ \gamma_{B*C}(b_1), \gamma_{B*C}(b_2), \cdots, \gamma_{B*C}(b_m) \end{array} \right\} \\
&\leq \max\{\gamma_A(ax), \gamma_{B*C}(axa)\} \\
&= \max\{\gamma_A(ax), \bigwedge_{\substack{axa \sum p_iq_i \\ \text{finite}}} \max\{\gamma_B(p_1), \gamma_B(p_2), \cdots, \\
&\qquad \gamma_B(p_m), \gamma_C(q_1), \gamma_C(q_2), \cdots, \gamma_C(q_m)\}\} \\
&\leq \max\{\gamma_A(ax), \max\{\gamma_B(a), \gamma_C(xa)\}\} \\
&= \max\{\gamma_A(a), \max\{\gamma_B(a), \gamma_C(a)\}\} \\
&= (\gamma_A) \wedge \gamma_B) \wedge \gamma_C)(a),
\end{aligned}$$

and so we have $A \cap B \cap C \subseteq A * B * C$.

(ii) $\Rightarrow$ (iii). Straightforward.

(iii) $\Rightarrow$ (i). Let $A = (\mu_A, \gamma_A)$ and $C = (\mu_C, \gamma_C)$ be any intuitionistic fuzzy right ideal and any intuitionistic fuzzy left ideal of R, respectively. Then since R itself is a fuzzy quasi-ideal of R, we have

$$A \cap C = A \cap R \cap C \subseteq A * 1_\sim * C \subseteq A * C.$$

It follows from Theorem 4.2 that R is regular. □

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