

ON SOME INTEGRAL INEQUALITIES WITH ITERATED INTEGRALS

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ABSTRACT. The main aim of the present paper is to establish some new Gronwall type inequalities involving iterated integrals and give some applications of the main results.

1. Introduction

The integral inequalities involving functions of one and more than one independent variables which provide explicit bounds on unknown functions play a fundamental role in the development of the theory of differential equations.

The Gronwall type integral inequalities provide a necessary tool for the study of the theory of differential equations, integral equations and inequalities of various types (see Gronwall [9] and Guiliano [10]). Some applications of this result to the study of stability of the solution of linear and nonlinear differential equations may be found in Bellman [3]. Some applications to the existence and uniqueness theory of differential equations may be found in Nemyckii-Stepanov [14], Bihari [4], and Langenhop [11]. During the past few years several authors (see references below and some of the references cited therein) have established several Gronwall type integral inequalities in two or more independent real variables. Of course, such results have application in the theory of partial differential equations and Volterra integral equations.

In [5] (see also [1, p.98]), Bykov and Salpagarov proved the following interesting integral inequality:

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LEMMA 1.1. Let $u(t)$, $b(t)$, $k(t, s)$ and $h(t, s, \sigma)$ be nonnegative continuous functions for $\alpha \leq \tau \leq s \leq t \leq \beta$. Suppose that

$$(1.1) \quad \begin{aligned} u(t) \leq & a + \int_{\alpha}^t b(s)u(s) ds + \int_{\alpha}^t \int_{\alpha}^s k(s, \tau)u(\tau) d\tau ds \\ & + \int_{\alpha}^t \int_{\alpha}^s \int_{\alpha}^{\tau} h(s, \tau, \sigma)u(\sigma) d\sigma d\tau ds \end{aligned}$$

for any $t \in [\alpha, \beta]$, where $a \geq 0$ is a constant. Then, for any $t \in [\alpha, \beta]$,

$$\begin{aligned} u(t) \leq & a \exp \left(\int_{\alpha}^t b(s) ds + \int_{\alpha}^t \int_{\alpha}^s k(s, \tau) d\tau ds \right. \\ & \left. + \int_{\alpha}^t \int_{\alpha}^s \int_{\alpha}^{\tau} h(s, \tau, \sigma) d\sigma d\tau ds \right). \end{aligned}$$

In this paper, we consider simple inequalities involving iterated integrals in the inequality (1.1) for functions when the function u in the right-hand side of the inequality (1.1) is replaced by the function $u \log u$ and the constant a is replaced by a nonnegative, nondecreasing function $a(t)$. We also provide some integral inequalities involving iterated integrals and some applications for the main results.

2. Some inequalities for Bykov type

Throughout the paper, all the functions which appear in the inequalities are assumed to be real-valued.

First, we state and prove some new interesting integral inequalities involving iterated integrals.

THEOREM 2.1. Let $u(t)$, $b(t)$, $k(t, s)$ and $h(t, s, \tau)$ be nonnegative continuous functions for $\alpha \leq \tau \leq s \leq t \leq \beta$. Suppose that

$$(2.1) \quad \begin{aligned} u(t) \leq & a + \int_{\alpha}^t b(s)u(s) \log u(s) ds \\ & + \int_{\alpha}^t \int_{\alpha}^s k(s, \tau)u(\tau) \log u(\tau) d\tau ds \\ & + \int_{\alpha}^t \int_{\alpha}^s \int_{\alpha}^{\tau} h(s, \tau, \sigma)u(\sigma) \log u(\sigma) d\sigma d\tau ds \end{aligned}$$

for any $t \in [\alpha, \beta]$, where $a > 1$ is a constant. Then, for any $t \in [\alpha, \beta]$,

$$(2.2) \quad u(t) \leq a^{\exp(A(t))},$$

where

$$(2.3) \quad \begin{aligned} A(t) = & \int_{\alpha}^t b(s) ds + \int_{\alpha}^t \int_{\alpha}^s k(s, \tau) d\tau ds \\ & + \int_{\alpha}^t \int_{\alpha}^s \int_{\alpha}^{\tau} h(s, \tau, \sigma) d\sigma d\tau ds. \end{aligned}$$

Proof. We denote the right-hand side of (2.1) by $v(t)$. Then $v(\alpha) = a$, the function $v(t)$ is nondecreasing in $t \in [\alpha, \beta]$, $u(t) \leq v(t)$ and

$$\begin{aligned} v'(t) &= b(t)u(t) \log u(t) + \int_{\alpha}^t k(t, \tau)u(\tau) \log u(\tau) d\tau \\ &+ \int_{\alpha}^t \int_{\alpha}^{\tau} h(t, \tau, \sigma)u(\sigma) \log u(\sigma) d\sigma d\tau \\ &= b(t)v(t) \log v(t) + \int_{\alpha}^t k(t, \tau)v(\tau) \log v(\tau) d\tau \\ &+ \int_{\alpha}^t \int_{\alpha}^{\tau} h(t, \tau, \sigma)v(\sigma) \log v(\sigma) d\sigma d\tau \\ &\leq B(t)v(t), \end{aligned}$$

where

$$\begin{aligned} B(t) &= b(t) \log v(t) + \int_{\alpha}^t k(t, \tau) \log v(\tau) d\tau \\ &+ \int_{\alpha}^t \int_{\alpha}^{\tau} h(t, \tau, \sigma) \log v(\sigma) d\sigma d\tau. \end{aligned}$$

That is,

$$(2.4) \quad \frac{v'(t)}{v(t)} \leq B(t).$$

By taking $t = s$ in (2.4) and then integrating it from α to any $t \in [\alpha, \beta]$, we have

$$(2.5) \quad \log v(t) \leq \log a + \int_{\alpha}^t B(s) ds.$$

Now, by a suitable application of Lemma 1.1 to (2.5), we get

$$(2.6) \quad \log v(t) \leq (\log a) \exp\left(\int_{\alpha}^t B(s) ds\right) = \log a^{\exp(A(t))},$$

where $A(t)$ is defined by (2.3). From (2.6), we observe that

$$(2.7) \quad v(t) \leq a^{\exp(A(t))}.$$

Now, by using (2.7) in $u(t) \leq v(t)$, we get the required inequality in (2.2). This completes the proof. \square

REMARK 2.1. If, in (2.1) and (2.2), the constant a is replaced by a nonnegative, nondecreasing function $a(t) \geq 1$, then the inequality (2.1) implies that, for all $\alpha \leq t \leq T \leq \beta$,

$$u(t) \leq a(T) + \int_{\alpha}^t b(s)u(s) \log u(s) ds + \int_{\alpha}^t \int_{\alpha}^s k(s, \tau)u(\tau) \log u(\tau) d\tau ds \\ + \int_{\alpha}^t \int_{\alpha}^s \int_{\alpha}^{\tau} h(s, \tau, \sigma)u(\sigma) \log u(\sigma) d\sigma d\tau ds.$$

Therefore, Theorem 2.1 implies $u(t) \leq (a(T))^{\exp(A(t))}$ and, for $T = t$, we obtain $u(t) \leq (a(t))^{\exp(A(t))}$.

LEMMA 2.2. Let $u(t)$, $a(t)$, $b(t)$, $k(t, s)$ and $h(t, s, \tau)$ be nonnegative continuous functions for $\alpha \leq \tau \leq s \leq t \leq \beta$. Suppose that

$$(2.8) \quad u(t) \leq a(t) + \int_{\alpha}^t b(s)u(s) ds + \int_{\alpha}^t \int_{\alpha}^s k(s, \tau)u(\tau) d\tau ds \\ + \int_{\alpha}^t \int_{\alpha}^s \int_{\alpha}^{\tau} h(s, \tau, \sigma)u(\sigma) d\sigma d\tau ds$$

for any $t \in [\alpha, \beta]$. Then, for any $t \in [\alpha, \beta]$,

$$(2.9) \quad u(t) \leq a(t) + \int_{\alpha}^t f(s) \exp\left(\int_s^t B(\delta) d\delta\right) ds,$$

where

$$(2.10) \quad f(t) = a(t)b(t) + \int_{\alpha}^t k(t, \tau)a(\tau) d\tau + \int_{\alpha}^t \int_{\alpha}^{\tau} h(t, \tau, \sigma)a(\sigma) d\sigma d\tau,$$

$$(2.11) \quad B(t) = b(t) + \int_{\alpha}^t k(t, \tau) d\tau + \int_{\alpha}^t \int_{\alpha}^{\tau} h(t, \tau, \sigma) d\sigma d\tau.$$

Proof. We denote the right-hand side of (2.8) by $a(t) + v(t)$. Then $v(\alpha) = 0$, the function $v(t)$ is nondecreasing in $t \in [\alpha, \beta]$, $u(t) \leq a(t) + v(t)$ and

$$\begin{aligned}
 v'(t) &= b(t)u(t) + \int_{\alpha}^t k(t, \tau)u(\tau) d\tau \\
 &+ \int_{\alpha}^t \int_{\alpha}^{\tau} h(t, \tau, \sigma)u(\sigma) d\sigma d\tau \\
 &\leq f(t) + B(t)v(t),
 \end{aligned}
 \tag{2.12}$$

where $f(t)$ and $B(t)$ are defined by (2.10) and (2.11), respectively. The condition (2.12) implies that

$$[v'(s) - B(s)v(s)] \exp\left(\int_s^t B(\delta) d\delta\right) \leq f(s) \exp\left(\int_s^t B(\delta) d\delta\right)$$

for $\alpha \leq s$ or

$$\frac{d}{ds} \left[v(s) \exp\left(\int_s^t B(\delta) d\delta\right) \right] \leq f(s) \exp\left(\int_s^t B(\delta) d\delta\right).$$

Integrating over s from α to any $t \in [\alpha, \beta]$, we have

$$v(t) \leq \int_{\alpha}^t f(s) \exp\left(\int_s^t B(\delta) d\delta\right) ds.
 \tag{2.13}$$

Now, by using (2.13) in $u(t) \leq a(t) + v(t)$, we get the required inequality in (2.9). This completes the proof. \square

THEOREM 2.3. *Let $u(t)$, $b(t)$, $k(t, s)$ and $\sigma(t)$ be nonnegative continuous functions for $\alpha \leq s \leq t \leq \beta$. Suppose $\sigma(t) \geq 1$ and*

$$\begin{aligned}
 u(t) &\leq \sigma(t) \left\{ a + \int_{\alpha}^t b(s)u(s) \log u(s) ds \right. \\
 &\quad \left. + \int_{\alpha}^t \int_{\alpha}^s k(s, \tau)u(\tau) \log u(s) d\tau ds \right\}
 \end{aligned}$$

for any $t \in [\alpha, \beta]$, where $a \geq 1$ is a constant. Then

$$u(t) \leq \sigma(t) \exp\left(B_1(t) + \int_{\alpha}^t f_1(s) \exp\left(\int_s^t f_2(\delta) d\delta\right) ds \right)
 \tag{2.14}$$

for any $t \in [\alpha, \beta]$, where

$$(2.15) \quad B_1(t) = a + \int_{\alpha}^t b(s)\sigma(s) \log \sigma(s) ds + \int_{\alpha}^t \int_{\alpha}^s k(s, \tau)\sigma(\tau) \log \sigma(\tau) d\tau ds,$$

$$(2.16) \quad f_1(t) = b(t)\sigma(t)B_1(t) + \int_{\alpha}^t k(t, \tau)\sigma(\tau)B_1(\tau) d\tau,$$

$$(2.17) \quad f_2(t) = b(t)\sigma(t) + \int_{\alpha}^t k(t, \tau)\sigma(\tau) d\tau.$$

Proof. We deduce from the hypothesis on $u(t)$ that $u(t) \leq \sigma(t)v(t)$, where the function $v(t)$ is defined by

$$v(t) = a + \int_{\alpha}^t b(s)u(s) \log u(s) ds + \int_{\alpha}^t \int_{\alpha}^s k(s, \tau)u(\tau) \log u(s) d\tau ds.$$

The function $v(t)$ is nondecreasing in $t \in [\alpha, \beta]$, $v(\alpha) = a$ and

$$\begin{aligned} v'(t) &= b(t)u(t) \log u(t) + \int_{\alpha}^t k(t, \tau)u(\tau) \log u(\tau) d\tau \\ &\leq \left(b(t)\sigma(t) \log(\sigma(t)v(t)) + \int_{\alpha}^t k(t, \tau)\sigma(\tau) \log(\sigma(\tau)v(\tau)) d\tau \right) v(t). \end{aligned}$$

This implies that

$$\frac{v'(s)}{v(s)} \leq b(s)\sigma(s) \log(\sigma(s)v(s)) + \int_{\alpha}^s k(s, \tau)\sigma(\tau) \log(\sigma(\tau)v(\tau)) d\tau.$$

Integrating over s from α to any $t \in [\alpha, \beta]$, we have

$$(2.18) \quad \log v(t) \leq B_1(t) + \int_{\alpha}^t b(s)\sigma(s) \log v(s) ds + \int_{\alpha}^t \int_{\alpha}^s k(s, \tau)\sigma(\tau) \log v(\tau) d\tau ds,$$

where $B_1(t)$ is defined by (2.15). Applying Lemma 2.2 in (2.17), we have

$$(2.19) \quad v(t) \leq \exp\left(B_1(t) + \int_{\alpha}^t f_1(s) \exp\left(\int_s^t f_2(\delta) d\delta \right) ds \right),$$

where $f_1(t), f_2(t)$ are defined in (2.16) and (2.17), respectively. Now, by using (2.19) in $u(t) \leq \sigma(t)v(t)$, we get the required inequality in (2.14). This completes the proof. \square

In the same manner, we can prove the following theorem:

THEOREM 2.4. *Let $u(t)$ be nonnegative, continuous functions for $\alpha \leq t \leq \beta$. Suppose that*

$$(2.20) \quad \begin{aligned} u(t) \leq a + \int_{\alpha}^t k(t, s)u(s) \log u(s) ds \\ + \int_{\alpha}^t \int_{\alpha}^s h(t, s, \sigma)u(\sigma) \log u(\sigma) d\sigma ds, \end{aligned}$$

where $a \geq 1$ is a constant and $k(t, s)$ and $h(t, s, \sigma)$ are nonnegative, continuous functions for $\alpha \leq \sigma \leq s \leq t \leq \beta$. Suppose the partial derivatives $\frac{\partial k}{\partial t}(t, s)$, $\frac{\partial h}{\partial t}(t, s, \sigma)$ exist and are nonnegative, continuous for $\alpha \leq \sigma \leq s \leq t \leq \beta$. Then, for any $t \in [\alpha, \beta]$,

$$(2.21) \quad u(t) \leq a^{\exp(B_2(t))},$$

where

$$\begin{aligned} B_2(t) = \int_{\alpha}^t k(s, s) ds + \int_{\alpha}^t \int_{\alpha}^s [k_s(s, \sigma) + h(s, s, \sigma)] d\sigma ds \\ + \int_{\alpha}^t \int_{\alpha}^s \int_{\alpha}^{\sigma} h_s(s, \sigma, \tau) d\tau d\sigma ds. \end{aligned}$$

REMARK 2.2. If, in (2.20) and (2.21), the constant a is replaced by a nonnegative, nondecreasing function $a(t) \geq 1$, then the inequality (2.20) implies that, for all $\alpha \leq t \leq T \leq \beta$,

$$\begin{aligned} u(t) \leq a(T) + \int_{\alpha}^t k(t, s)u(s) \log u(s) ds \\ + \int_{\alpha}^t \int_{\alpha}^s h(t, s, \sigma)u(\sigma) \log u(\sigma) d\sigma ds. \end{aligned}$$

Therefore, Theorem 2.4 implies $u(t) \leq (a(T))^{\exp(B_2(t))}$ and, for $T = t$, we obtain $u(t) \leq (a(t))^{\exp(B_2(t))}$.

3. Some further inequalities

In this section, we present a generalization of the result obtained in Theorem 2.4 and some new integral inequalities involving iterated integrals. Let $\alpha < \beta$ and set $J_i = \{(t_1, t_2, \dots, t_i) \in R^i : \alpha \leq t_i \leq \dots \leq t_1 \leq \beta\}$ for $i = 1, \dots, n$.

THEOREM 3.1. Let $u(t)$ and $b(t)$ be nonnegative continuous functions in $J = [\alpha, \beta]$ and $a \geq 1$ be a constant. Suppose that

$$(3.1) \quad \begin{aligned} &u(t) \\ &\leq a + \int_{\alpha}^t k_1(t, t_1)u(t_1) \log u(t_1) dt_1 + \cdots \\ &\quad + \int_{\alpha}^t \left(\int_{\alpha}^{t_1} \cdots \left(\int_{\alpha}^{t_{n-1}} k_n(t, t_1, \dots, t_n)u(t_n) \log u(t_n) dt_n \right) \cdots \right) dt_1 \end{aligned}$$

for any $t \in J$, where $k_i(t, t_1, \dots, t_i)$ are nonnegative, continuous functions in J_{i+1} for $i = 1, 2, \dots, n$.

Suppose the partial derivative $\frac{\partial k_i}{\partial t}(t, t_1, \dots, t_i)$ exists and are nonnegative, continuous in J_{i+1} for $i = 1, 2, \dots, n$, which are nondecreasing in $t \in J$ for any fixed $(t_1, \dots, t_i) \in J_i$ for $i = 1, \dots, n$. Then, for any $t \in J$,

$$(3.2) \quad u(t) \leq a^{\exp(B_3(t))},$$

where $B_3(t) = \int_{\alpha}^t (R[1](\tau) + Q[1](\tau)) d\tau$, and

$$\begin{aligned} R[1](t) &= k_1(t, t) + \int_{\alpha}^t k_2(t, t, t_2) dt_2 \\ &\quad + \sum_{i=3}^n \int_{\alpha}^t \left(\int_{\alpha}^{t_2} \cdots \left(\int_{\alpha}^{t_{i-1}} k_i(t, t, t_2, \dots, t_i) dt_i \right) \cdots \right) dt_2, \\ Q[1](t) &= \int_{\alpha}^t \frac{\partial k_1}{\partial t}(t, t_1) dt_1 \\ &\quad + \sum_{i=2}^n \int_{\alpha}^t \left(\int_{\alpha}^{t_1} \cdots \left(\int_{\alpha}^{t_{i-1}} \frac{\partial k_i}{\partial t}(t, t_1, \dots, t_i) dt_i \right) \cdots \right) dt_1. \end{aligned}$$

Proof. We denote the right-hand side of (3.1) by $v(t)$. Then, for $\alpha \leq t \leq \beta$, (3.1) implies $v(\alpha) = a$, the function $v(t)$ is nondecreasing continuous, $u(t) \leq v(t)$ and we have

$$(3.3) \quad v'(t) = R[u \log u](t) + Q[u \log u](t),$$

where

$$\begin{aligned}
 & R[u \log u](t) \\
 = & k_1(t, t)u(t) \log u(t) + \int_{\alpha}^t k_2(t, t, t_2)u(t_2) \log u(t_2)dt_2 \\
 & + \sum_{i=3}^n \int_{\alpha}^t \left(\int_{\alpha}^{t_2} \cdots \left(\int_{\alpha}^{t_{i-1}} k_i(t, t, t_2, \dots, t_i)u(t_i) \log u(t_i) dt_i \right) \cdots \right) dt_2, \\
 & Q[u \log u](t) \\
 = & \int_{\alpha}^t \frac{\partial k_1}{\partial t}(t, t_1)u(t_1) \log u(t_1) dt_1 \\
 & + \sum_{i=2}^n \int_{\alpha}^t \left(\int_{\alpha}^{t_1} \cdots \left(\int_{\alpha}^{t_{i-1}} \frac{\partial k_i}{\partial t}(t, t_1, \dots, t_i)u(t_i) \log u(t_i) dt_i \right) \cdots \right) dt_1.
 \end{aligned}$$

We note that $R[u \log u]$ and $Q[u \log u]$ are linear functionals and

$$R[u_1 \log u_1] \leq R[u_2 \log u_2], \quad Q[u_1 \log u_1] \leq Q[u_2 \log u_2]$$

if $u_1 \log u_1 \leq u_2 \log u_2$ for any $t \in J$, and

$$R[u \log u] \leq R[\log u]u, \quad Q[u \log u] \leq Q[\log u]u$$

if $\log u(t)$ is nonnegative in J and $u(t)$ is nondecreasing in J . The equality (3.3) implies the estimate

$$(3.4) \quad v'(t) \leq \{R[\log v](t) + Q[\log v](t)\}v(t).$$

From (3.4), we have

$$(3.5) \quad \frac{v'(t)}{v(t)} \leq R[\log v](t) + Q[\log v](t).$$

By taking $t = s$ in (3.5) and then integrating it from 0 to any $t \in J$, we obtain

$$(3.6) \quad \log v(t) \leq \log a + \int_{\alpha}^t (R[\log v](s) + Q[\log v](s)) ds.$$

Now, we denotes the right-hand side of (3.6) by $v_1(t)$. Then $v_1(\alpha) = \log a$, $\log v(t) \leq v_1(t)$, the function $v_1(t)$ is nondecreasing in $t \in [\alpha, \beta]$ and

$$(3.7) \quad v'_1(t) \leq \{R[1](t) + Q[1](t)\}v_1(t).$$

The condition (3.7) implies that

$$\left(v_1'(s) - \{R[1](s) + Q[1](s)\}v_1(s) \right) \exp\left(\int_s^t \{R[1](\tau) + Q[1](\tau)\} d\tau \right) \leq 0$$

for $\alpha \leq s$ or

$$\frac{d}{ds} \left[v_1(s) \exp\left(\int_s^t \{R[1](\tau) + Q[1](\tau)\} d\tau \right) \right] \leq 0.$$

Integrating over s from α to any $t \in [\alpha, \beta]$, we have

$$(3.8) \quad v_1(t) \leq \log a \exp(B_3(t)) = \log a^{\exp(B_3(t))},$$

where $B_3(t) = \int_\alpha^t \{R[1](\tau) + Q[1](\tau)\} d\tau$. The desired inequality in (3.2) now follows by using $\log v(t) \leq v_1(t)$ in (3.8) and then $u(t) \leq v(t)$. This completes the proof. \square

THEOREM 3.2. *Let $u(t)$ and $b(t)$ be nonnegative continuous functions in $J = [\alpha, \beta]$ and $a \geq 1$ be a constant. Suppose that*

$$(3.9) \quad \begin{aligned} & u(t) \\ & \leq a + \int_\alpha^t k_1(t, t_1) u(t_1) \log u(t_1) dt_1 + \cdots \\ & \quad + \int_\alpha^t \left(\int_\alpha^{t_1} \cdots \left(\int_\alpha^{t_{n-1}} k_n(t, t_1, \dots, t_n) u(t_n) \log u(t_n) dt_n \right) \cdots \right) dt_1 \end{aligned}$$

for any $t \in J$, where $k_i(t, t_1, \dots, t_i)$ are nonnegative, continuous functions in J_{i+1} for $i = 1, 2, \dots, n$, which are nondecreasing in $t \in J$ for all fixed $(t_1, \dots, t_i) \in J_i$ for $i = 1, \dots, n$. Then, for any $t \in J$,

$$(3.10) \quad u(t) \leq a^{\exp(B_4(t))},$$

where $B_4(t) = \int_\alpha^t \widehat{R}[1](t, \tau) d\tau$, and

$$\begin{aligned} \widehat{R}[1](t, \tau) &= k_1(t, \tau) + \int_\alpha^t k_2(t, \tau, t_2) dt_2 \\ & \quad + \sum_{i=3}^n \int_\alpha^t \left(\int_\alpha^{t_2} \cdots \left(\int_\alpha^{t_{i-1}} k_i(t, \tau, t_2, \dots, t_i) dt_i \right) \cdots \right) dt_2. \end{aligned}$$

Proof. For any fixed $T \in (\alpha, \beta]$ and $\alpha \leq t \leq T$, we define a function $w(t)$ by

$$\begin{aligned} w(t) &= a + \int_{\alpha}^t k_1(T, t_1)u(t_1) \log u(t_1) dt_1 + \dots \\ &\quad + \int_{\alpha}^t \left(\int_{\alpha}^{t_1} \dots \left(\int_{\alpha}^{t_{n-1}} k_n(T, t_1, \dots, t_n)u(t_n) \log u(t_n) dt_n \right) \dots \right) dt_1. \end{aligned}$$

Then $w(\alpha) = a$, the function $w(t)$ is nondecreasing continuous and $u(t) \leq w(t)$. Since $\frac{\partial k_i}{\partial t}(T, t_1, \dots, t_i) = 0$ for all $i = 1, \dots, n$ and $t \in J = [\alpha, \beta]$, we have

$$(3.11) \quad w'(t) = \widehat{R}[u \log u](T, t),$$

where

$$\begin{aligned} &\widehat{R}[u \log u](T, t) \\ &= k_1(T, t)u(t) \log u(t) + \int_{\alpha}^t k_2(T, t, t_2)u(t_2) \log u(t_2) dt_2 \\ &\quad + \sum_{i=3}^n \int_{\alpha}^t \left(\int_{\alpha}^{t_2} \dots \left(\int_{\alpha}^{t_{i-1}} k_i(T, t, t_2, \dots, t_i)u(t_i) \log u(t_i) dt_i \right) \dots \right) dt_2. \end{aligned}$$

The equality (3.11) implies the estimate

$$(3.12) \quad w'(t) \leq \{\widehat{R}[\log w](T, t)\}w(t).$$

From (3.12), we have

$$(3.13) \quad \frac{w'(t)}{w(t)} \leq \widehat{R}[\log w](T, t).$$

By taking $t = s$ in (3.13) and then integrating it from α to any $t \in J$, we obtain

$$(3.14) \quad \log w(t) \leq \log a + \int_{\alpha}^t \widehat{R}[\log w](T, s) ds.$$

Now, we denote the right-hand side of (3.14) by $w_1(t)$. Then $w_1(\alpha) = \log a$, $\log v(t) \leq w_1(t)$, the function $w_1(t)$ is nondecreasing in $t \in [\alpha, \beta]$ and

$$(3.15) \quad w_1'(t) \leq \{\widehat{R}[1](T, t)\}w_1(t).$$

The inequality (3.15) implies the estimate

$$\left(w_1'(s) - \{\widehat{R}[1](T, t)\}w_1(s) \right) \exp\left(\int_s^t \{\widehat{R}[1](T, \tau)\} d\tau \right) \leq 0$$

for $\alpha \leq s$ or

$$\frac{d}{ds} \left[w_1(s) \exp\left(\int_s^t \{\widehat{R}[1](T, \tau)\} d\tau \right) \right] \leq 0.$$

Integrating over s from α to any $t \in J = [\alpha, \beta]$, we have

$$(3.16) \quad w_1(t) \leq \log a \exp(B_4(t)) = \log a^{\exp(B_4(t))},$$

where $B_4(t) = \int_\alpha^t \{\widehat{R}[1](T, \tau)\} d\tau$. In particular, for $T = t$, we find the desired inequality in (3.10) now follows by using $\log w(t) \leq w_1(t)$ in (3.8) and then $u(t) \leq w(t)$. This completes the proof. \square

THEOREM 3.3. *Let u, f_1, \dots, f_n be nonnegative, continuous functions in $J = [\alpha, \beta]$. Suppose that*

$$(3.17) \quad \begin{aligned} &u(t) \\ &\leq a + \int_\alpha^t f_1(t_1)u(t_1) \log u(t_1) dt_1 + \dots \\ &+ \int_\alpha^t f_1(t_1) \left(\int_\alpha^{t_1} f_2(t_2) \dots \left(\int_\alpha^{t_{n-1}} f_n(t_n)u(t_n) \log u(t_n) dt_n \right) \dots \right) dt_1 \end{aligned}$$

for any $t \in J$, where $a \geq 1$ is a constant. Then

$$(3.18) \quad u(t) \leq R_1(t)$$

for any $t \in J$, where

$$R_n(t) = \left[\frac{1}{a} - \int_\alpha^t (2^{n-1}f_1(s) + 2^{n-2}f_2(s) + \dots + f_n(s)) ds \right]^{-1}$$

for any $t \in [\alpha, \beta_1)$, where β_1 is chosen so that the expression between $[\dots]$ is positive in the subinterval $[\alpha, \beta_1)$, and

$$R_i(t) = a + \int_\alpha^t R_{i+1}(s) \left(\sum_{k=1}^i 2^{i-k} f_k(s) \right) ds$$

for any $t \in J$ and $i = n - 1, \dots, 1$.

Proof. We set

$$u_1(t) = a + L_1[u \log u](t), \quad u_{j+1}(t) = u_j^2 + L_{j+1}[u \log u](t)$$

for any $t \in J$ and $j = 1, \dots, n - 1$, where

$$\begin{aligned} &L_k[u \log u](t) \\ &= \int_{\alpha}^t f_k(t_k)u(t_k) \log u(t_k) dt_k + \dots \\ &+ \int_{\alpha}^t f_k(t_k) \left(\int_{\alpha}^{t_k} f_{k+1}(t_{k+1}) \dots \left(\int_{\alpha}^{t_k} f_n(t_n)u(t_n) \log u(t_n) dt_n \right) \dots \right) dt_k \end{aligned}$$

for any $t \in J$ and $k = 1, \dots, n$. Now, (3.17) implies

$$(3.19) \quad u(t) \leq u_1(t).$$

Taking into account that $u_k(t) \leq u_{k+1}(t)$, $(L_k[u \log u])' \leq f_k(u^2(t) + L_{k+1}[u \log u])$ for $k = 1, \dots, n - 1$ and

$$(L_n[u \log u])' = f_n(t)u(t) \log u(t),$$

for $k = 2, \dots, n - 1$, we successively have

$$(3.20) \quad \begin{cases} u_1'(t) &= (L_1[u \log u](t))' = f_1[u^2(t) + L_2[u \log u]] \\ &\leq f_1[u_1^2(t) + L_2[u \log u]] = f_1u_2, \\ u_k'(t) &\leq (2^{k-1}f_1 + 2^{k-2}f_2 + \dots + f_k)u_{k+1}, \\ u_n'(t) &\leq (2^{n-1}f_1 + 2^{n-2}f_2 + \dots + f_n)u_n^2(t). \end{cases}$$

According to (3.20), the function $z = -u_n^{-1}$ satisfies

$$(3.21) \quad z'(t) = u_n^{-2}u_n' \leq 2^{n-1}f_1 + 2^{n-2}f_2 + \dots + f_n.$$

By taking $t = s$ in (3.21) and then integrating it from α to any $t \in [\alpha, \beta_1)$, we have

$$(3.22) \quad u_n(t) \leq \left[\frac{1}{a} - \int_{\alpha}^t (2^{n-1}f_1(s) + 2^{n-2}f_2(s) + \dots + f_n(s)) ds \right]^{-1},$$

where β_1 is chosen so that the expression between $[\dots]$ is positive in the subinterval $[\alpha, \beta_1)$. Define a function $R_n(t)$ by the right side of (3.22). Then (3.20) and (3.22) gives

$$(3.23) \quad u_i(t) \leq a + \int_{\alpha}^t R_{i+1} \left(\sum_{k=1}^i 2^{i-k} f_k(s) \right) ds$$

for $i = n - 1, \dots, 1$. For $i = 1$, (3.19) and (3.23) imply (3.18). This completes the proof. □

4. Applications

In this section, we show that our main results are useful in showing the global existence of solutions to certain integro-differential equations of the form

$$(4.1) \quad x'(t) = F\left(t, s, x(t), \int_{t_0}^t f(t, s, x(s)) ds\right)$$

for any $t \in J = [\alpha, \beta]$ with the given initial condition

$$(4.2) \quad x(t_0) = x_0,$$

where $f \in C(J^2 \times R, R)$, $F \in C(J^2 \times R^2, R)$ and $x_0 \geq 1$ is constant. Assume that

$$(4.3) \quad \int_{t_0}^t |F(s, \sigma, u, v)| ds \leq \int_{t_0}^t (k(t, s)|u| \log |u| + h_1(t)|v|) ds,$$

$$(4.4) \quad |f(t, s, u)| \leq h_2(t, s)|u| \log |u|,$$

$$(4.5) \quad h_1(t)h_2(s, \sigma) \leq h(t, s, \sigma),$$

where the functions $k(t, s)$ and $h(t, s, \sigma)$ are defined as in Theorem 2.4 for $\alpha \leq \sigma \leq s \leq t \leq \beta$.

If $x(t)$ is a solution of the equation (4.1) with (4.2), the solution $x(t)$ can be written as

$$(4.6) \quad x(t) = x_0 + \int_{t_0}^t F\left(s, \sigma, x(\sigma), \int_{t_0}^s f(s, \sigma, x(\sigma)) d\sigma\right) ds$$

for any $t \in J$. Using (4.3)~(4.5) in (4.6) and making the change of variables, we have

$$(4.7) \quad \begin{aligned} |x(t)| &\leq |x_0| + \int_{t_0}^t k(t, s)|x(s)| \log |x(s)| ds \\ &\quad + \int_{t_0}^t \int_{t_0}^s h(t, s, \sigma)|x(\sigma)| \log |x(\sigma)| d\sigma ds. \end{aligned}$$

Now, a suitable application of Theorem 2.4 to (4.7) yields

$$\log |x(t)| \leq (\log |x_0|) \exp(B_2(t)),$$

which implies that $x(t)$ is bounded.

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