J. Korean Math. Soc. **43** (2006), No. 3, pp. 539–552

LINEAR PRESERVERS OF BOOLEAN NILPOTENT MATRICES

SEOK-ZUN SONG, KYUNG-TAE KANG, AND YOUNG-BAE JUN

ABSTRACT. For an $n \times n$ Boolean matrix A, A is called *nilpotent* if $A^m = O$ for some positive integer m. We consider the set of $n \times n$ nilpotent Boolean matrices and we characterize linear operators that strongly preserve nilpotent matrices over Boolean algebras.

1. Introduction and preliminaries

There is a great deal of literature on the study of matrix theory over a finite Boolean algebra (see [1], [2], [4]-[7]). But many results in Boolean matrix theory are stated only for binary Boolean matrices because there exists a semi-ring isomorphism between the matrices over the Boolean algebra of subsets of a k element set and the k tuples of binary Boolean matrices. The isomorphism allows many questions concerning matrices over an arbitrary finite Boolean algebra to be referred to the binary Boolean case. In many instances, the results in the general case are not immediately obvious because the above mentioned isomorphism was not well known.

Botta et al. [3] obtained characterizations of the linear operators which preserve nilpotent matrices over fields. In this paper, we will characterize linear operators that strongly preserve nilpotent matrices over general Boolean algebras.

For a fixed positive integer k, let $\mathbb{B}_k$ be the *Boolean algebra* of subsets of a k-element set $\mathbb{S}_k$ and $\sigma_1, \sigma_2, \ldots, \sigma_k$ denote the singleton subsets of $\mathbb{S}_k$. Union is denoted by $+$ and intersection by juxtaposition; 0 denotes the null set and 1 the set $\mathbb{S}_k$. Under these two operations, $\mathbb{B}_k$ is a commutative, anti-negative semi-ring (that is, only zero element has an

Received January 6, 2005.

2000 Mathematics Subject Classification: Primary 15A03, 15A04, 15A23.

Key words and phrases: Boolean algebra, nilpotent matrix, constituent, linear operator.

additive inverse); all of its elements, except 0 and 1, are zero-divisors. In particular, if $k = 1$, $\mathbb{B}_1$ is called the *binary Boolean algebra.*

Let $\mathbb{M}_n(\mathbb{B}_k)$ denote the set of all $n \times n$ matrices over a general Boolean algebra $\mathbb{B}_k$ with $k \geq 1$. We denote the $n \times n$ identity matrix by I_n, the $n \times n$ zero matrix by O_n and the $n \times n$ matrix all of whose entries are 1 by J_n. For a matrix A in $\mathbb{M}_n(\mathbb{B}_k)$, $|A|$ is denoted as the number of nonzero elements in A.

The $n \times n$ matrix all of whose entries are zero except its (i,j)-th, which is 1, is denoted E_{ij}. We call E_{ij} a *cell.* Let $\mathbb{E}_n = \{E_{ij} \mid i,j = 1,\ldots,n\}$ denote the set of all cells. When $i \neq j$, we say E_{ij} is an *off-diagonal* cell; E_{ii} is a *diagonal* cell. A *line* is a row or a column. A set of cells is *collinear* if they are all in the same line.

For any matrix $A = [a_{ij}] \in \mathbb{M}_n(\mathbb{B}_k)$, the p^{th} *constituent*, A_p, of A is the $n \times n$ binary Boolean matrix whose (i,j)-th entry is 1 if and only if $a_{ij} \supseteq \sigma_p$. Via the constituents, A can be written uniquely as

$$A = \sum_{p=1}^{k} \sigma_p A_p,$$

which is called the *canonical form* of A (see [7]).

It follows from the uniqueness of the decomposition and the fact that the singletons are mutually orthogonal idempotents that for all matrices $A, B \in \mathbb{M}_n(\mathbb{B}_k)$ and all $\alpha \in \mathbb{B}_k$,

$$(1.1) \quad (AB)_p = A_p B_p, \quad (A+B)_p = A_p + B_p, \quad \text{and} \quad (\alpha A)_p = \alpha_p A_p$$

for all $1 \leq p \leq k$.

LEMMA 1.1. ([5]) *For any matrix $A \in \mathbb{M}_n(\mathbb{B}_k)$ with $k \geq 1$, A is invertible if and only if all its constituents are permutation matrices. In particular, if A is invertible, then $A^{-1} = A^t$.*

A matrix A in $\mathbb{M}_n(\mathbb{B}_k)$ is called *nilpotent* if $A^m = O_n$ for some integer $m \geq 1$, and we denote $\mathbb{N}_n(\mathbb{B}_k)$ as the set of all nilpotent matrices in $\mathbb{M}_n(\mathbb{B}_k)$. We also shall denote the set of all matrices in which all diagonal entries are zero by $\mathbb{Z}_n(\mathbb{B}_k)$. We can easily show that all off-diagonal cells in $\mathbb{M}_n(\mathbb{B}_k)$ are nilpotent but all diagonal cells are not nilpotent.

THEOREM 1.2. *For a matrix $A \in \mathbb{M}_n(\mathbb{B}_k)$, A is nilpotent in $\mathbb{M}_n(\mathbb{B}_k)$ if and only if all its constituents are nilpotent in $\mathbb{M}_n(\mathbb{B}_1)$.*

Proof. Let $A = \sum_{p=1}^{k} \sigma_p A_p$ be a matrix in $\mathbb{M}_n(\mathbb{B}_k)$. Since $\sigma_p \sigma_q = \sigma_p$ or 0 according as $p = q$ or $p \neq q$, we have

$$A^m = \sum_{p=1}^{k} \sigma_p (A_p)^m \tag{1.2}$$

for all integer $m \geq 1$. For a nilpotent matrix A in $\mathbb{M}_n(\mathbb{B}_k)$, assume that there exists a p^{th} constituent, A_p of A such that A_p is not nilpotent in $\mathbb{M}_n(\mathbb{B}_1)$ so that $(A_p)^m \neq O_n$ for all integer $m \geq 1$. Therefore we have that an (i,j)-th entry of $(A_p)^m$ is 1. By (1.2), the (i,j)-th entry of A^m must contain σ_p for all integer $m \geq 1$, a contradiction. Hence all constituents of A are nilpotent in $\mathbb{M}_n(\mathbb{B}_1)$.

Conversely, assume that all constituents of A are nilpotent in $\mathbb{M}_n(\mathbb{B}_1)$. Then there exist positive integers $m_1, \ldots, m_k$ such that $(A_p)^{m_p} = O_n$ for all $p = 1, \ldots, k$. Let $m = \max\{m_1, \ldots, m_k\}$. Then we have $(A_p)^m = O_n$ and so $A^m = O_n$ by (1.2). Therefore A is nilpotent in $\mathbb{M}_n(\mathbb{B}_k)$. □

PROPOSITION 1.3. *For any integer $k \geq 1$, we have $\mathbb{N}_n(\mathbb{B}_k) \subseteq \mathbb{Z}_n(\mathbb{B}_k)$.*

Proof. Let $A = [a_{ij}] \notin \mathbb{Z}_n(\mathbb{B}_k)$. Then there exists a diagonal entry a_{ii} of A such that $a_{ii} \neq 0$ for some $i = 1, \ldots, n$ so that $a_{ii} \supseteq \sigma_p$ for some $p = 1, \ldots, k$. Therefore the (i,i)-th entry of A^m must contain σ_p so that $A^m \neq O_n$ for all integer $m \geq 1$. This implies that $A \notin \mathbb{N}_n(\mathbb{B}_k)$. □

2. Nilpotent matrix preservers over binary Boolean algebra

In this section we obtain characterizations of the linear operators that preserve nilpotent matrices over the binary Boolean algebra $\mathbb{B}_1$.

A mapping $T : \mathbb{M}_n(\mathbb{B}_k) \to \mathbb{M}_n(\mathbb{B}_k)$ is said to be a *linear operator* on $\mathbb{M}_n(\mathbb{B}_k)$ if $T(aA+bB) = aT(A)+bT(B)$ for all A, B in $\mathbb{M}_n(\mathbb{B}_k)$ and for all a, b in $\mathbb{B}_k$. A linear operator T on $\mathbb{M}_n(\mathbb{B}_k)$ is said to be *strongly preserve* $\mathbb{N}_n(\mathbb{B}_k)$ (or T *strongly preserves nilpotent matrices*) if $A \in \mathbb{N}_n(\mathbb{B}_k)$ if and only if $T(A) \in \mathbb{N}_n(\mathbb{B}_k)$.

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are matrices in $\mathbb{M}_n(\mathbb{B}_k)$, we shall use the notation $A \geq B$(or $B \leq A$) if $a_{ij} \supseteq b_{ij}$ (equivalently $a_{ij} + b_{ij} = a_{ij}$) for all $i, j = 1, \ldots, n$. This provides a reflexive and transitive relation on $\mathbb{M}_n(\mathbb{B}_k)$. If A and B are matrices in $\mathbb{M}_n(\mathbb{B}_k)$ with $A \geq B$, it follows from the linearity of T that $T(A) \geq T(B)$ for any linear operator T on $\mathbb{M}_n(\mathbb{B}_k)$.

PROPOSITION 2.1. *If $A \in \mathbb{N}_n(\mathbb{B}_k)$ and $A \geq B$, we have $B \in \mathbb{N}_n(\mathbb{B}_k)$.*

Proof. It follows from $O_n = A^m \geq B^m$ for some integer $m \geq 1$. □

LEMMA 2.2. *Let $T : \mathbb{M}_n(\mathbb{B}_1) \to \mathbb{M}_n(\mathbb{B}_1)$ be a linear operator which strongly preserves $\mathbb{N}_n(\mathbb{B}_1)$. Then T is nonsingular.*

Proof. Suppose that $T(E) = O_n$ for some cell $E \in \mathbb{E}_n$. Then we have that E is an off-diagonal cell since every diagonal cells are not nilpotent. Consider $X = E + E^t$. Then we can easily show that $X \notin \mathbb{N}_n(\mathbb{B}_1)$ while $T(X) = T(E + E^t) = T(E^t) \in \mathbb{N}_n(\mathbb{B}_1)$, a contradiction. Hence $T(E) \neq O_n$ for all cells $E \in \mathbb{E}_n$. The Lemma follows. □

LEMMA 2.3. *Let $T : \mathbb{M}_n(\mathbb{B}_1) \to \mathbb{M}_n(\mathbb{B}_1)$ be a linear operator which strongly preserves $\mathbb{N}_n(\mathbb{B}_1)$. Then T is invertible on $\mathbb{Z}_n(\mathbb{B}_1)$ and T^{-1} is also a linear operator on $\mathbb{Z}_n(\mathbb{B}_1)$ and strongly preserves $\mathbb{N}_n(\mathbb{B}_1)$. Furthermore T permutes cells in $\mathbb{Z}_n(\mathbb{B}_1)$.*

Proof. It had been proved in [2] that for a finite semiring, there is a power of T which is idempotent. Let $L = T^p$ be idempotent for some integer $p \geq 1$. Then L still strongly preserves $\mathbb{N}_n(\mathbb{B}_1)$. By Lemma 2.2, we have that L is nonsingular.

To show that $T : \mathbb{Z}_n(\mathbb{B}_1) \to \mathbb{Z}_n(\mathbb{B}_1)$ is invertible, we will claim that $L : \mathbb{Z}_n(\mathbb{B}_1) \to \mathbb{Z}_n(\mathbb{B}_1)$ is the identity map. Let E be an off-diagonal cell in $\mathbb{Z}_n(\mathbb{B}_1)$. Since T and L are nonsingular, we have $T(E) \neq O_n$ and $L(E) \neq O_n$. It follows from Proposition 2.1 that neither $T(E)$ nor $L(E)$ can dominate any diagonal cell because $T(E)$ and $L(E)$ are nilpotent. Thus there exists at least one off-diagonal cell F in $\mathbb{Z}_n(\mathbb{B}_1)$ such that $L(E) \geq F$. If $E \neq F$, then we have $E + F^t \in \mathbb{N}_n(\mathbb{B}_1)$ so that $L(E + F^T) \in \mathbb{N}_n(\mathbb{B}_1)$. Now,

$$\begin{aligned} L(E + F^t) = L(E) + L(F^t) &= L^2(E) + L(F^t) \\ &\geq L(F) + L(F^t) = L(F + F^t). \end{aligned}$$

By Proposition 2.1, $L(F + F^t) \in \mathbb{N}_n(\mathbb{B}_1)$ while $F + F^t \notin \mathbb{N}_n(\mathbb{B}_1)$, a contradiction. Therefore $F = E$ and hence $L(E)$ dominates E only, that is, $L(E) = E$. Therefore L must be the identity map on $\mathbb{Z}_n(\mathbb{B}_1)$ and so T is invertible on $\mathbb{Z}_n(\mathbb{B}_1)$.

Let $A, B \in \mathbb{Z}_n(\mathbb{B}_1)$ and $a, b \in \mathbb{B}_1$ be arbitrary. Then we have

$$T(aT^{-1}(A) + bT^{-1}(B)) = aA + bB,$$

equivalently

$$T^{-1}(aA + bB) = aT^{-1}(A) + bT^{-1}(B).$$

Thus T^{-1} is a linear operator on $\mathbb{Z}_n(\mathbb{B}_1)$. It is easy to show that T^{-1} strongly preserves $\mathbb{N}_n(\mathbb{B}_1)$.

Let E be a cell in $\mathbb{Z}_n(\mathbb{B}_1)$. Then there exists at least one cell F in $\mathbb{Z}_n(\mathbb{B}_1)$ such that $T(E) \geq F$ or equivalently, $E \geq T^{-1}(F)$. Since T^{-1} is nonsingular, it follows that $E = T^{-1}(F)$, equivalently, $T(E) = F$. Since T is invertible on $\mathbb{Z}_n(\mathbb{B}_1)$, it follows that T permutes cells in $\mathbb{Z}_n(\mathbb{B}_1)$. □

COROLLARY 2.4. *Let $T : \mathbb{M}_n(\mathbb{B}_1) \to \mathbb{M}_n(\mathbb{B}_1)$ be a linear operator which strongly preserves $\mathbb{N}_n(\mathbb{B}_1)$. Then we have $T(A^t) = T(A)^t$ for all $A \in \mathbb{Z}_n(\mathbb{B}_1)$.*

Proof. Let E be any cell in $\mathbb{Z}_n(\mathbb{B}_1)$. Consider a matrix $E + E^t$. Then we have $E + E^t \notin \mathbb{N}_n(\mathbb{B}_1)$ so that $T(E) + T(E^t) = T(E + E^t) \notin \mathbb{N}_n(\mathbb{B}_1)$. By Lemma 2.3, both $T(E)$ and $T(E^t)$ are different cells in $\mathbb{Z}_n(\mathbb{B}_1)$. This shows that $T(E^t) = T(E)^t$ because $T(E) + T(E^t) \notin \mathbb{N}_n(\mathbb{B}_1)$. It follows that $T(A^t) = T(A)^t$ for all $A \in \mathbb{Z}_n(\mathbb{B}_1)$. □

A matrix $E \in \mathbb{Z}_n(\mathbb{B}_k)$ is called an *s-star matrix* if $|E| = s$ and all its nonzero entries lie on a row or column for $2 \leq s \leq n-1$. Then we can easily show that any s-star matrix is nilpotent for all $s = 2, \ldots, n-1$.

PROPOSITION 2.5. *If A is a 2-star matrix in $\mathbb{Z}_n(\mathbb{B}_1)$, there exists a permutation matrix P such that $PAP^t = E_{12} + E_{13}$ or $E_{21} + E_{31}$.*

Proof. Since A is a 2-star matrix, its form has $A = E_{ij} + E_{ik}$ or $E_{ij} + E_{kj}$, where i, j and k are all distinct. For the case of $E = E_{ij} + E_{ik}$, let α be a permutation of $\{1, 2, \ldots, n\}$ such that $\alpha(i) = 1$, $\alpha(j) = 2$, $\alpha(k) = 3$ and $\alpha(l) = l$ for all $l \in \{1, 2, \ldots, n\} \setminus \{i, j, k, 1, 2, 3\}$. Let $P = \sum_{s=1}^{n} E_{\alpha(s)s}$. Then P is the permutation matrix corresponding to α and

$$\begin{aligned} PAP^t &= \left(\sum_{s=1}^{n} E_{\alpha(s)s}\right)(E_{ij} + E_{ik})\left(\sum_{s=1}^{n} E_{s\alpha(s)}\right) \\ &= E_{\alpha(i)\alpha(j)} + E_{\alpha(i)\alpha(k)} = E_{12} + E_{13}. \end{aligned}$$

Similarly, if $E = E_{ij} + E_{kj}$, we obtain that $PAP^t = E_{21} + E_{31}$ for some permutation matrix P. □

COROLLARY 2.6. *Let A be a matrix in $\mathbb{Z}_n(\mathbb{B}_1)$ with $|A| = 2$. If A is not a 2-star matrix, then there exists a permutation matrix P such that $PAP^t = E_{12} + E$, where $E = E_{21}$, E_{23}, E_{31} or E_{34}.*

Proof. The proof follows from a similar method to Proposition 2.5. □

LEMMA 2.7. *If T is a linear operator on $\mathbb{M}_n(\mathbb{B}_1)$ that strongly preserves $\mathbb{N}_n(\mathbb{B}_1)$, then T preserves 2-star matrices.*

Proof. Let A be a 2-star matrix. By Proposition 2.5, we can assume that $A = E_{12} + E_{13}$ or $E_{21} + E_{31}$. For the case of $A = E_{12} + E_{13}$, we have $|T(A)| = 2$ by Lemma 2.3. Assume that $T(A)$ is not a 2-star matrix. By Corollary 2.6, there exists a permutation matrix P such that $PT(A)P^t = E_{12} + E$, where $E = E_{21}$, E_{23}, E_{31} or E_{34}. Since permutation similarity preserves nilpotent matrices, without loss of generality, we may assume that $T(A) = E_{12} + E$. Let

$$M = \begin{cases} E_{12} + E_{21} & \text{if } E = E_{21}, \\ E_{12} + E_{23} + E_{31} & \text{if } E = E_{23} \text{ or } E_{31}, \\ E_{12} + E_{23} + E_{34} + E_{41} & \text{if } E = E_{34}, \end{cases}$$

so that M is not nilpotent with $M \geq T(A)$. If we let $N = M \setminus T(A)$, then both $N + T(E_{12})$ and $N + T(E_{13})$ are nilpotent. By Lemma 2.3, both $B_1 = T^{-1}(N) + E_{12}$ and $B_2 = T^{-1}(N) + E_{13}$ are nilpotent. It follows that

$$B_1 + T^{-1}(N) = B_1 \quad \text{and} \quad B_2 + T^{-1}(N) = B_2. \tag{2.1}$$

Let $B = T^{-1}(N) + E_{12} + E_{13}$ so that $B = B_1 + B_2$. Now we will show that

$$B^m = B_1^m + B_2^m \tag{2.2}$$

for all integer $m \geq 1$. We shall prove (2.2) by induction on m. If $m = 1$, the result is obvious. We may assume that (2.2) holds for $m \geq 1$. Then by (2.1), we have

$$\begin{aligned} B^{m+1} &= BB^m \\ &= B(B_1^m + B_2^m) \\ &= (B_1 + B_2)(B_1^m + B_2^m) \\ &= B_1^{m+1} + B_2^{m+1} + B_2B_1^m + B_1B_2^m \\ &= B_1^{m+1} + B_2^{m+1} + T^{-1}(N)B_1^m + E_{13}B_1^m + T^{-1}(N)B_2^m + E_{12}B_2^m \\ &= (B_1 + T^{-1}(N))B_1^m + (B_2 + T^{-1}(N))B_2^m + E_{13}B_1^m + E_{12}B_2^m \\ &= B_1^{m+1} + B_2^{m+1} + E_{13}B_1^m + E_{12}B_2^m. \end{aligned}$$

Now, we will show that

$$B_1^{m+1} + E_{12}B_2^m = B_1^{m+1} \quad \text{and} \quad B_2^{m+1} + E_{13}B_1^m = B_2^{m+1}.$$

Notice that

$$E_{13}B_1^m = E_{13}(T^{-1}(N) + E_{12})^m = \sum_{k=0}^{m} E_{13}(T^{-1}(N))^k E_{12}^{m-k}.$$

Let $X = E_{13}(T^{-1}(N))^k E_{12}^{m-k}$. Then we have that

$$X = \begin{cases} O_n & \text{if } k \leq m-2, \\ E_{13}(T^{-1}(N))^{m-1}E_{12} & \text{if } k = m-1, \\ E_{13}(T^{-1}(N))^m & \text{if } k = m. \end{cases}$$

If $X = E_{13}(T^{-1}(N))^m$, then it is a term of the expansion of B_2^{m+1} so that $B_2^{m+1} + X = B_2^{m+1}$. If $X = E_{13}(T^{-1}(N))^{m-1}E_{12}$, then we have $(T^{-1}(N))^{m-1} \geq E_{31}$. Let $C = [c_{ij}] = T^{-1}(N)$. Then the $(3,1)$-th entry of C^{m-1} is 1 so that there exist indices $i_1, i_2, \ldots, i_{m-2} \in \{1, 2, \ldots, n\}$ such that $c_{3i_1}c_{i_1i_2}\cdots c_{i_{m-2}1} = 1$. Therefore, we have $c_{3i_1} = c_{i_1i_2} = \cdots = c_{i_{m-2}1} = 1$ so that $T^{-1}(N) = C \geq E_{3i_1} + E_{i_1i_2} + \cdots + E_{i_{m-2}1}$. Thus we have

$$B_2 = T^{-1}(N) + E_{13} \geq E_{3i_1} + E_{i_1i_2} + \cdots + E_{i_{m-2}1} + E_{13}.$$

Since $E_{3i_1} + E_{i_1i_2} + \cdots + E_{i_{m-2}1} + E_{13}$ is not nilpotent, B_2 is not nilpotent by Proposition 2.1. This contradicts to the fact that B_2 is nilpotent. Therefore, we have established that $B_2^{m+1} + E_{13}B_1^m = B_2^{m+1}$. Similarly, we obtain that $B_1^{m+1} + E_{12}B_2^m = B_1^{m+1}$. Thus,

$$B^{m+1} = B_1^{m+1} + B_2^{m+1} + E_{13}B_1^m + E_{12}B_2^m = B_1^{m+1} + B_2^{m+1}.$$

Since B_1 and B_2 are nilpotent, there exist integers $m_1, m_2 \geq 1$ such that $B_i^{m_i} = O_n$ for $i = 1, 2$. Let $m = \max\{m_1, m_2\}$. Then we have $B^m = B_1^m + B_2^m = O_n + O_n = O_n$. Thus, B is nilpotent and so

$$T(B) = T(T^{-1}(N) + E_{12} + E_{13}) = N + E_{12} + E_{13} = M$$

is also nilpotent. This contradicts to the fact that M is not nilpotent. Therefore $T(A)$ is a 2-star matrix. Similarly, if $A = E_{21} + E_{31}$, we obtain that $T(A)$ is a 2-star matrix. Therefore T preserves 2-star matrices. □

COROLLARY 2.8. *If T is a linear operator on $\mathbb{M}_n(\mathbb{B}_1)$ that strongly preserves $\mathbb{N}_n(\mathbb{B}_1)$, then T preserves all s-star matrices for each $s = 2, 3, \ldots, n-1$.*

Proof. By Lemma 2.7, T preserves 2-star matrices. Assume that T preserves k-star matrices for some $k = 2, 3, \ldots, n-2$. Let A be any $(k+1)$-star matrix. Without loss of generality, we may assume that $A = E_{ii_1} + \cdots + E_{ii_k} + E_{ii_{k+1}}$, where $i \neq i_1, \ldots, i_k, i_{k+1}$. Let $B = E_{ii_1} + \cdots + E_{ii_k}$ so that B is a k-star matrix. By the induction hypothesis, we have $T(B)$ is a k-star matrix. Thus the form of $T(B)$ is either $E_{jj_1} + \cdots + E_{jj_k}$ or $E_{j_1j} + \cdots + E_{j_kj}$, where $j \neq j_1, \ldots, j_k$. Let $T(B) = E_{jj_1} + \cdots + E_{jj_k}$. Without loss of generality, we can take

$T(E_{ii_l}) = E_{jj_l}$ for each $l = 1, 2, \ldots, k$. Suppose that $T(A)$ is not $(k+1)$-star matrix. Then we have $T(E_{ii_{k+1}}) = E_{st}$, where $s \neq t$ and $s \neq j$. Thus, $T(E_{ii_1} + E_{ii_{k+1}}) = E_{jj_1} + E_{st}$ and $T(E_{ii_2} + E_{ii_{k+1}}) = E_{jj_2} + E_{st}$ are 2-star matrices because $E_{ii_1} + E_{ii_{k+1}}$ and $E_{ii_2} + E_{ii_{k+1}}$ are 2-star matrices. Since $s \neq j$, we have $j_1 = t$ and $j_2 = t$ so that $j_1 = j_2$, a contradiction. Hence $T(A)$ is a $(k+1)$-star matrix. Therefore T preserves s-star matrices for all $s = 2, 3, \ldots, n-1$. □

An $(n-1)$-star matrix $A \in \mathbb{Z}_n(\mathbb{B}_k)$ is called a *row matrix* (or *column matrix*) if all its nonzero entries lie on a row (or column). Let $R_i = \sum_{j \neq i} E_{ij}$ and $C_j = \sum_{i \neq j} E_{ij}$ for $i, j = 1, \ldots, n$. Then R_i is a row matrix and C_j is a column matrix for all $i, j = 1, \ldots, n$. Let $\mathbb{R} = \{R_i | i = 1, \ldots, n\}$ and $\mathbb{C} = \{C_j | j = 1, \ldots, n\}$.

PROPOSITION 2.9. *Let T be a linear operator on $\mathbb{M}_n(\mathbb{B}_1)$ that strongly preserves $\mathbb{N}_n(\mathbb{B}_1)$. Then*

(1) *T maps $\mathbb{R}$ to $\mathbb{R}$ and maps $\mathbb{C}$ to $\mathbb{C}$, or*
(2) *T maps $\mathbb{R}$ to $\mathbb{C}$ and maps $\mathbb{C}$ to $\mathbb{R}$.*

Proof. Let R_i be any row matrix. By Corollary 2.8, $T(R_i)$ is a row matrix or a column matrix. Assume that $T(R_i)$ is a row matrix. Say $T(R_i) = R_k$ for some $k = 1, \ldots, n$. Suppose that $T(R_j) = C_l$ for some $j \neq i$ and for some column matrix C_l. Since $|R_i + R_j| = 2n - 2$, it follows from Lemma 2.3 that $|R_k + C_l| = |T(R_i + R_j)| = 2n - 2$ so that $R_k = C_l^t$. By Corollary 2.4, $T(R_j^t) = T(R_j)^t = C_l^t = R_k = T(R_i)$ so that $R_j^t = R_i$. This is impossible. Thus T maps $\mathbb{R}$ to $\mathbb{R}$. By a similar method, T maps $\mathbb{C}$ to $\mathbb{C}$.

Similarly, if T maps a row matrix to a column matrix, we obtain that T maps $\mathbb{R}$ to $\mathbb{C}$ and maps $\mathbb{C}$ to $\mathbb{R}$. □

LEMMA 2.10. *Let T be a linear operator on $\mathbb{M}_n(\mathbb{B}_1)$ that strongly preserves $\mathbb{N}_n(\mathbb{B}_1)$. Then there exists a permutation matrix P such that $T(X) = PXP^t$ or $T(X) = PX^tP^t$ for all $X \in \mathbb{Z}_n(\mathbb{B}_1)$.*

Proof. Suppose that T strongly preserves $\mathbb{N}_n(\mathbb{B}_1)$. By Lemma 2.9, T maps $\mathbb{R}$ to $\mathbb{R}$ and maps $\mathbb{C}$ to $\mathbb{C}$, or T maps $\mathbb{R}$ to $\mathbb{C}$ and maps $\mathbb{C}$ to $\mathbb{R}$.

First, we assume that T maps $\mathbb{R}$ to $\mathbb{R}$ and maps $\mathbb{C}$ to $\mathbb{C}$. Then we have that $T(R_i) = R_{\alpha(i)}$ and $T(C_j) = C_{\beta(j)}$, where α and β are some permutations of $\{1, 2, \ldots, n\}$. Thus, for any cell $E_{ij} \in \mathbb{Z}_n(\mathbb{B}_1)$, we can write $T(E_{ij}) = E_{\alpha(i)\beta(j)}$. Let $P = \sum_{i=1}^{n} E_{\alpha(i)i}$ and $Q = \sum_{j=1}^{n} E_{j\beta(j)}$ be the permutation matrices corresponding to α and β, respectively. Then for

any matrix $X = [x_{ij}] \in \mathbb{Z}_n(\mathbb{B}_1)$, we have that $T(X) = \sum_{i,j=1}^{n} x_{ij} E_{\alpha(i)\beta(j)} = PXQ$.

Now, we will show that $Q = P^t$, equivalently $QP = I_n$. By Corollary 2.4, we have $PX^tQ = T(X^t) = T(X)^t = Q^tX^tP^t$ so that $QPX^tQP = X^t$. Suppose that $QP \neq I_n$. Since QP is a permutation matrix, there exists an (i,j)-th entry of QP is 1 for some $i \neq j$. Then we have $QPE_{ij}{}^tQP = E_{ij}$. This contradicts to $QPX^tQP = X^t$. Hence we have $QP = I_n$ so that $T(X) = PXP^t$ for all $X \in \mathbb{Z}_n(\mathbb{B}_1)$.

Next, we assume that T maps $\mathbb{R}$ to $\mathbb{C}$ and maps $\mathbb{C}$ to $\mathbb{R}$. By the similar argument, we obtain that $T(X) = PX^tP^t$ for all $X \in \mathbb{Z}_n(\mathbb{B}_1)$. □

For two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ in $\mathbb{M}_n(\mathbb{B}_k)$, $A \circ B$ denote the *Hadamard* (or *Schur*) *product*, the (i,j)-th entry of $A \circ B$ is $a_{ij}\, b_{ij}$. Set $K_n = J_n \setminus I_n$.

THEOREM 2.11. *Let T be a linear operator on $\mathbb{M}_n(\mathbb{B}_1)$. Then T strongly preserves $\mathbb{N}_n(\mathbb{B}_1)$ if and only if there exist a permutation matrix P and matrices D_i which are not nilpotent with $i = 1, \ldots, n$ such that either*

(1) $T(X) = P(X \circ K_n)P^t + \sum_{i=1}^{n} x_{ii} D_i$ *for all* $X \in \mathbb{M}_n(\mathbb{B}_1)$, *or*

(2) $T(X) = P(X^t \circ K_n)P^t + \sum_{i=1}^{n} x_{ii} D_i$ *for all* $X \in \mathbb{M}_n(\mathbb{B}_1)$.

Proof. Suppose that T strongly preserves $\mathbb{N}_n(\mathbb{B}_1)$. By Lemma 2.10, there exists a permutation matrix P such that $T(X) = PXP^t$ or $T(X) = PX^tP^t$ for all $X \in \mathbb{Z}_n(\mathbb{B}_1)$.

Assume that $T(X) = PXP^t$ for all $X \in \mathbb{Z}_n(\mathbb{B}_1)$. For any matrix X in $\mathbb{M}_n(\mathbb{B}_1)$, we have that $X \circ K_n \in \mathbb{Z}_n(\mathbb{B}_1)$ so that $T(X \circ K_n) = P(X \circ K_n)P^t$. Let $D_i = T(E_{ii})$ for each $i = 1, 2, \ldots, n$. It follows from $E_{ii} \notin \mathbb{N}_n(\mathbb{B}_1)$ that D_i is not a nilpotent matrix. Thus $T(X \circ I_n) = \sum_{i=1}^{n} x_{ii} D_i$. Since $X = (X \circ K_n) + (X \circ I_n)$, we have that

$$T(X) = T(X \circ K_n) + T(X \circ I_n) = P(X \circ K_n)P^t + \sum_{i=1}^{n} x_{ii} D_i$$

for all $X \in \mathbb{M}_n(\mathbb{B}_1)$.

Similarly, if $T(X) = PX^tP^t$, then we obtain that

$$T(X) = P(X^t \circ K_n)P^t + \sum_{i=1}^{n} x_{ii} D_i$$

for all $X \in \mathbb{M}_n(\mathbb{B}_1)$.

Conversely, let $X \in \mathbb{N}_n(\mathbb{B}_1)$. By Proposition 1.3, $X \circ I_n = O_n$ so that $x_{ii} = 0$ for all $i = 1, 2, \ldots, n$. It follows from $X \circ K_n = X$ that $T(X) = PXP^t$ or $T(X) = PX^tP^t$. Therefore we have $T(X) \in \mathbb{N}_n(\mathbb{B}_1)$ because similarity and transposition strongly preserves $\mathbb{N}_n(\mathbb{B}_1)$.

Let $T(X) \in \mathbb{N}_n(\mathbb{B}_1)$. Then we can easily show that $x_{ii} = 0$ for all $i = 1, 2, \ldots, n$. Thus we have $X \circ I_n = O_n$ so that $X = X \circ K_n$. Therefore we have that $T(X) = PXP^t$ or $T(X) = PX^tP^t$. It follows that $X \in \mathbb{N}_n(\mathbb{B}_1)$ because similarity and transposition strongly preserves $\mathbb{N}_n(\mathbb{B}_1)$. □

3. Nilpotent matrix preservers over general Boolean algebra

In this section, we study nilpotent matrices over general Boolean algebras $\mathbb{B}_k$ with $k \geq 1$. Furthermore, using Lemma 2.10, we obtain characterizations of linear operators which strongly preserve nilpotent matrices over general Boolean algebras.

If T is a linear operator on $\mathbb{M}_n(\mathbb{B}_k)$ with $k \geq 1$, for each $1 \leq p \leq k$ define its p^{th} *constituent operator*, T_p, by

$$T_p(B) = (T(B))_p$$

for all $B \in \mathbb{M}_n(\mathbb{B}_k)$. By the linearity of T, we have

$$T(A) = \sum_{p=1}^{k} \sigma_p T_p(A_p) \tag{3.1}$$

for any matrix $A \in \mathbb{M}_n(\mathbb{B}_k)$ ([7]).

LEMMA 3.1. *If T is a linear operator on $\mathbb{M}_n(\mathbb{B}_k)$ which strongly preserves $\mathbb{N}_n(\mathbb{B}_k)$, then each p^{th} constituent operator, T_p, strongly preserves $\mathbb{N}_n(\mathbb{B}_1)$.*

Proof. Let A be any matrix in $\mathbb{M}_n(\mathbb{B}_1)$. Obviously, A is the matrix in $\mathbb{M}_n(\mathbb{B}_k)$ such that $A_p = A$ for all $p = 1, 2, \ldots, k$. Thus, we have that $A = \sum_{p=1}^{k} \sigma_p A_p = \sum_{p=1}^{k} \sigma_p A$. If $A \in \mathbb{N}_n(\mathbb{B}_1)$, then $A \in \mathbb{N}_n(\mathbb{B}_k)$ by Theorem 1.2. Since T preserves $\mathbb{N}_n(\mathbb{B}_k)$, $T(A) = \sum_{p=1}^{k} \sigma_p T_p(A_p) \in \mathbb{N}_n(\mathbb{B}_k)$. Again by Theorem 1.2, each $T_p(A_p) \in \mathbb{N}_n(\mathbb{B}_1)$ so that $T_p(A) \in \mathbb{N}_n(\mathbb{B}_1)$ for all $p = 1, 2, \ldots, k$.

Conversely, if $T_p(A) \in \mathbb{N}_n(\mathbb{B}_1)$ for each $p = 1, 2, \ldots, k$, then $T(A) = \sum_{p=1}^{k} \sigma_p T_p(A_p)$ and so $T(A) \in \mathbb{N}_n(\mathbb{B}_k)$ by Theorem 1.2. Hence $A \in \mathbb{N}_n(\mathbb{B}_k)$. By Theorem 1.2, we have that $A(= A_p) \in \mathbb{N}_n(\mathbb{B}_1)$. □

For any fixed invertible matrix U in $\mathbb{M}_n(\mathbb{B}_k)$, the operator $A \to UAU^t$ is called a *similarity operator*. And the operator $B \to B^t$ is called a *transposition operator* on $\mathbb{M}_n(\mathbb{B}_k)$ ([7]). We can easily show that any similarity operator and transposition operator on $\mathbb{M}_n(\mathbb{B}_k)$ are linear operators which strongly preserve nilpotent matrices. Also, we can restate Lemma 2.10 as follows: the linear operators on $\mathbb{Z}_n(\mathbb{B}_1)$ that strongly preserve $\mathbb{N}_n(\mathbb{B}_1)$ are compositions of transpositions and similarity operators. But for a general Boolean algebra $\mathbb{B}_k$ with $k \geq 2$, the following example shows that there exists another linear operator on $\mathbb{Z}_n(\mathbb{B}_k)$ that strongly preserves $\mathbb{N}_n(\mathbb{B}_k)$ which is neither a transposition operator nor a similarity operator.

EXAMPLE 3.2. Let

$$U = \begin{bmatrix} \sigma_1 & \sigma_2 & \sigma_3 \\ \sigma_2 & \sigma_3 & \sigma_1 \\ \sigma_3 & \sigma_1 & \sigma_2 \end{bmatrix} \in \mathbb{M}_3(\mathbb{B}_3).$$

By Lemma 1.1, U is an invertible matrix in $\mathbb{M}_3(\mathbb{B}_3)$ with $U^{-1} = U^t$. Define an operator T on $\mathbb{Z}_3(\mathbb{B}_3)$ by

$$T(X) = U(\sigma_1 X_1 + \sigma_2 X_2^t + \sigma_3 X_3)U^t$$

for all $X = \sum_{p=1}^{3} \sigma_p X_p \in \mathbb{Z}_3(\mathbb{B}_3)$. Then we can easily show that T is a linear operator on $\mathbb{Z}_3(\mathbb{B}_3)$ which is neither a transposition operator nor a similarity operator. It is easy to show that T strongly preserves $\mathbb{N}_3(\mathbb{B}_3)$. □

LEMMA 3.3. *Let T be a linear operator on $\mathbb{M}_n(\mathbb{B}_k)$ that strongly preserves $\mathbb{N}_n(\mathbb{B}_k)$. Then there exists an invertible matrix U in $\mathbb{M}_n(\mathbb{B}_k)$ such that*

$$T(X) = U\Big(\sum_{p=1}^{k} \sigma_p Y_p\Big)U^t$$

for all $X \in \mathbb{Z}_n(\mathbb{B}_k)$, where $Y_p = X_p$ or $Y_p = X_p^t$ for each $p = 1, \ldots, k$.

Proof. Assume that T strongly preserves $\mathbb{N}_n(\mathbb{B}_k)$. By Lemma 3.1, each p^{th} constituent operator, T_p, on $\mathbb{M}_n(\mathbb{B}_1)$ strongly preserves $\mathbb{N}_n(\mathbb{B}_1)$.

By Lemma 2.10, each T_p has the form

$$T_p(X_p) = Q_p X_p Q_p^t \quad \text{or} \quad T_p(X_p) = Q_p X_p^t Q_p^t$$

for all $X = \sum_{p=1}^{k} \sigma_p X_p \in \mathbb{Z}_n(\mathbb{B}_k)$, where each Q_p is a permutation matrix for all $p = 1, \ldots, k$. By (3.1), we have $T(X) = \sum_{p=1}^{k} \sigma_p Q_p Y_p Q_p^t$, where $Y_p = X_p$ or $Y_p = X_p^t$ for each $p = 1, \ldots, k$, equivalently

$$T(X) = \Big(\sum_{p=1}^{k} \sigma_p Q_p\Big)\Big(\sum_{p=1}^{k} \sigma_p Y_p\Big)\Big(\sum_{p=1}^{k} \sigma_p Q_p\Big)^t.$$

If we let $U = \Big(\sum_{p=1}^{k} \sigma_p Q_p\Big)$, then U is invertible in $\mathbb{M}_n(\mathbb{B}_k)$ by Lemma 1.1, and hence the result is satisfied. □

PROPOSITION 3.4. *Let D be a matrix in $\mathbb{M}_n(\mathbb{B}_k)$ all of whose constituents are not nilpotent in $\mathbb{M}_n(\mathbb{B}_1)$. Then xD is not a nilpotent matrix for all nonzero $x \in \mathbb{B}_k$.*

Proof. Suppose that $xD \in \mathbb{N}_n(\mathbb{B}_k)$ for some nonzero $x \in \mathbb{B}_k$. Then there exists $\sigma_p \in \mathbb{B}_k$ such that $x \geq \sigma_p$ so that $xD \geq \sigma_p D$. By Proposition 2.1, we have $\sigma_p D$ is nilpotent. But $\sigma_p D = D_p$ is not nilpotent, a contradiction. Hence xD is not a nilpotent matrix for all nonzero $x \in \mathbb{B}_k$. □

THEOREM 3.5. *Let T be a linear operator on $\mathbb{M}_n(\mathbb{B}_k)$ with $k \geq 1$. Then T strongly preserves $\mathbb{N}_n(\mathbb{B}_k)$ if and only if there exist an invertible matrix U in $\mathbb{M}_n(\mathbb{B}_k)$ and matrices D_i all of whose constituents are not nilpotent for $i = 1, \ldots, n$ such that*

$$T(X) = U\Big(\sum_{p=1}^{k} (\sigma_p Y_p \circ K_n)\Big)U^t + \sum_{i=1}^{n} x_{ii} D_i$$

for all $X \in \mathbb{M}_n(\mathbb{B}_k)$, where $Y_p = X_p$ or $Y_p = X_p^t$ for each $p = 1, \ldots, k$.

Proof. Assume that T strongly preserves $\mathbb{N}_n(\mathbb{B}_k)$. By Lemma 3.3, there exists an invertible matrix U in $\mathbb{M}_n(\mathbb{B}_k)$ such that

$$T(X) = U\Big(\sum_{p=1}^{k} \sigma_p Y_p\Big)U^t$$

for all $X \in \mathbb{Z}_n(\mathbb{B}_k)$, where $Y_p = X_p$ or $Y_p = X_p^t$ for each $p = 1, \ldots, k$.

For any matrix X in $\mathbb{M}_n(\mathbb{B}_k)$, we have that $X \circ K_n \in \mathbb{Z}_n(\mathbb{B}_k)$ so that $T(X \circ K_n) = U\Big(\sum_{p=1}^{k}(\sigma_p Y_p \circ K_n)\Big)U^t$, where $Y_p = X_p$ or $Y_p = X_p^t$ for each $p = 1, \ldots, k$. Let $D_i = T(E_{ii})$ for each $i = 1, 2, \ldots, n$. Suppose that a p^{th} constituent, $(D_i)_p$, of D_i is nilpotent in $\mathbb{M}_n(\mathbb{B}_1)$. Then we have $(D_i)_p = \sigma_p D_i$ is nilpotent. This contradicts to the fact that $T(\sigma_p E_{ii}) = \sigma_p D_i$ is not nilpotent. Therefore all constituents of D_i are not nilpotent in $\mathbb{M}_n(\mathbb{B}_1)$. Furthermore, by Proposition 3.4, we obtain that xD_i is not nilpotent for all nonzero $x \in \mathbb{B}_k$. It follows from $D_i = T(E_{ii})$ that $T(X \circ I_n) = \sum_{i=1}^{n} x_{ii} D_i$. Since $X = (X \circ K_n) + (X \circ I_n)$, we have that

$$T(X) = T(X \circ K_n) + T(X \circ I_n) = U\Big(\sum_{p=1}^{k}(\sigma_p Y_p \circ K_n)\Big)U^t + \sum_{i=1}^{n} x_{ii} D_i$$

for all $X \in \mathbb{M}_n(\mathbb{B}_k)$, where $Y_p = X_p$ or $Y_p = X_p^t$ for each $p = 1, \ldots, k$.

Conversely, let $X \in \mathbb{N}_n(\mathbb{B}_k)$. By Proposition 1.3, $X \circ I_n = O_n$ so that $x_{ii} = 0$ for all $i = 1, \ldots, n$. Therefore we have $T(X) = U\Big(\sum_{p=1}^{k}(\sigma_p Y_p \circ K_n)\Big)U^t$, where $Y_p = X_p$ or $Y_p = X_p^t$ for each $p = 1, \ldots, k$. It follows from Theorem 1.2 that $Y_p \in \mathbb{N}_n(\mathbb{B}_1)$ for all $p = 1, \ldots, k$ because $X \in \mathbb{N}_n(\mathbb{B}_k)$. Thus we have $\sum_{p=1}^{k}(\sigma_p Y_p \circ K_n) \in \mathbb{N}_n(\mathbb{B}_k)$. Since similarity operator strongly preserves $\mathbb{N}_n(\mathbb{B}_k)$, we obtain that $T(X) \in \mathbb{N}_n(\mathbb{B}_k)$.

Let $T(X) \in \mathbb{N}_n(\mathbb{B}_k)$. By Proposition 3.4, we have $x_{ii} = 0$ for all $i = 1, \ldots, n$ so that $T(X) = U\Big(\sum_{p=1}^{k}(\sigma_p Y_p \circ K_n)\Big)U^t$. Since similarity operator strongly preserves $\mathbb{N}_n(\mathbb{B}_k)$, we have $\sum_{p=1}^{k}(\sigma_p Y_p \circ K_n) \in \mathbb{N}_n(\mathbb{B}_k)$. Since $x_{ii} = 0$ for all $i = 1, \ldots, n$, we have $Y_p \in \mathbb{N}_n(\mathbb{B}_1)$ from Theorem 1.2 for each $p = 1, \ldots, k$, equivalently $X_p \in \mathbb{N}_n(\mathbb{B}_1)$. Therefore, we obtain that $X = \sum_{p=1}^{k} \sigma_p X_p \in \mathbb{N}_n(\mathbb{B}_k)$ by Theorem 1.2. □

Thus we obtain characterizations of linear operators which strongly preserve nilpotent matrices over general Boolean algebras.

References

[1] L. B. Beasley and N. J. Pullman, *Boolean rank preserving operators and Boolean rank-1 spaces*, Linear Algebra Appl. **59** (1984), 55–77.

[2] ———, *Operators that preserve semiring matrix functions*, Linear Algebra Appl. **99** (1988), 199–216.

[3] P. Botta, S. Pierce, and W. Watkins, *Linear transformations that preserve the nilpotent matrices*, Pacific J. Math. **104** (1983), no. 1, 39–46.
[4] K. H. Kim, *Boolean matrix theory and applications*, Pure and Applied Mathematics, Vol. 70, Marcel Dekker, New York, 1982.
[5] S. Kirkland and N. J. Pullman, *Linear operators preserving invariants of nonbinary matrices*, Linear Multilinear Algebra **33** (1993), 295–300.
[6] S. -Z. Song, L. B. Beasley, G. -S. Cheon, and Y. -B. Jun, *Rank and perimeter preservers of Boolean rank-1 matrices*, J. Korean Math. Soc. **41** (2004), no. 2, 397–406.
[7] S. -Z. Song and S. -G. Lee, *Column ranks and their preservers of general Boolean matrices*, J. Korean Math. Soc. **32** (1995), no. 3, 531–540.

Seok-Zun Song and Kyung-Tae Kang
Department of Mathematics
Cheju National University
Jeju 690-756, Korea
E-mail: szsong@cheju.ac.kr
kangkt@cheju.ac.kr

Young-Bae Jun
Department of Mathematics Education
Gyeongsang National University
Jinju 660-701, Korea
E-mail: ybjun@nongae.gsnu.ac.kr