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A STUDY ON SOME PERIODIC TIME VARYING BILINEAR MODEL

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ABSTRACT. We consider a class of bilinear models with periodic regime switching and find easy-to-check sufficient conditions that ensures the existence of a stationary process obtained from given difference equation. Existence of a higher order moments is examined.

1. Introduction

Many real time series exhibit nonlinear features which cannot be adequately modeled by linear models. Therefore during the last two decades, nonlinear time series models have gained much attention and many types of nonlinear models are suggested and applied successively in various fields. A particular class of stationary and nonlinear models is the class of bilinear models. Usefulness of bilinear model is demonstrated by many authors (see, for example, Granger and Anderson [4], Subba Rao and Gabr [9], Terdik [10]). In addition to nonlinearity, it is observed that the structure of a lot of real time series encountered in many applications vary over time. For such series, nonlinear model with time varying coefficients may be a more realistic model. Bilinear time series with time varying coefficients is a general extension of deterministic coefficients bilinear models and may be a useful tool in describing the behavior of a wide class of nonlinear time series (Hallin [5], Bibi and Oyet [1]).

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A general time varying bilinear time series is given by

$$(1.1)\quad X_t = \sum_{i=1}^{q} a_i(t)X_{t-i} + \sum_{j=1}^{p} c_j(t)e_{t-j} + \sum_{i=1}^{P}\sum_{j=1}^{Q} b_{ij}(t)X_{t-i}e_{t-j} + e_t.$$

By setting $b_{ij}(t) = 0$ for all i, j, we obtain the time varying ARMA(p, q) models. The statistical and probabilistic properties for some subclasses of time invariant bilinear models have been presented in many studies(e.g., Subba Rao [8], Liu and Brockwell [7], Terdik [10] etc.). On the contrary, very few works have been presented on the time varying bilinear models because of the lack of stationarity and ergodicity in these models.

In this paper we will consider a class of periodic bilinear model with $m(\geq 2)$ switching regimes that admit the following representation:

$$(1.2)\qquad X_t = \sum_{i=1}^{q} a_i(t)X_{t-i} + \sum_{i=2}^{p} b_i(t)X_{t-i}e_{t-1} + e_t,$$

where p and q are known integers such that $p \geq 2,\ q \geq 0$. $\{e_t\}$ is a sequence of independent and identically distributed random variables with mean 0. The coefficients $\{a_i(t)\}$ and $\{b_i(t)\}$ switch between m-regimes i.e. $a_i(t) = \sum_{j=0}^{m-1} a_{ij} I_{\Delta^{(j)}}(t)$, $b_i(t) = \sum_{j=0}^{m-1} b_{ij} I_{\Delta^{(j)}}(t)$ and $\Delta^{(j)} = \{mk + j | k = \ldots, -1, 0, 1, 2, \ldots\}$, $j = 0, 1, \ldots, m-1$. $I_{\Delta^{(j)}}(t)$ denotes the indicator function of $\Delta^{(j)}$.

If $a_i(t) = 0$ for all $i = 1, 2, \ldots, q$, then (1.2) is reduced to the process given by

$$(1.3)\qquad X_t = \sum_{i=2}^{p} b_i(t)X_{t-i}e_{t-1} + e_t.$$

Our aim is to find sufficient conditions that ensure the existence of a stationary process which is connected to the equation (1.2) and (1.3). Existence of higher order moments is also examined.

2. Main results

We first consider the model $\{X_t\}$ generated by

$$(2.1)\qquad X_t = \sum_{i=1}^{q} a_i(t)X_{t-i} + \sum_{i=2}^{p} b_i(t)X_{t-i}e_{t-1} + e_t.$$

We can rewrite (2.1) as follows:

$$X(t) = D(t)X(t-1) + Ce_t \tag{2.2}$$

and $X_t = C'X(t)$ where $X(t) = (X_t, X_{t-1}, \ldots, X_{t-p+1})', C = (1, 0, \ldots, 0)'$, $D(t) = A(t) + B(t)e_{t-1}$ where $A(t)$ and $B(t)$ are proper matrices given below:

$$A(t) = \begin{pmatrix} a_1(t) & a_2(t) & a_3(t) & \cdots & a_{p-1}(t) & a_p(t) \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ & & \vdots & & & \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

and

$$B(t) = \begin{pmatrix} 0 & b_2(t) & b_3(t) & \cdots & b_p(t) \\ 0 & 0 & 0 & \cdots & 0 \\ & & \vdots & & \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Here we assume, without loss of generality, that $p \geq q$, $a_{q+1}(t) = \cdots = a_p(t) = 0$ and $a_i(t)$ and $b_i(t)$ are periodic sequences with period $m (\geq 2)$.

Let

$$F(t) = F(t, e_{t-1}, \ldots, e_{t-m}) = D(t)D(t-1)\cdots D(t-m+1)$$

and

$$G(t) = G(t, e_t, \ldots, e_{t-m+1}) = \sum_{n=1}^{m-1} (\Pi_{j=0}^{n-1} D(t-j))Ce_{t-n} + Ce_t.$$

Then

$$X(t) = F(t)X(t-m) + G(t). \tag{2.3}$$

Note that $F(t), F(t-m), F(t-2m), \ldots$ is a sequence of independent and identically distributed random matrices, and so is $G(t), G(t-m), G(t-2m), \ldots$, but $F(t)$ and $G(t)$ are not independent.

We define that

$$\gamma = \lim_{n\to\infty} \frac{1}{n} \log \|F(nm)\cdots F(m)\|, \tag{2.4}$$

$$Y(t) = \sum_{k=1}^{\infty} \{\Pi_{i=0}^{k-1} F(t-im)\} G(t-km) + G(t), \tag{2.5}$$

and

$$\mathbf{x}(t) = \sum_{n=1}^{\infty} \{\Pi_{j=0}^{n-1} D(t-j)\} C e_{t-n} + C e_t. \tag{2.6}$$

THEOREM 2.1. *Suppose that* $E|e_t| < \infty$ *and* $\gamma < 0$ *in* (2.4). *Then* (1) $Y(t)$ *converges almost surely and is the unique causal solution of* (2.3). (2) $\mathbf{x}(t)$ *in* (2.6) *is a unique solution of* (2.2). *In fact,* $Y(t)$ *and* $\mathbf{x}(t)$ *have the same limit a.e. and for each* i, $0 \le i \le m-1$, $\{(Y(mt+i), \ldots, Y(m(t-1)+i+1))\}$ *is stationary.* (3) *If* $E\|F(mn_0)\cdots F(m)\|^2 < 1$ *for some* n_0, *then* $\mathbf{x}(t)$ *is square integrable.*

PROOF. (1) Since $\{F(t-nm)\}_{n=0}^{\infty}$ is a sequence of independent and identically distributed random matrices, an application of the subadditive ergodic theorem yields that

$$\begin{aligned} \gamma &= \lim_{n\to\infty} \frac{1}{n} \log \|F(nm)\cdots F(m)\| \\ &= \lim_{n\to\infty} \frac{1}{n} \log \|F(t)\cdots F(t-mn)\| \ a.s. \end{aligned}$$

for some constant $\gamma < 0$. If $E|e_t| < \infty$, then $E\log^+\|G(t)\| < \infty$. Following Theorem 1 in Brandt [3], (2.3) has a stationary and ergodic solution. By simple calculation, we can prove that $Y(t)$ is the unique stationary solution of (2.3).

(2) From (2.4) and $\gamma < 0$, we obtain that for $n = mk+i$, $0 \le i \le m-1$

$$\begin{aligned} & \limsup \frac{1}{n} \log \|D(t)\cdots D(t-n+1)\| \\ = & \limsup \frac{1}{n} \log \|D(t)\cdots D(t-mk+1)D(t-mk) \\ & \qquad \cdots D(t-mk-i+1)\| \\ \le & \limsup \frac{1}{n}\Big(\log \|F(t)\cdots F(t-(k-1)m)\| \\ & + \log \|D(t-mk)\cdots D(t-mk-i)\|\Big) \\ < & \ 0 \ a.e. \end{aligned}$$

for sufficiently large n, and hence the convergence of $\mathbf{x}(t)$ follows from Cauchy's criterion. Suppose that $\mathbf{y}(t)$ is another solution of (2.2). Then

$$\begin{aligned} \|\mathbf{x}(t) - \mathbf{y}(t)\| &\le \|\Pi_{j=0}^{l} D(t-j)\| \|\mathbf{x}(t-l) - \mathbf{y}(t-l)\| \\ &\to 0 \text{ as } l \to \infty \ a.e. \end{aligned}$$

Now, define

$$Y(t,n) = \sum_{k=1}^{n} F(t)\cdots F(t-(k-1)m)G(t-km) + G(t)$$

$$\mathbf{x}(t,n) = \sum_{k=1}^{n} \Pi_{j=0}^{k-1} D(t-j)Ce_{t-k} + Ce_t.$$

Then we have that

$$\begin{aligned} & \|Y(t,n) - \mathbf{x}(t, mn+s)\| \\ \leq & \ \| \sum_{k=nm+1}^{nm+s} \Pi_{j=0}^{k-1} D(t-j)e_{t-k}\| \\ \leq & \ \|F(t)\cdots F(t-nm)\| \cdot K \to 0 \text{ as } n \to \infty, \end{aligned}$$

where $0 \leq s \leq m-1$ and K is a constant.

(3) From assumption, we have that

$$(E\|D(t)\cdots D(t-n_0+1)\|^2)^{\frac{1}{2}} = r < 1.$$

We prove that $\mathbf{x}(t)$ converges absolutely in the space of square integrable random variables, i.e. for each t,

$$\sum_{k=0}^{\infty} [E\|\Pi_{j=0}^{k-1} D(t-j)Ce_{t-k}\|^2]^{\frac{1}{2}} < \infty.$$

Since $D(t)$ depends only on e_{t-1}, we have the following inequality:

$$\begin{aligned} & (E\|\Pi_{j=0}^{k-1} D(t-j)Ce_{t-k}\|^2)^{\frac{1}{2}} \\ \leq & \ (E\|\Pi_{j=0}^{k-2} D(t-j)\|^2)^{\frac{1}{2}} (E\|D(t-k+1)Ce_{t-k}\|^2)^{\frac{1}{2}}. \end{aligned}$$

Let

$$M_k = \sum_{l=kn_0+1}^{(k+1)n_0} (E\|\Pi_{j=kn_0}^{l-1} D(t-j)\|^2)^{\frac{1}{2}}, \ k = 1, 2, \ldots.$$

From stationarity of $\{(D(t-kn_0), \ldots, D(t-(k+1)n_0+1))\}_{k=0}^{\infty}$, $M = M_k, k = 0, 1, 2, \ldots$ and let $r = (E\|D(t-kn_0)\cdots D(t-(k+1)n_0+1)\|^2)^{\frac{1}{2}}$. Independence of $\{D(t)\}$ implies that

$$\begin{aligned} & \sum_{k=0}^{\infty} (E\|\Pi_{j=0}^{k-1} D(t-j)\|^2)^{\frac{1}{2}} \\ \leq & \ 1 + M_1 + rM_2 + r^2 M_3 + \cdots \\ = & \ 1 + M(1 + r + r^2 + \cdots) < \infty \end{aligned}$$

and the proof is completed. □

In most cases of interest, γ cannot be calculated explicitly when $p > 1$. However, relation (2.4) offers a potential method for determining the value of γ, via Monte-Carlo simulations of the random matrices $F(nm)$.

For any square matrix M, the spectral radius of M is denoted by $\rho(M) = \max_i\{|\lambda_i(M)|\}$ where $\lambda_i(M)$ is the i^{th} eigenvalue of M. $|M|$ is a matrix of absolute values of entries of M. That is if $M = (m_{ij})$, then $|M| = (|m_{ij}|)$.

For the remaining part of this section, we consider the case that $A(t) \equiv 0$ in (2.1) To simplify notation, we assume that $m = 2$. Then the equation (2.1) can be written as

$$X_t = \sum_{i=2}^{p} b_i(t) X_{t-i} e_{t-1} + e_t. \tag{2.7}$$

Here $b_i(t) = \sum_{j=0}^{1} b_{ij} I_{\Delta^{(j)}}(t)$ and $\Delta^{(j)} = \{2k+j | k = \ldots, -1, 0, 1, 2, \ldots\}$, $i = 2, \ldots, p$, $j = 0, 1$.

Let $F(t) = D(t)D(t-1)$ and $G(t) = D(t)Ce_{t-1} + Ce_t$. $X(t) = F(t)$ $X(t-2) + G(t)$ where $D(t) = A + B(t)e_{t-1}$ with $B(t) = \sum_{j=0}^{1} B_j I_{\Delta^{(j)}}(t)$. Here A, B_0, and B_1 are proper matrices given below.

$$A = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ & & & \vdots & & \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

and for $j = 0, 1$

$$B_j = \begin{pmatrix} 0 & b_{2,j} & b_{3,j} & \cdots & b_{p,j} \\ 0 & 0 & 0 & \cdots & 0 \\ & & \vdots & & \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

LEMMA 2.1. (Kesten and Spitzer [6]) *Suppose that $\{A_n\}$ is a sequence of independent and identically distributed nonnegative random matrices, then*

$$\lim_{n\to\infty} \frac{1}{n} \log \|A_n A_{n-1} \cdots A_1\| \le \log \rho(E[A_1])$$

LEMMA 2.2. (Geršgorin) *Let $M = (m_{ij})$ be a $n \times n$ square matrix. Then*

$$\rho(M) \leq \min_{d_1,\dots,d_n} \max_{1\leq i\leq n} \frac{1}{d_i} \sum_{j=1}^{n} d_j |m_{ij}|$$

and

$$\rho(M) \leq \min_{d_1,\dots,d_n>0} \max_{1\leq j\leq n} d_j \sum_{i=1}^{n} \frac{1}{d_i} |m_{ij}|.$$

Let $E|e_t| = \mu < \infty$ and $E(e_t^2) = \sigma^2 < \infty$.

THEOREM 2.2. *Consider $\{X_t\}$ given in (2.7).*

(1) *If $E\log^+|e_t| < \infty$ and $\rho E|D(t)D(t-1)| < 1$, then the conclusion of theorem 2.1 (1) and (2) holds.*

(2) *If $\max\{\sigma^2 \sum_{i=2}^{p} b_{i,0}^2, \sigma^2 \sum_{i=2}^{p} b_{i,1}^2\} < 1$, then $Var(X_t)$ is uniformly bounded.*

PROOF. (1) Since for fixed t, a sequence $\{F(t+2k)\}, k = 1, 2, \dots$ is independent and identically distributed, result follows from Lemma 2.1 and Theorem 2.1.

(2) Let $r_t = E(X_t^2), R(t) = (r_t, \dots, r_{t-p+1})'$. Then

$$r_t = \sigma^2 \sum_{i=2}^{p} \{b_i(t)^2 r_{t-i}\} + \sigma^2$$

and

$$R(t) = A(t)R(t-1) + C\sigma^2,$$

where

$$A(t) = \begin{pmatrix} 0 & \sigma^2 b_2(t)^2 & \sigma^2 b_3(t)^2 & \cdots & \sigma^2 b_p(t)^2 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \vdots & & \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

If t is even, then

$$A(t)A(t-1) = (a_{ij})$$

$$= \begin{pmatrix} \sigma^2 b_{2,0}^2 & \sigma^2 b_{3,0}^2 & \cdots & \sigma^2 b_{p-1,0}^2 & \sigma^2 b_{p,0}^2 & 0 \\ 0 & \sigma^2 b_{2,1}^2 & \cdots & \sigma^2 b_{p-2,1}^2 & \sigma^2 b_{p-1,1}^2 & \sigma^2 b_{p,1}^2 \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ & & \vdots & & & \\ 0 & 0 & \cdots & 1 & 0 & 0 \end{pmatrix}.$$

The fact that $r_t < \infty$ uniformly in t can be obtained from $\rho(A(t)A(t-1)) < 1$.

Now choose $\alpha > 0$ so that $\max\{\sigma^2 \sum_{i=2}^p b_{i,0}^2, \sigma^2 \sum_{i=2}^p b_{i,1}^2\} + \alpha = 1$. Choose $d_1 > \frac{p-1}{2} > 0$ and fix. and define $d_{i+1} = d_i + \frac{\alpha}{p}$, $i = 1, 2, \dots, p-1$. Then $\frac{p-1}{2} < d_1 < d_2 < \cdots < d_p < 1$ and we can prove that

$$\frac{1}{d_1} \sum_{j=1}^{p-1} d_j \cdot \sigma^2 b_{j+1,0}^2 < 1,$$

$$\frac{1}{d_2} \sum_{j=2}^{p} d_j \cdot \sigma^2 b_{j,1}^2 < 1.$$

Therefore we have that, by Lemma 2.3,

$$\rho(A(t)A(t-1)) \le \max_{1 \le i \le n} \frac{1}{d_i} \sum_{j=1}^{p} d_j a_{ij} < 1$$

and conclusion follows. For the case that t is odd, result can be derived by the same manner. □

COROLLARY 2.1. *If* $\max\{\mu \sum_{i=2}^p |b_{i,0}|, \mu \sum_{i=2}^p |b_{i,1}|\} < 1$, *then the conclusion of Theorem 2.1* (1) *and* (2) *holds.*

PROOF. If t is even, then $E|D(t)D(t-1)|$ is given by

$$E|D(t)D(t-1)|$$
$$= \begin{pmatrix} |b_{2,0}|\mu & |b_{3,0}|\mu & \cdots & |b_{p-1,0}|\mu & |b_{p,0}|\mu & 0 \\ 0 & |b_{2,1}|\mu & \cdots & |b_{p-2,1}|\mu & |b_{p-1,1}|\mu & |b_{p,1}|\mu \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ & & \vdots & & & \\ 0 & 0 & \cdots & 1 & 0 & 0 \end{pmatrix}.$$

The results is obtained by adopting the similar method as that of the second part of the proof of Theorem 2.2. □

COROLLARY 2.2. *If*

$$(\mu \sum_{i=1}^{[p/2]} b_{2i,0} + 1)(\mu \sum_{i=1}^{[p/2]} b_{2i,1} + 1) + \mu^2 \sum_{i=1}^{[(p-1)/2]} b_{2i+1,0} \cdot \sum_{i=1}^{[(p-1)/2]} b_{2i+1,1} < 2,$$

then $\rho E|D(t)D(t-1)| < 1$ *and the conclusion of Theorem 2.1* (1) *and* (2) *holds. Here* $[x]$ *denotes the integer part of* x.

PROOF. The conclusion follows from Theorem 2.1 and the fact that all roots of $x^p - a_1x^{p-1} - a_2x^{p-2} - \cdots - a_p = 0$ lie inside the unit circle if $|a_1| + \cdots + |a_p| < 1$. □

REMARK 1. No conditions in Theorem 2.2(2), Corollary 2.1 and Corollary 2.2 are superior to any other else.

REMARK 2. Consider the model given in (2.7) with $m \geq 3$. By the same manner used in the proof of Corollaries 2.1 and 2.2, we can show that $\max_{0\leq j\leq m-1}\{\sigma^2 \sum_{i=2}^{p} b_{i,j}^2\} < 1$ yields that $Var(X_t) < M < \infty$ for some M and $\max_{0\leq j\leq m-1}\{\mu \sum_{i=2}^{p} |b_{i,j}|\} < 1$ implies that $\gamma < 0$ and therefore the conclusion of Theorem 2.1 (1) and (2) holds.

REMARK 3. In Bibi and Ho [2], it is proved that $\{X_t\}$ in (2.7) admits a unique stable solution if and only if $\rho(\Gamma_0\Gamma_1) < 1$, where $\Gamma_i = A^{\otimes 2} + B_i^{\otimes 2}$, $i = 0, 1$, where $A^{\otimes 2}$ denotes the Kronecker product of A by itself. But $\rho(\Gamma_1\Gamma_2) < 1$ involves the computation of the eigenvalues of a $p^2 \times p^2$ matrix, which is intensive for large p. Existence of higher order moments are also studied.

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