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THE BIVARIATE F_3-BETA DISTRIBUTION

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Abstract. A new bivariate beta distribution based on the Appell function of the third kind is introduced. Various representations are derived for its product moments, marginal densities, marginal moments, conditional densities and conditional moments. The method of maximum likelihood is used to derive the associated estimation procedure as well as the Fisher information matrix.

1. Introduction

There have been very few bivariate beta distributions proposed in the statistics literature, see Chapter 9 in Hutchinson and Lai [6], Chapter 4 in Arnold *et al.* [3] and Chapter 49 in Kotz *et al.* [7] for good reviews. The most recent bivariate beta distribution discussed in Olkin and Liu [9] is actually a particular case of a multivariate beta distribution introduced by Libby and Novick [8]. These distributions have attracted useful applications in several areas; for example, in the modeling of the proportions of substances in a mixture, brand shares, i.e., the proportions of brands of some consumer product that are bought by customers (Chatfield [4]), proportions of the electorate voting for the candidate in a two-candidate election (Hoyer and Mayer [5]) and the dependence between two soil strength parameters (A-Grivas and Asaoka [1]). They have also been used extensively as a prior in Bayesian statistics (see, for example, Apostolakis and Moieni [2]).

In this note, we introduce a new bivariate beta distribution and study its properties. The joint pdf of this new distribution is taken to be

$$f(x,y) = \frac{C x^{\beta-1} y^{\beta'-1} (1-x-y)^{\gamma-\beta-\beta'-1}}{(1-ux)^{\alpha}(1-vy)^{\alpha'}} \tag{1}$$

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for $0 < x < 1$, $0 < y < 1$, $0 < x+y < 1$, $-1 < u < 1$, $-1 < v < 1$, $\alpha > 0$, $\alpha' > 0$, $\beta > 0$, $\beta' > 0$ and $\gamma > \beta + \beta'$, where C denotes the normalizing constant. Application of equation (3.1.2.7) in Prudnikov *et al.* [10, volume 1] shows that one can determine C as

$$\frac{1}{C} = \frac{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta-\beta')}{\Gamma(\gamma)} F_3\left(\alpha, \alpha', \beta, \beta'; \gamma; u, v\right), \tag{2}$$

where F_3 is the Appell function of the third kind defined by

$$F_3\left(a, a', b, b'; c; z, \xi\right) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(a)_k (a')_l (b)_k (b')_l z^k \xi^l}{(c)_{k+l} k! l!},$$

where $(f)_k = f(f+1)\cdots(f+k-1)$ denotes the ascending factorial. Because of this, we refer to (1) as the *F_3–beta distribution.* Note that if $u = 0$ and $v = 0$ then (1) reduces to the usual bivariate beta distribution with parameters β, β' and $\gamma-\beta-\beta'$. The four additional parameters u, v, α and α' can be interpreted by examining the behavior of (1) near the boundaries of the simplex $\{(x,y) : 0 < x < 1, 0 < y < 1, 0 < x+y < 1\}$. Letting $x \to 0$, note that

$$f(x,y) \sim C x^{\beta-1} \frac{y^{\beta'-1}(1-y)^{\gamma-\beta-\beta'-1}}{(1-vy)^{\alpha'}}.$$

Thus, the distribution of y along the vertical boundary of the simplex belongs to Libby and Novick [8]'s generalized beta family with the parameters β', $\gamma-\beta-\beta'$, v and α'. The parameters v and α' provide the scale variation from the standard beta. Letting $y \to 0$, note that

$$f(x,y) \sim C y^{\beta'-1} \frac{x^{\beta-1}(1-x)^{\gamma-\beta-\beta'-1}}{(1-ux)^{\alpha}}.$$

Thus, the distribution of x along the horizontal boundary of the simplex belongs to Libby and Novick [8]'s generalized beta family with the parameters β, $\gamma-\beta-\beta'$, u and α. The parameters u and α provide the scale variation from the standard beta.

Let us now briefly discuss the shape of (1). The derivatives of $\log f$ with respect to x and y are

$$\frac{\partial \log f}{\partial x} = \frac{\beta-1}{x} - \frac{\gamma-\beta-\beta'-1}{1-x-y} + \frac{\alpha u}{1-ux} \tag{3}$$

and

$$\frac{\partial \log f}{\partial y} = \frac{\beta'-1}{y} - \frac{\gamma-\beta-\beta'-1}{1-x-y} + \frac{\alpha' v}{1-vy}, \tag{4}$$

respectively. Setting (3) and (4) to zero, one notes that the critical points of (1) are given by the simultaneous solutions of the two quadratic equations

$$\begin{aligned}&\left(\gamma-\beta^{'}-\alpha-2\right)ux^2\\&+\left\{-(\beta-1)u(1-y)+\alpha u(1-y)-\gamma+\beta^{'}+2\right\}x\\&+(\beta-1)(1-y)=0\end{aligned}$$

and

$$\begin{aligned}&\left(\gamma-\beta-\alpha^{'}-2\right)vy^2\\&+\left\{-\left(\beta^{'}-1\right)v(1-x)+\alpha^{'}v(1-x)-\gamma+\beta+2\right\}y\\&+\left(\beta^{'}-1\right)(1-x)=0.\end{aligned}$$

Thus, (1) can exhibit up to two critical points. Figures 1, 2 and 3 below illustrate some possible shapes of (1).

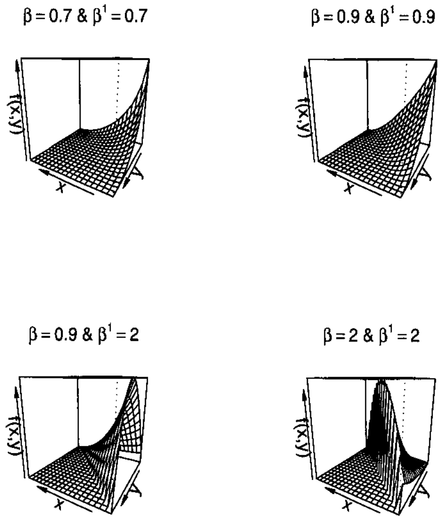

FIGURE 1. Plots of the pdf of (1) for $\gamma = 5$, $\alpha = 2$, $u = 0.5$, $v = 0.5$ and selected values of $(\beta, \beta^{'})$.

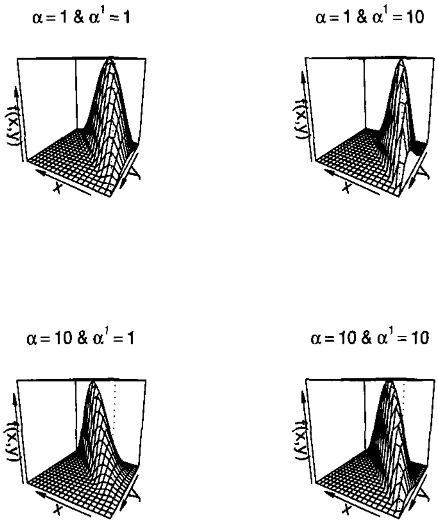

FIGURE 2. Plots of the pdf of (1) for $\beta = 3$, $\beta' = 3$, $\gamma = 9$, $u = 0.5$, $v = 0.5$ and selected values of (α, α').

The rest of this note is organized as follows. In Sections 2 to 4, various representations are derived for the product moments, marginal densities, marginal moments, conditional densities and the conditional moments associated with (1). The associated estimation procedure by the method of maximum likelihood as well as the Fisher information matrix are presented in Section 5.

2. Product moments

Theorems 1 and 2 derive two representations for the product moments of (1). The first is expressed in terms of the Appell function of the third kind while the second representation is an infinite series of Gauss hypergeometric functions.

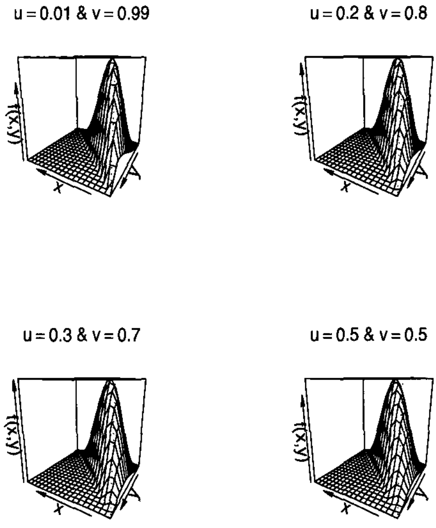

FIGURE 3. Plots of the pdf of (1) for $\beta = 3$, $\beta' = 3$, $\gamma = 9$, $\alpha = 2$, $\alpha' = 2$ and selected values of (u, v).

THEOREM 1. *The product moment of X and Y associated with* (1) *is given by*

$$E\left(X^m Y^n\right) = \frac{C\Gamma(m+\beta)\Gamma(n+\beta')\Gamma(\gamma-\beta-\beta')}{\Gamma(m+n+\gamma)} \times F_3\left(\alpha, \alpha', m+\beta, n+\beta'; m+n+\gamma; u, v\right) \tag{5}$$

for any real $m > 0$ and $n > 0$.

PROOF. One can write

$$E\left(X^m Y^n\right) = C \int_0^1 \int_0^{1-x} \frac{x^{m+\beta-1} y^{n+\beta'-1} (1-x-y)^{\gamma-\beta-\beta'-1}}{(1-ux)^{\alpha}(1-vy)^{\alpha'}} dy dx. \tag{6}$$

The result of the theorem follows by applying equation (3.1.2.7) in Prudnikov *et al.* [10, volume 1] to calculate the integral in (6). □

THEOREM 2. *The product moment of X and Y associated with* (1) *is given by*

$$E\left(X^mY^n\right) = CB\left(n+\beta',\gamma-\beta-\beta'\right) \times \sum_{k=0}^{\infty} \frac{(n+\beta')_k(\alpha')_k v^k}{(n+\gamma-\beta)_k k!} B\left(m+\beta, n+k+\gamma-\beta\right) \times {}_2F_1\left(m+\beta,\alpha;m+n+k+\gamma;u\right) \tag{7}$$

for any real $m > 0$ *and* $n > 0$, *where* ${}_2F_1$ *is the Gauss hypergeometric function defined by*

$${}_2F_1\left(a,b;c;x\right) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!}$$

for $\mid x \mid < 1$.

PROOF. Consider the integral with respect to y in (6). Applying equation (2.2.6.15) in Prudnikov *et al.* [10, volume 1], one can reduce (6) to

$$E\left(X^mY^n\right) = CB\left(n+\beta',\gamma-\beta-\beta'\right) \int_0^1 \frac{x^{m+\beta-1}(1-x)^{n+\gamma-\beta-1}}{(1-ux)^{\alpha}} \times {}_2F_1\left(n+\beta',\alpha';n+\gamma-\beta;(1-x)v\right) dx \tag{8}$$

Using the definition of the Gauss hypergeometric function, (8) can be rewritten as

$$\begin{aligned} E\left(X^mY^n\right) &= CB\left(n+\beta',\gamma-\beta-\beta'\right) \int_0^1 \frac{x^{m+\beta-1}(1-x)^{n+\gamma-\beta-1}}{(1-ux)^{\alpha}} \\ &\quad \times \sum_{k=0}^{\infty} \frac{(n+\beta')_k(\alpha')_k(1-x)^k v^k}{(n+\gamma-\beta)_k k!} dx \\ &= CB\left(n+\beta',\gamma-\beta-\beta'\right) \sum_{k=0}^{\infty} \frac{(n+\beta')_k(\alpha')_k v^k}{(n+\gamma-\beta)_k k!} \\ &\quad \times \int_0^1 \frac{x^{m+\beta-1}(1-x)^{n+k+\gamma-\beta-1}}{(1-ux)^{\alpha}} dx. \end{aligned} \tag{9}$$

The result in (7) follows by another application of equation (2.2.6.15) in Prudnikov *et al.* [10, volume 1] to calculate the integral in (9). □

3. Marginal pdfs and moments

Theorems 3 and 4 derive the marginal pdfs and marginal moments of (1). Expressions for the pdfs involve the Gauss hypergeometric function while the moments are expressed in terms of the Appell function.

THEOREM 3. *If X and Y have the joint pdf (1) then the marginal pdfs are given by*

$$\begin{aligned} f_X(x) &= CB\left(\beta', \gamma-\beta-\beta'\right) x^{\beta-1}(1-x)^{\gamma-\beta-1}(1-ux)^{-\alpha} \\ &\quad \times {}_2F_1\left(\beta', \alpha'; \gamma-\beta; (1-x)v\right) \end{aligned} \tag{10}$$

and

$$\begin{aligned} f_Y(y) &= CB\left(\beta, \gamma-\beta-\beta'\right) y^{\beta'-1}(1-y)^{\gamma-\beta'-1}(1-vy)^{-\alpha'} \\ &\quad \times {}_2F_1\left(\beta, \alpha; \gamma-\beta'; (1-y)u\right) \end{aligned} \tag{11}$$

for $0 < x < 1$ and $0 < y < 1$.

PROOF. The marginal pdf of X can be written as

$$f_X(x) = C\frac{x^{\beta-1}}{(1-ux)^{\alpha}}\int_0^{1-x}\frac{y^{\beta'-1}(1-x-y)^{\gamma-\beta-\beta'-1}}{(1-vy)^{\alpha'}}dy. \tag{12}$$

The result in (10) follows by applying equation (2.2.6.15) in Prudnikov *et al.* [10, volume 1] to calculate the integral in (12). The result in (11) follows similarly. □

THEOREM 4. *The moments of the marginal pdfs in* (10) *and* (11) *are given by*

$$\begin{aligned} E\left(X^m\right) &= \frac{C\Gamma(m+\beta)\Gamma(\beta')\Gamma(\gamma-\beta-\beta')}{\Gamma(m+\gamma)} \\ &\quad \times F_3\left(\alpha, \alpha', m+\beta, \beta'; m+\gamma; u, v\right) \end{aligned}$$

and

$$E\left(X^n\right) = \frac{C\Gamma(\beta)\Gamma(n+\beta')\Gamma(\gamma-\beta-\beta')}{\Gamma(n+\gamma)} F_3\left(\alpha, \alpha', \beta, n+\beta'; n+\gamma; u, v\right)$$

for any real $m > 0$ and $n > 0$.

PROOF. set $m = 0$ $(n = 0)$ into (5) and simplify. □

The pdfs in (10) and (11) belong to a generalized beta family. If $u = v = 0$ then both (10) and (11) reduce to standard beta pdfs. Writing (10) and (11) as

$$f_X(x) = CB\left(\beta', \gamma - \beta - \beta'\right) \times \sum_{k=0}^{\infty} \frac{(\beta')_k (\alpha')_k v^k}{(\gamma - \beta)_k k!} x^{\beta-1} (1-x)^{k+\gamma-\beta-1} (1-ux)^{-\alpha}$$

and

$$f_Y(y) = CB\left(\beta, \gamma - \beta - \beta'\right) \times \sum_{k=0}^{\infty} \frac{(\beta)_k (\alpha)_k u^k}{(\gamma - \beta')_k k!} y^{\beta'-1} (1-y)^{k+\gamma-\beta'-1} (1-vy)^{-\alpha'},$$

respectively, one notes that the marginal pdfs are infinite mixtures of pdfs which belong to Libby and Novick [8]'s generalized beta family. Also, if $u = 0$ (respectively, $v = 0$) then (11) (respectively, (10)) reduces to Libby and Novick [8]'s generalized beta family.

4. Conditional pdfs and moments

Theorems 5 and 6 derive the conditional pdfs and conditional moments of (1). Expressions for both the pdfs and the moments involve the Gauss hypergeometric function.

THEOREM 5. *If X and Y have the joint pdf (1) then the conditional pdf of X given $Y = y$ is given by*

$$f_{X|Y}(x \mid y) = \frac{x^{\beta-1}(1-x-y)^{\gamma-\beta-\beta'-1}(1-ux)^{-\alpha}}{B\left(\beta, \gamma-\beta-\beta'\right)(1-y)^{\gamma-\beta'-1}\, {}_2F_1\left(\beta, \alpha; \gamma-\beta'; (1-y)u\right)} \tag{13}$$

for $0 < x < 1$. The conditional pdf of Y given $X = x$ is given by

$$f_{Y|X}(y \mid x) = \frac{y^{\beta'-1}(1-x-y)^{\gamma-\beta-\beta'-1}(1-vy)^{-\alpha'}}{B\left(\beta', \gamma-\beta-\beta'\right)(1-x)^{\gamma-\beta-1}\, {}_2F_1\left(\beta', \alpha'; \gamma-\beta; (1-x)v\right)} \tag{14}$$

for $0 < y < 1$.

PROOF. follows immediately from (1) and Theorem 3. □

THEOREM 6. *The moments of the conditional pdfs in* (13) *and* (14) *are given by*

$$E\left(X^m \mid y\right) = \frac{B\left(m+\beta, \gamma-\beta-\beta^{'}\right) {}_2F_1\left(m+\beta, \alpha; m+\gamma-\beta^{'}; (1-y)u\right)}{B\left(\beta, \gamma-\beta-\beta^{'}\right) {}_2F_1\left(\beta, \alpha; \gamma-\beta^{'}; (1-y)u\right)} \tag{15}$$

and

$$E\left(Y^n \mid x\right) = \frac{B\left(n+\beta^{'}, \gamma-\beta-\beta^{'}\right) {}_2F_1\left(n+\beta^{'}, \alpha^{'}; n+\gamma-\beta; (1-x)v\right)}{B\left(\beta^{'}, \gamma-\beta-\beta^{'}\right) {}_2F_1\left(\beta^{'}, \alpha^{'}; \gamma-\beta; (1-x)v\right)} \tag{16}$$

for any real $m > 0$ *and* $n > 0$.

PROOF. Using (13), one can write

$$\begin{aligned} & E\left(X^m \mid y\right) \\ &= \frac{1}{B\left(\beta, \gamma-\beta-\beta^{'}\right)(1-y)^{\gamma-\beta^{'}-1} {}_2F_1\left(\beta, \alpha; \gamma-\beta^{'}; (1-y)u\right)} \\ &\quad \times \int_0^{1-y} x^{m+\beta-1}(1-x-y)^{\gamma-\beta-\beta^{'}-1}(1-ux)^{-\alpha}\, dx. \end{aligned}$$

The result in (15) follows by applying equation (2.2.6.15) in Prudnikov *et al.* [10, volume 1] to calculate the integral in (17). The result in (16) follows similarly. □

5. Estimation

The basic model of the paper is (1) parameterized by $(\alpha, \alpha^{'}, \beta, \beta^{'}, \gamma, u, v)$. Here, we consider estimation of the seven parameters by the method of maximum likelihood. We also compute the associated Fisher information matrix.

Suppose $(x_1, y_1), \ldots, (x_n, y_n)$ is a random sample from (1). The log-likelihood can be expressed as:

$$\begin{aligned}
&\log L\left(\alpha, \alpha^{'}, \beta, \beta^{'}, \gamma, u, v\right) \\
&= n \log C + (\beta - 1) \sum_{j=1}^{n} \log x_j + \left(\beta^{'} - 1\right) \sum_{j=1}^{n} \log y_j \\
&+ \left(\gamma - \beta - \beta^{'} - 1\right) \sum_{j=1}^{n} \log\left(1 - x_j - y_j\right) \\
&- \alpha \sum_{j=1}^{n} \log\left(1 - u x_j\right) - \alpha^{'} \sum_{j=1}^{n} \log\left(1 - v y_j\right).
\end{aligned}$$

The first-order derivatives of this with respect to the seven parameters are:

$$\frac{\partial \log L}{\partial \alpha} = \frac{n}{C} \frac{\partial C}{\partial \alpha} - \sum_{j=1}^{n} \log\left(1 - u x_j\right),$$

$$\frac{\partial \log L}{\partial \alpha^{'}} = \frac{n}{C} \frac{\partial C}{\partial \alpha^{'}} - \sum_{j=1}^{n} \log\left(1 - v y_j\right),$$

$$\frac{\partial \log L}{\partial \beta} = \frac{n}{C} \frac{\partial C}{\partial \beta} + \sum_{j=1}^{n} \log x_j - \sum_{j=1}^{n} \log\left(1 - x_j - y_j\right),$$

$$\frac{\partial \log L}{\partial \beta^{'}} = \frac{n}{C} \frac{\partial C}{\partial \beta^{'}} + \sum_{j=1}^{n} \log y_j - \sum_{j=1}^{n} \log\left(1 - x_j - y_j\right),$$

$$\frac{\partial \log L}{\partial \gamma} = \frac{n}{C} \frac{\partial C}{\partial \gamma} + \sum_{j=1}^{n} \log\left(1 - x_j - y_j\right),$$

$$\frac{\partial \log L}{\partial u} = \frac{n}{C} \frac{\partial C}{\partial u} + \alpha \sum_{j=1}^{n} \frac{x_j}{1 - u x_j}$$

and

$$\frac{\partial \log L}{\partial v} = \frac{n}{C} \frac{\partial C}{\partial v} + \alpha^{'} \sum_{j=1}^{n} \frac{y_j}{1 - v y_j}.$$

The maximum likelihood estimators of $(\alpha, \alpha^{'}, \beta, \beta^{'}, \gamma, u, v)$ are the simultaneous solutions of the equations $\partial \log L / \partial \alpha = 0$, $\partial \log L / \partial \alpha^{'} = 0$, $\partial \log L / \partial \beta = 0$, $\partial \log L / \partial \beta^{'} = 0$, $\partial \log L / \partial \gamma = 0$, $\partial \log L / \partial u = 0$ and

$\partial \log L/\partial v = 0$. The associated Fisher information matrix requires the second-order derivatives of $\log L$ which can be calculated as:

$$\frac{\partial^2 \log L}{\partial \alpha \partial u} = \frac{n}{C}\frac{\partial^2 C}{\partial \alpha \partial u} - \frac{n}{C^2}\frac{\partial C}{\partial \alpha}\frac{\partial C}{\partial u} + \sum_{j=1}^{n} \frac{x_j}{1 - u x_j},$$

$$\frac{\partial^2 \log L}{\partial \alpha' \partial v} = \frac{n}{C}\frac{\partial^2 C}{\partial \alpha' \partial v} - \frac{n}{C^2}\frac{\partial C}{\partial \alpha'}\frac{\partial C}{\partial v} + \sum_{j=1}^{n} \frac{y_j}{1 - v y_j},$$

$$\frac{\partial^2 \log L}{\partial u^2} = \frac{n}{C}\frac{\partial^2 C}{\partial u^2} - \frac{n}{C^2}\left(\frac{\partial C}{\partial u}\right)^2 + \alpha \sum_{j=1}^{n} \frac{x_j^2}{(1 - u x_j)^2}$$

and

$$\frac{\partial^2 \log L}{\partial v^2} = \frac{n}{C}\frac{\partial^2 C}{\partial v^2} - \frac{n}{C^2}\left(\frac{\partial C}{\partial v}\right)^2 + \alpha' \sum_{j=1}^{n} \frac{y_j^2}{(1 - v y_j)^2}.$$

All of the remaining second-order derivatives of $\log L$ can be expressed as:

$$\frac{\partial^2 \log L}{\partial \theta_i \partial \theta_j} = \frac{n}{C}\frac{\partial^2 C}{\partial \theta_i \partial \theta_j} - \frac{n}{C^2}\frac{\partial C}{\partial \theta_i}\frac{\partial C}{\partial \theta_j}.$$

Thus, the elements of the Fisher information matrix follow by noting that

$$E\left(\frac{X}{1 - uX}\right) = \frac{C\left(\alpha + 1, \alpha', \beta + 1, \beta', \gamma, u, v\right)}{C\left(\alpha, \alpha', \beta, \beta', \gamma, u, v\right)},$$

$$E\left(\frac{Y}{1 - vY}\right) = \frac{C\left(\alpha, \alpha' + 1, \beta, \beta' + 1, \gamma, u, v\right)}{C\left(\alpha, \alpha', \beta, \beta', \gamma, u, v\right)},$$

$$E\left(\frac{X^2}{(1 - uX)^2}\right) = \frac{C\left(\alpha + 2, \alpha', \beta + 2, \beta', \gamma, u, v\right)}{C\left(\alpha, \alpha', \beta, \beta', \gamma, u, v\right)}$$

and

$$E\left(\frac{Y^2}{(1 - vY)^2}\right) = \frac{C\left(\alpha, \alpha' + 2, \beta, \beta' + 2, \gamma, u, v\right)}{C\left(\alpha, \alpha', \beta, \beta', \gamma, u, v\right)}.$$

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