Commun. Korean Math. Soc. **21** (2006), No. 2, pp. 337–345

THE ACTION OF IMAGE OF BRAIDING UNDER THE HARER MAP

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ABSTRACT. John Harer conjectured that the canonical map from braid group to mapping class group induces zero homology homomorphism. To prove the conjecture it suffices to show that this map preserves the first Araki-Kudo-Dyer-Lashof operation. To get information on this homology operation we need to investigate the image of braiding under the Harer map. The main result of this paper is to give both algebraic and geometric interpretations of the image of braiding under the Harer map. For this we need to calculate long chains of consecutive actions of Dehn twists on the fundamental group of surface.

1. Introduction

Let B_n be the Artin's braid group of n strings. Let $\Gamma_{g,1}$ be the mapping class group of self-homeomorphisms of the surface $S_{g,1}$ which is a compact orientable surface of genus g with one boundary component. It has been well-known that $\Gamma_{g,1}$ is generated by $2g+1$ Dehn twists $a_1, a_2, b_1, b_2, \ldots, b_g, w_1, \ldots, w_{g-1}$([5], [9]).

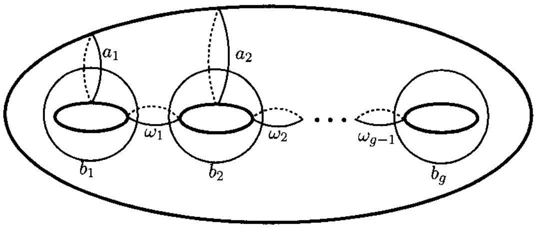


Figure 1.

Received February 28, 2006.
2000 Mathematics Subject Classification: 55R35, 20F36, 14H10, 55S12.
Key words and phrases: braid group, mapping class group, Harer conjecture, (r, s)-braiding, double loop space, actions of Dehn twists.
This work was supported by Inha University research grant 22791-01.

Let $\beta_1, \ldots, \beta_{n-1}$ be the canonical generators of B_n. Then there is an obvious map

$$\phi : B_{2g} \longrightarrow \Gamma_{g,1}$$

which maps $\beta_1 \mapsto b_1$, $\beta_1 \mapsto w_1$, $\beta_3 \mapsto b_2$, $\beta_4 \mapsto w_2$, ..., that is,

$$\phi(\beta_i) = \begin{cases} b_{\frac{i+1}{2}} & \text{if } i \text{ is odd} \\ w_{\frac{i}{2}} & \text{if } i \text{ is even.} \end{cases}$$

This map, called Harer map, is a well-defined group homomorphism since two Dehn twists α and β which transversally intersect at one point satisfy the braid relation $\alpha\beta\alpha = \beta\alpha\beta$.

It was conjectured in the middle of 1980's by Harer that the homology homomorphism $\phi_* : H_*(B_\infty; \mathbb{Z}/2) \longrightarrow H_*(\Gamma_\infty; \mathbb{Z}/2)$ is zero, where $B_\infty = \varinjlim B_n$, $\Gamma_\infty = \varinjlim \Gamma_{n,1}$. This problem, called Harer conjecture, was first raised by Ed Miller and Fred Cohen. John Harer made that conjecture from his experience and observations through many hand calculations of homology groups of mapping class groups. This problem is still open though several great mathematicians have tried to solve it.

The homology of the infinite braid group is well-known ([2]):

$$H_*(B_\infty; \mathbb{Z}/2) \cong \mathbb{Z}/2[a_1, a_2, \ldots], \quad \text{where} \quad \deg\ a_i = 2^i - 1.$$

Moreover, we have that $Q_1(a_i) = a_{i+1}$, where Q_1 is the first Araki-Kudo-Dyer-Lashof operation ([3]). Since $H_1(\Gamma_\infty; \mathbb{Z}/2) = 0$, in order to prove the Harer conjecture, it suffices to show that the homology homomorphism induced by the Harer map preserves the first Araki-Kudo-Dyer-Lashof operation Q_1 which arises from the double loop space structure of BB_∞^+ and also of $B\Gamma_\infty^+$.

A monoidal category $\mathfrak{C}$ is called a braided monoidal category if for each pair of objects A, B of $\mathfrak{C}$, there is a natural commutativity isomorphism $\beta_{A,B} : A \otimes B \longrightarrow B \otimes A$ satisfying :

(a) $\beta_{A,E} = \beta_{E,A} = 1_A$ for each object A and the identity object E

(b) $\beta_{A\otimes B,C} = (\beta_{A,C} \otimes 1_B) \circ (1_B \otimes \beta_{B,C})$
$\beta_{A,B\otimes C} = (1_B \otimes \beta_{A,C}) \circ (\beta_{A,B} \otimes 1_C)$.

The isomorphism $\beta_{A,B}$ is called a braiding. The naturality of braiding means that for any morphisms $f : A \longrightarrow A'$, $g : B \longrightarrow B'$ we have

$$(g \otimes f) \circ \beta_{A,B} = \beta_{A',B'} \circ (f \otimes g).$$

It was shown by Z. Fiedorowicz that the group completion of the classifying space of a braided monoidal category is homotopy equivalent to a double loop space, and its converse also holds. The first Araki-Kudo-Dyer-Lashof operation acting on the homology of a double loop space rises from the braid structure. Hence in order to understand this homology operation we need to research into the braiding.

Let $\mathfrak{B}$ be the collection (or disjoint union) of braid groups, then it may be regarded as a braided monoidal category whose objects are nonnegative integers. It is equipped with the obvious braiding $\beta_{r,s} \in B_{r+s}$ which is regarded as an isomorphism from $r+s$ to $s+r$. For better understanding of the image of the homology homomorphism induced by the map ϕ, we need to investigate the image of the braiding under ϕ.

Let $\beta_{2g,2g} \in B_{4g}$ be the canonical $(2g, 2g)$-braiding. Let $\phi(\beta_{2g,2g}) = \bar{\beta}_{g,g} \in \Gamma_{2g,1}$. Let $\pi_1 S_{g,1} = F_{\{x_1,y_1,\ldots,x_g,y_g\}} = F_{2g}$. Then $\Gamma_{g,1}$ may be regarded as a subgroup of the automorphism group of $\pi_1 S_{g,1}$ which consists of the automorphisms fixing the fundamental relator $R = [y_1, x_1][y_2, x_2] \cdots [y_g, x_g]$ representing the closed curve along the boundary of $S_{g,1}$.

The main result of this paper is to show what $\bar{\beta}_{g,g}$ really is in Aut F_{2g}, that is, to show how $\bar{\beta}_{g,g}$ acts on $F_{\{x_1,y_1,\ldots,x_{2g},y_{2g}\}}$. We also give geometric descriptions of the resulting loops of the action. These are important informations for understanding the nature of the Harer homology homomorphism. The proof is obtained by tracking the transformations of various loops and long hand calculations. We may also perform the algebraic calculations by using a computer program. In the calculations it is very important to set up suitable orientations of generating loops of $\pi_1 S_{g,1}$ which are compatible, in our calculations, with the actions of Dehn twists.

2. Braid groups and mapping class groups

Let B_n be the braid group of n strings generated by $\beta_1, \ldots, \beta_{n-1}$ which satisfy the braid relation

$$\beta_i\beta_{i+1}\beta_i = \beta_{i+1}\beta_i\beta_{i+1} \quad (1 \leq i \leq n-2).$$

Let $\mathfrak{B} = \coprod_{n\geq 0} B_n$ be the disjoint union of braid groups which may be regarded as the category whose objects are nonnegative integers. $\mathfrak{B}$ is a monoidal braided category whose monoidal structure is the juxtaposition of two braids. The (r, s)-braiding $\beta_{r,s} \in B_{r+s}$ which is regarded as a morphism $\beta_{r,s} : r + s \longrightarrow s + r$ is the braid joining the first r points on the top and the last r points on the bottom, and also the last s points on the top and the first s points on the bottom.

The braiding $\beta_{r,s} \in B_{r+s}$ is expressed in terms of the generators as

$$\beta_{r,s} = (\beta_s\beta_{s-1} \cdots \beta_1)(\beta_{s+1}\beta_s \cdots \beta_2) \cdots (\beta_{r+s-1} \cdots \beta_r).$$

Let $S_{g,1}$ be the surface obtained from the compact orientable surface S_g of genus g by removing an open disk from S_g. Let $\Gamma_{g,1}$ be the mapping class group of orientation preserving self-homeomorphisms of $S_{g,1}$ which fix the boundary pointwise.

$\Gamma_{g,1}$ may be regarded as a subgroup of $\mathrm{Aut}\pi_1 S_{g,1}$ which consists of automorphisms fixing the fundamental relator $R = [y_1, x_1] \cdots [y_g, x_g]$, where $x_1, y_1, \ldots, x_g, y_g$ are generators of the free group $\pi_1 S_{g,1} = F_{\{x_1,y_1,\ldots,x_g,y_g\}} = F_{2g}$.

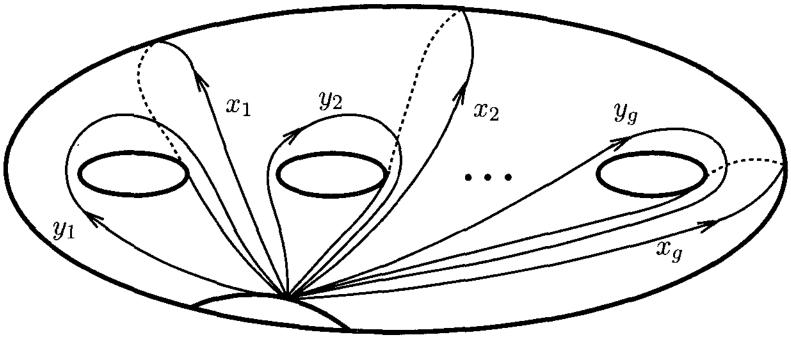


Figure 2. The generators $x_1, y_1, \ldots, x_g, y_g$ of $\pi_1 S_{g,1}$

The fundamental relator $R = [y_1, x_1] \cdots [y_g, x_g]$ represents the loop along the boundary. It is here important to set up correct orientations of generating loops and other loops in $\pi_1 S_{g,1}$ which are compatible with the action of $2g+1$ Dehn twists $a_1, a_2, b_1, b_2, \ldots, b_g, w_1, w_2, \ldots, w_{g-1}$ in $\Gamma_{g,1}$. (See Figure 1.)

Let z_i be the loop in $\pi_1 S_{g,1}$ parallel to the Dehn twist w_i, then we have

$$z_i = x_i^{-1} y_{i+1} x_{i+1} y_{i+1}^{-1}.$$

The following figure shows this.

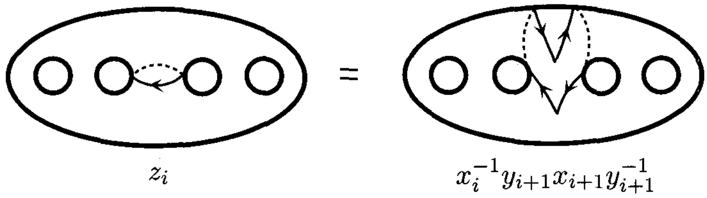


Figure 3.

3. The action of the image of braiding under the Harer map

Let $a_1, a_2, b_1, b_2, \ldots, b_g, w_1, w_2, \ldots, w_{g-1}$ be the standard Dehn twists as in Figure 1. The Harer map $\phi : B_{2g} \to \Gamma_{g,1}$ is defined by

$$\phi(\beta_i) = \begin{cases} b_{\frac{i+1}{2}} & \text{if } i \text{ is odd} \\ w_{\frac{i}{2}} & \text{if } i \text{ is even.} \end{cases}$$

The Harer conjecture is that ϕ induces a trivial homology homomorphism for $\mathbb{Z}/2$ - coefficient. In order to prove the conjecture it suffices to show that the homomorphism ϕ_* preserves the first Araki-Kudo-Dyer-Lashof operation Q_1 which arises from the braid structure of the braided monoidal category $\mathfrak{B}$ of braid groups. In order to understand the image of Harer homology homomorphism we need to investigate the image of braidings, that is, for braiding $\beta_{2g,2g} \in B_{4g}$ we are going to figure out what $\phi(\beta_{2g,2g}) = \bar{\beta}_{g,g} \in \Gamma_{2g,1}$ is as an automorphism of $\pi_1 S_{g,1}$. We first need to understand the actions of the standard Dehn twists.

LEMMA 3.1. *Let $a_1, a_2, b_1, b_2, \ldots, b_{2g}, w_1, w_2, \ldots, w_{2g-1}$ be the Dehn twists as in Figure 1 which generate $\Gamma_{2g,1}$. Then they act on $\pi_1 S_{2g,1}$ as follows :*

$$\begin{aligned} a_i &: y_i \longmapsto y_i x_i^{-1} \\ b_i &: x_i \longmapsto x_i y_i \\ w_i &: \begin{cases} x_i & \longmapsto z_i^{-1} y_{i+1} x_{i+1} y_{i+1}^{-1} \\ y_i & \longmapsto y_i z_i \\ y_{i+1} & \longmapsto z_i^{-1} y_{i+1}, \end{cases} \end{aligned}$$

where $z_i = x_i^{-1} y_{i+1} x_{i+1} y_{i+1}^{-1}$.

The automorphisms in this lemma fix the generators that do not appear in the list. The proof of this lemma is obtained by drawing the loops transformed by the corresponding Dehn twists and express the resulting loops in terms of the generators of $\pi_1 S_{g,1}$.

The following lemma is useful in making the geometric interpretation of loops which will later arise as results of action of Dehn twists.

LEMMA 3.2. *Let $R_1 = [y_1, x_1] \cdots [y_g, x_g]$ and $R_2 = [y_{g+1}, x_{g+1}] \cdots [y_{2g}, x_{2g}]$ be elements of $\pi_1 S_{2g,1}$, then the following loops represent them :*

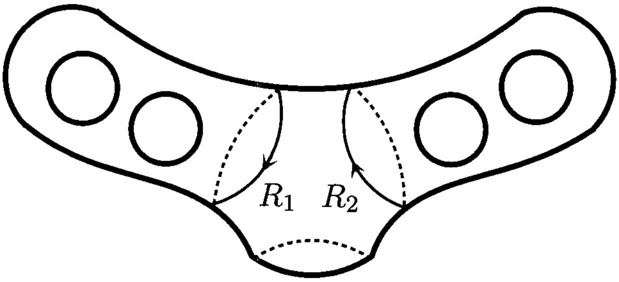


Figure 4.

Let $\beta_{2g,2g} \in F_{4g}$ be the usual $(2g,2g)$-braiding. For the map $\phi : B_{4g} \rightarrow \Gamma_{2g,1}$, let $\bar{\beta}_{g,g} = \phi(\beta_{2g,2g}) \in \Gamma_{2g,1}$, then it is regarded as an element in $\mathrm{Aut}\pi_1 S_{2g,1}$. We like to figure out what $\bar{\beta}_{g,g}$ is. Here is our main result.

THEOREM 3.3. *$\bar{\beta}_{g,g} = (w_g b_g \cdots w_1)(b_{g+1} w_g \cdots w_2) \cdots (b_{2g} w_{2g-1} \cdots w_g)$ acts on $\pi_1 S_{2g,1} = F_{\{x_1,y_1,\ldots,x_g,y_g\}}$ as follows :*

$$
\begin{aligned}
y_i &\longmapsto y_{g+i} \\
y_{g+i} &\longmapsto y_i^{\bar{Y}_g} \\
x_i &\longmapsto Y_1^{\bar{Y}_i} x_{g+i} \\
x_{g+i} &\longmapsto Y_{i+1}^{\bar{Y}_g} R_2^{-1} Y_{i+1}^{-1} x_i \bar{Y}_g,
\end{aligned}
$$

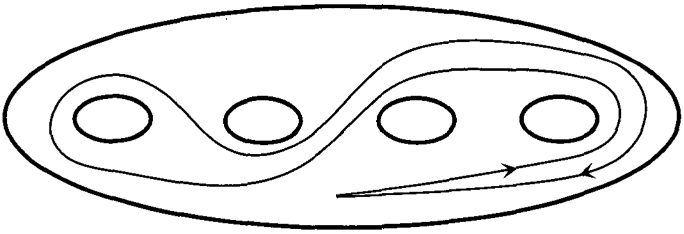
where $R_2 = [y_{g+1}, x_{g+1}] \cdots [y_{2g}, x_{2g}]$, $Y_i = y_i \cdots y_g$ and $\bar{Y}_i = y_{g+1} \cdots y_{g+i}$. Here i varies from 1 to g. Y_{i+1} is regarded as 1 if $i = g$. The notation α^β means $\beta^{-1}\alpha\beta$.

The proof of this theorem is obtained by long hand calculations tracking many consecutive actions of Dehn twists according to the actions described in Lemma 3.1. Note that $\bar{\beta}_{g,g}$ is actually a kind of $(2g,2g)$-braiding in the category which is a disjoint union of mapping class groups $\Gamma_{2g,1}$.

The geometric description of the action of $\bar{\beta}_{g,g}$ is as important as the algebraic description. Let us consider the case where $g = 2$, that is, consider the geometric expression of the action of $\bar{\beta}_{2,2}$ on the generators of $\pi_1 S_{4,1}$.

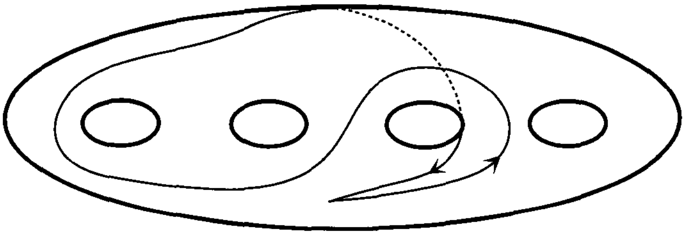
LEMMA 3.4. *The action of $\bar{\beta}_{2,2} = w_2 b_2 w_1 b_1 b_3 w_2 b_2 w_1 w_3 b_3 w_2 b_2 b_4 w_3 b_3 w_2$ is geometrically described in the following figure.*

$y_3 \longrightarrow$



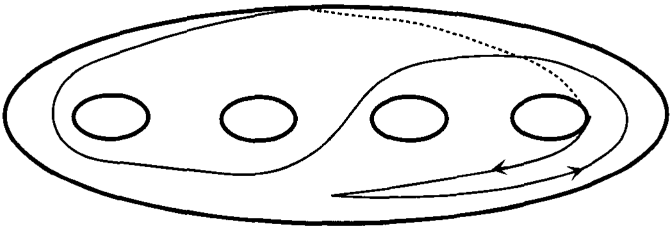
$y_4^{-1}y_3^{-1}y_1y_3y_4$

$x_1 \longrightarrow$



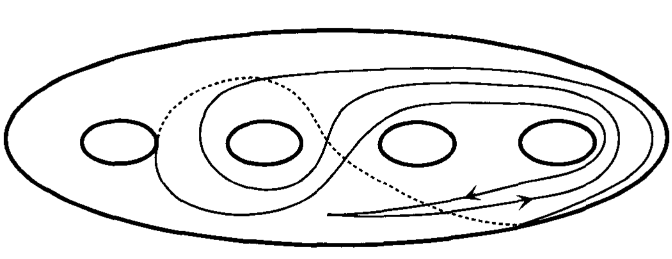
$y_3^{-1}y_1y_2y_3x_3$

$x_2 \longrightarrow$

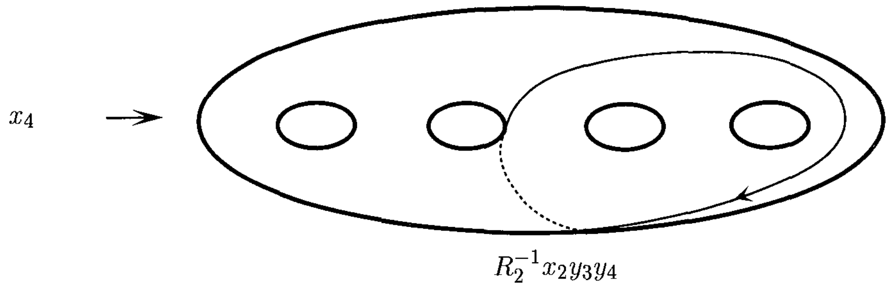


$y_4^{-1}y_3^{-1}y_1y_2y_3y_4x_4$

$x_3 \longrightarrow$



$y_4^{-1}y_3^{-1}y_2y_3y_4R_2^{-1}y_2^{-1}x_1y_3y_4$



where $R_2 = [y_{g+1}, x_{g+1}] \cdots [y_{2g}, x_{2g}]$.

It is easy to see these actions for more general case extending the above action to general genus g, except the action on x_{g+i} (for $1 \le i \le g-1$). Note that the action on x_{2g} is a little different from that on x_{g+i} for $1 \le i \le g-1$. Here is the geometric description of the action of $\beta_{g,g}$ on x_{g+i}.

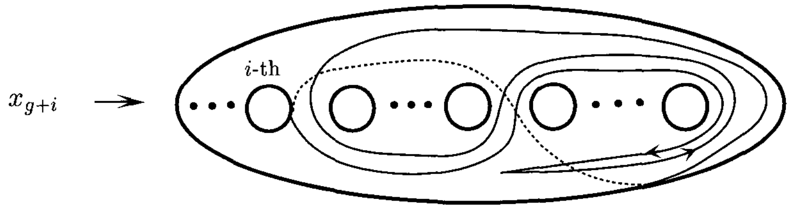


Figure 5.

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