

LONG-TIME PROPERTIES OF PREY-PREDATOR SYSTEM WITH CROSS-DIFFUSION

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ABSTRACT. Using calculus inequalities and embedding theorems in R^1 , we establish W_2^1 -estimates for the solutions of prey-predator population model with cross-diffusion and self-diffusion terms. Two cases are considered ; (i) $d_1 = d_2$, $\alpha_{12} = \alpha_{21} = 0$, and (ii) $0 < \alpha_{21} < 8\alpha_{11}$, $0 < \alpha_{12} < 8\alpha_{22}$. It is proved that solutions are bounded uniformly pointwise, and that the uniform bounds remain independent of the growth of the diffusion coefficient in the system. Also, convergence results are obtained when $t \rightarrow \infty$ via suitable Liapunov functionals.

1. Introduction

In recent years Cross-Diffusion systems have been drawing great deal of attention in the field of strongly coupled parabolic and elliptic equations. There are many established results on the Lotka-Volterra competition model with cross-diffusion in the literatures as [5], [6], [8], [11]–[13], [17]–[19], [21]–[25]. For the cross-diffusion systems with prey-predator type reaction functions, there are a few results mainly on the steady-state problems with the elliptic systems, see [1], [9], [10], [15], [20].

In this paper we are interested in the time-dependent properties of the following Cross-Diffusion system with prey-predator type of reactions :

$$(1.1) \quad \begin{cases} u_t = \Delta[(d_1 + \alpha_{11}u + \alpha_{12}v)u] + u(a_1 - b_1u - c_1v) & \text{in } \Omega \times (0, \infty), \\ v_t = \Delta[(d_2 + \alpha_{21}u + \alpha_{22}v)v] + v(a_2 + b_2u - c_2v) & \text{in } \Omega \times (0, \infty), \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(x, 0) = u_0(x) \geq 0, \quad v(x, 0) = v_0(x) \geq 0 & \text{in } \bar{\Omega}, \end{cases}$$

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where $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain. The coefficients α_{ij} 's are nonnegative constants for $i, j = 1, 2$. And d_i, b_i, c_i ($i = 1, 2$), and a_1 are positive constants. Only a_2 may be nonpositive. Throughout this paper we assume that the initial functions $u_0(x), v_0(x)$ are not identically zero.

In system (1.1) u and v are nonnegative functions which represent the population densities of the prey and predator species, respectively, which are interacting and migrating in the same habitat Ω . By using the strong maximum principle and the Hopf boundary lemma for parabolic equations, it is shown that $u(x, t) > 0$ and $v(t, x) > 0$ in $[0, 1] \times (0, \infty)$. The coefficients d_1 and d_2 are the diffusion rates of the two species, respectively. The positive constant a_1 means that the prey is assumed to be sharing limited resource so that its population can increase a bit in the absence of predator. If $a_2 > 0$ the predator is assumed to have another source of food supply than the prey, sufficient to increase the predator population somewhat in the absence of prey. If $a_2 \leq 0$ the predator population will be decreasing in the absence of prey. The coefficients b_1 and c_2 account for the competitions within the prey species and predator species, respectively. c_1 represents the death rate of the prey due to the encounter with predator. And, b_2 is the growth rate of the predator due to their prey consumption. The positive cross-diffusion rates α_{12} and α_{21} mean that the prey tends to avoid higher density of the predator species and vice versa by diffusing away. The tendency to move in the direction of lower density of own species is represented by the self-diffusion rates α_{11} and α_{22} for the prey and predator, respectively. For details in the biological background, we refer the reader to the monograph of Okubo and Levin [16].

The local existence of solutions to (1.1) was established by Amann [2], [3], [4] which deal with more general form of equations :

$$(1.2) \quad \begin{cases} u_t = \Delta[(d_1 + \alpha_{11}u + \alpha_{12}v)u] + u f(x, u, v) & \text{in } \Omega \times (0, \infty), \\ v_t = \Delta[(d_2 + \alpha_{21}u + \alpha_{22}v)v] + v g(x, u, v) & \text{in } \Omega \times (0, \infty), \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(x, 0) = u_0(x) \geq 0, \quad v(x, 0) = v_0(x) \geq 0 & \text{in } \bar{\Omega}, \end{cases}$$

where f and g are functions in $C^\infty(\bar{\Omega} \times \mathbb{R}^2, \mathbb{R})$. According to his results the system (1.2) has a unique nonnegative solution $u(\cdot, t), v(\cdot, t)$ in $C([0, T], W_p^1(\Omega)) \cap C^\infty((0, T), C^\infty(\Omega))$, where $T \in (0, \infty]$ is the maximal existence time for the solution u, v . The following result is also due to Amann [3].

THEOREM 1. *Let u_0 and v_0 be in $W_p^1(\Omega)$. The system (1.2) possesses a unique nonnegative maximal smooth solution $u(x, t), v(x, t) \in C([0, T], W_p^1(\Omega)) \cap C^\infty(\bar{\Omega} \times (0, T))$ for $0 \leq t < T$, where $p > n$ and $0 < T \leq \infty$. If the solution satisfies the estimates $\sup_{0 < t < T} \|u(\cdot, t)\|_{W_p^1(\Omega)} < \infty$, $\sup_{0 < t < T} \|v(\cdot, t)\|_{W_p^1(\Omega)} < \infty$, then $T = +\infty$. If, in addition, u_0 and v_0 are in $W_p^2(\Omega)$ then $u(x, t), v(x, t) \in C([0, \infty), W_p^2(\Omega))$, and $\sup_{0 \leq t < \infty} \|u(\cdot, t)\|_{W_p^2(\Omega)} < \infty$, $\sup_{0 \leq t < \infty} \|v(\cdot, t)\|_{W_p^2(\Omega)} < \infty$.*

The system (1.2) is a special case of the concrete example (7), (8) in Introduction of [3], and the results stated in Theorem 1 is from the Theorem in Introduction of [3]. The results in Theorem 1 mean that once we establish the uniform W_p^1 -bound, (with $p > n$), independent of the maximal existence time T for the solutions, the global existence of the solutions will follow. And also the uniform L_∞ -bound of the solutions will be obtained from the Sobolev embedding theorems.

In this paper we consider the following two cases for the system (1.1) in the spatial domain $\Omega = [0, 1] \subset \mathbb{R}^1$:

- Case(A) $d_1 = d_2$ and $\alpha_{11} = \alpha_{22} = 0$,
- Case(B) $0 < \alpha_{21} < 8\alpha_{11}$ and $0 < \alpha_{12} < 8\alpha_{22}$.

The system (1.1) is rewritten in each case as follows :

$$\begin{aligned}
 \text{(A)} \quad & \begin{cases} u_t = (du + \alpha_{12}uv)_{xx} + u(a_1 - b_1u - c_1v) & \text{in } [0, 1] \times (0, \infty), \\ v_t = (dv + \alpha_{21}uv)_{xx} + v(a_2 + b_2u - c_2v) & \text{in } [0, 1] \times (0, \infty), \\ u_x(x, t) = v_x(x, t) = 0 & \text{at } x = 0, 1, \\ u(x, 0) = u_0(x) \geq 0, \quad v(x, 0) = v_0(x) \geq 0 & \text{in } [0, 1], \end{cases} \\
 \text{(B)} \quad & \begin{cases} u_t = (d_1u + \alpha_{11}u^2 + \alpha_{12}uv)_{xx} + u(a_1 - b_1u - c_1v) & \text{in } [0, 1] \times (0, \infty), \\ v_t = (d_2v + \alpha_{21}uv + \alpha_{22}v^2)_{xx} + v(a_2 + b_2u - c_2v) & \text{in } [0, 1] \times (0, \infty), \\ u_x(x, t) = v_x(x, t) = 0 & \text{at } x = 0, 1, \\ u(x, 0) = u_0(x) \geq 0, \quad v(x, 0) = v_0(x) \geq 0 & \text{in } [0, 1], \end{cases}
 \end{aligned}$$

where α_{ij} , d , d_i , a_1 , b_i , c_i are all positive constants for $i, j = 1, 2$, and a_2 is a real constant. Throughout this paper we assume that the initial functions $u_0(x)$, $v_0(x)$ are not identically zero and contained in the function space $W_2^1([0, 1])$.

We first prove the uniform boundedness of the global solutions of the systems (A) and (B) in Case(A) and Case(B), respectively, by applying the calculus inequalities of Gagliardo-Nirenberg type. The main frame of derivation of estimates will follow the papers [22] and [23] of the author

which deal with cross-diffusion systems with competition type reactions ($f = a_1 - b_1u - c_1v$, $g = a_2 - b_2u - c_2v$) in Case(A) and Case(B), respectively. Due to the difference in the reaction functions we have to take some condition on the coefficient b_2 , the growth rate of the predator due to their prey consumption. In order to obtain L_1 -estimates for the solutions we need to assume that b_2 is not too large compared to b_1 , c_1 , c_2 so that

$$0 < b_2 < c_1 + 2 \min\{b_1, c_2\}.$$

In each step of estimates of the solution we look for the contribution of the diffusion coefficients d , d_1 , d_2 and conclude that the uniform bound of the solution is independent of d , d_1 , d_2 if $d, d_1, d_2 \geq 1$. Using this result we obtain convergence results on the solution for large d , d_1 , d_2 . There we also have to adopt different forms of Liapunov functionals from the ones used in [22] and [23], because of the differences in the asymptotic behaviors of the solutions.

Here we state the main theorems of this paper.

THEOREM 2. *Assume that $0 < b_2 < c_1 + 2 \min\{b_1, c_2\}$ for the system (A). Suppose that the initial functions u_0, v_0 are in $W_2^2([0, 1])$, and let $(u(x, t), v(x, t))$ be the maximal solution obtained as in Theorem 1. Then there exist positive constants t_0 , $M' = M'(d, \alpha_{12}, \alpha_{21}, a_i, b_i, c_i, i = 1, 2)$, and $M = M(d, \alpha_{12}, \alpha_{21}, a_i, b_i, c_i, i = 1, 2)$ such that*

$$\begin{aligned} \max\{\|u(\cdot, t)\|_{1,2}, \|v(\cdot, t)\|_{1,2} : t \in (t_0, T)\} &\leq M', \\ \max\{u(x, t), v(x, t) : (x, t) \in [0, 1] \times (t_0, T)\} &\leq M, \end{aligned}$$

and $T = +\infty$. In the case $d \geq 1$, the constant M is independent of $d \geq 1$, that is, $M = M(\alpha_{12}, \alpha_{21}, a_i, b_i, c_i, i = 1, 2)$.

THEOREM 3. *Assume that $0 < b_2 < c_1 + 2 \min\{b_1, c_2\}$ for the system (B) in Case(B). For the maximal solution $(u(x, t), v(x, t))$ obtained as in Theorem 1 there exist positive constants t_0 , $M' = M'(d_i, \alpha_{ij}, a_i, b_i, c_i, i, j = 1, 2)$, and $M = M(d_i, \alpha_{ij}, a_i, b_i, c_i, i, j = 1, 2)$ such that*

$$\begin{aligned} \max\{\|u(\cdot, t)\|_{1,2}, \|v(\cdot, t)\|_{1,2} : t \in (t_0, T)\} &\leq M', \\ \max\{u(x, t), v(x, t) : (x, t) \in [0, 1] \times (t_0, T)\} &\leq M, \end{aligned}$$

and $T = +\infty$. In the case that $d_1, d_2 \geq 1$, and $\underline{d} \leq \frac{d_2}{d_1} \leq \bar{d}$, where \underline{d}, \bar{d} are positive constants, the constants M', M are independent of $d_1, d_2 \geq 1$, that is, M' and M are depending only on $\underline{d}, \bar{d}, \alpha_{ij}, a_i, b_i, c_i, i, j = 1, 2$.

Before we state the convergence results for the cross-diffusion prey-predator systems, let us briefly mention the asymptotic behavior of the

solution $(u(t), v(t))$ of the kinetic system of prey-predator type in the following :

$$(k) \begin{cases} u_t = u(a_1 - b_1u - c_1v) & \text{for } t \in (0, \infty), \\ v_t = v(a_2 + b_2u - c_2v) & \text{for } t \in (0, \infty), \\ u(0) = u_0 \geq 0, \quad v(0) = v_0 \geq 0, \end{cases}$$

The asymptotic behaviors of the solutions of system (k) are classified into the three cases when :

$$(i) \quad -\frac{a_1 b_2}{b_1 c_2} < \frac{a_2}{c_2} < \frac{a_1}{c_1}, \quad (ii) \quad \frac{a_1}{c_1} < \frac{a_2}{c_2}, \quad (iii) \quad \frac{a_2}{c_2} < -\frac{a_1 b_2}{b_1 c_2}.$$

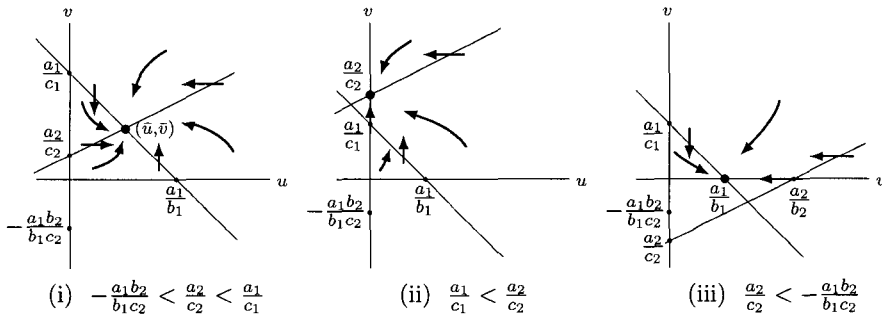


FIGURE 1. The unique nonnegative stable steady-state of the kinetic system (k) in each case of (i), (ii), and (iii)

In each case above the kinetic system (k) has a unique nonnegative stable steady-state as illustrated in Figure 1. The positive steady-state (\bar{u}, \bar{v}) in the case (i) in Figure 1 is given by

$$(\bar{u}, \bar{v}) = \left(\frac{a_1 c_2 - a_2 c_1}{b_1 c_2 + b_2 c_1}, \frac{a_2 b_1 + a_1 b_2}{b_1 c_2 + b_2 c_1} \right).$$

In case (i) for system (A) we obtain the following convergence results saying that under some condition a cross-diffusion prey-predator system has the same asymptotic property as its kinetic system :

THEOREM 4. Suppose that $-\frac{a_1 b_2}{b_1 c_2} < \frac{a_2}{c_2} < \frac{a_1}{c_1}$, and $0 < b_2 < c_1 + 2 \min\{b_1, c_2\}$ for the system (A). Let u_0, v_0 be in $W_2^2([0, 1])$. If $d \geq 1$ satisfies that

$$(1.3) \quad (b_2^2 \alpha_{12}^2 \bar{u}^2 + c_1^2 \alpha_{21}^2 \bar{v}^2) M^2 < 4b_2 c_1 \bar{u} \bar{v} d^2,$$

where M is the positive constant in Theorem 2 (independent of $d \geq 1$), then the solution $(u(x, t), v(x, t))$ converges to (\bar{u}, \bar{v}) uniformly in $[0, 1]$ as $t \rightarrow \infty$, and (\bar{u}, \bar{v}) is globally asymptotically stable.

Similar convergence result is proved for system (B) in case (i). And also in cases (ii) and (iii) we obtain convergence results for each system (A) and (B).

This paper consists of seven sections : Section 1. Introduction. In Section 2, 3, and 4 the convergence results in cases (i),(ii), and (iii) are proved, respectively, by using the uniform boundedness results in Theorems 2 and 3 for systems (A) and (B). Section 5. Calculus inequalities. Section 6 and 7. Uniform boundedness results (Theorems 2 and 3) for systems (A) and (B), respectively.

2. Convergence in Case (i)

In this section, we prove the convergence result in Theorem 4 for system (A) in case (i). And also the convergence result for system (B) in case (i) will be stated in Theorem 5 and proved.

PROOF OF THEOREM 4. In this proof we consider the case (i) when $-\frac{a_1 b_2}{b_1 c_2} < \frac{a_2}{c_2} < \frac{a_1}{c_1}$. Using the functional $H(u, v)$ defined below we observe the convergence of global solutions of the cross-diffusion prey-predator system (A) :

$$(2.1) \quad H(u, v) = \int_0^1 \left\{ b_2 \left(u - \bar{u} - \bar{u} \log \frac{u}{\bar{u}} \right) + c_1 \left(v - \bar{v} - \bar{v} \log \frac{v}{\bar{v}} \right) \right\} dx,$$

where $(\bar{u}, \bar{v}) = \left(\frac{a_1 c_2 - a_2 c_1}{b_1 c_2 + b_2 c_1}, \frac{a_2 b_1 + a_1 b_2}{b_1 c_2 + b_2 c_1} \right)$ is the positive stable steady-state of the kinetic system (k) in the case (i) as shown in Figure 1. $H(u, v)$ is always nonnegative and is zero only if $u \equiv \bar{u}$ and $v \equiv \bar{v}$. In order to prove the convergence of the solution first we observe the time derivative of $H(u(t), v(t))$ for the system (A) :

$$(2.2) \quad \begin{aligned} & \frac{dH(u(t), v(t))}{dt} \\ &= \int_0^1 \left\{ b_2 \left(1 - \frac{\bar{u}}{u} \right) u_t + c_1 \left(1 - \frac{\bar{v}}{v} \right) v_t \right\} dx \\ &= \int_0^1 \left\{ b_2 \left(1 - \frac{\bar{u}}{u} \right) (d u + \alpha_{12} u v)_{xx} + c_1 \left(1 - \frac{\bar{v}}{v} \right) (d v + \alpha_{21} u v)_{xx} \right\} dx \\ & \quad + \int_0^1 \left\{ b_2 (u - \bar{u}) f + c_1 (v - \bar{v}) g \right\} dx \\ &= - \int_0^1 \left\{ \frac{b_2 \bar{u}}{u^2} (d + \alpha_{12} v) u_x^2 + \left(\frac{b_2 \alpha_{12} \bar{u}}{u} + \frac{c_1 \alpha_{21} \bar{v}}{v} \right) u_x v_x \right. \\ & \quad \left. + \frac{c_1 \bar{v}}{v^2} (d + \alpha_{21} u) v_x^2 \right\} dx \\ & \quad - \int_0^1 \left\{ b_1 b_2 (u - \bar{u})^2 + c_1 c_2 (v - \bar{v})^2 \right\} dx, \end{aligned}$$

where we denoted $f = a_1 - b_1u - c_1v$ and $g = a_2 + b_2u - c_2v$ and used the fact that

$$\begin{aligned} & b_2(u - \bar{u})f + c_1(v - \bar{v})g \\ &= b_2(u - \bar{u})(a_1 - b_1u - c_1v) + c_1(v - \bar{v})(a_2 + b_2u - c_2v) \\ &= -b_2(u - \bar{u})(b_1(u - \bar{u}) + c_1(v - \bar{v})) \\ &\quad + c_1(v - \bar{v})(b_2(u - \bar{u}) - c_2(v - \bar{v})) \\ &= -b_1b_2(u - \bar{u})^2 - c_1c_2(v - \bar{v})^2. \end{aligned}$$

Now we remind the uniform boundedness result for the solution of the system (A) in the case $d \geq 1$ as in Theorem 2 that there exist positive constants t_0 and $M = M(\alpha_{12}, \alpha_{21}, a_i, b_i, c_i, i = 1, 2)$ such that

$$(2.3) \quad 0 \leq u(x, t), v(x, t) \leq M \quad \text{for every } (x, t) \in [0, 1] \times (t_0, \infty).$$

From the proof of Theorem 2 we can choose the constant M depending on the initial functions u_0, v_0 so that the inequalities in (2.3) hold for all $t \geq 0$. Using (2.3) and condition (1.3) in the hypothesis of the present theorem (Theorem 4) for every constant γ such that $0 < \gamma < \frac{4b_2c_1\bar{u}\bar{v}d^2 - (b_2^2\alpha_{12}^2\bar{u}^2 + c_1^2\alpha_{21}^2\bar{v}^2)M^2}{4M^2(b_2\bar{u}(d + \alpha_{12}M) + c_1\bar{v}(d + \alpha_{21}M))}$ we have the following inequality :

$$(2.4) \quad \begin{aligned} & \frac{b_2\bar{u}}{u^2}(d + \alpha_{12}v)u_x^2 + \left(\frac{b_2\alpha_{12}\bar{u}}{u} + \frac{c_1\alpha_{21}\bar{v}}{v}\right)u_xv_x \\ & + \frac{c_1\bar{v}}{v^2}(d + \alpha_{21}u)v_x^2 \geq \gamma\{u_x^2 + v_x^2\}, \end{aligned}$$

since

$$\begin{aligned} & \left(\frac{b_2\alpha_{12}\bar{u}}{u} + \frac{c_1\alpha_{21}\bar{v}}{v}\right)^2 - 4\left\{\frac{b_2\bar{u}}{u^2}(d + \alpha_{12}v) - \gamma\right\}\left\{\frac{c_1\bar{v}}{v^2}(d + \alpha_{21}u) - \gamma\right\} \\ \leq & \frac{b_2^2\alpha_{12}^2\bar{u}^2}{u^2} + \frac{c_1^2\alpha_{21}^2\bar{v}^2}{v^2} - \frac{4b_2c_1\bar{u}\bar{v}d^2}{u^2v^2} \\ & + 4\gamma\left\{\frac{b_2\bar{u}}{u^2}(d + \alpha_{12}M) + \frac{c_1\bar{v}}{v^2}(d + \alpha_{21}M)\right\} \\ \leq & \frac{1}{u^2v^2}[(b_2^2\alpha_{12}^2\bar{u}^2 + c_1^2\alpha_{21}^2\bar{v}^2)M^2 - 4b_2c_1\bar{u}\bar{v}d^2 \\ & + 4\gamma M^2\{b_2\bar{u}(d + \alpha_{12}M) + c_1\bar{v}(d + \alpha_{21}M)\}] \\ < & 0. \end{aligned}$$

From (2.2) and (2.4) we have for $t \geq 0$

$$\begin{aligned} \frac{dH(u(t), v(t))}{dt} & \leq -\gamma \int_0^1 \{u_x^2 + v_x^2\} dx \\ & \quad - \int_0^1 \{b_1b_2(u - \bar{u})^2 + c_1c_2(v - \bar{v})^2\} dx \leq 0. \end{aligned}$$

We notice that $\frac{dH(u(t), v(t))}{dt} = 0$ only if $u(x, t) \equiv \bar{u}$ and $v(x, t) \equiv \bar{v}$.

Thus it is shown that $H(u(t), v(t)) \searrow 0$ as $t \rightarrow \infty$. And we obtain the L_2 convergences, $|u(t) - \bar{u}|_2 \rightarrow 0, |v(t) - \bar{v}|_2 \rightarrow 0$ as $t \rightarrow \infty$ by using the uniform boundedness of $(u(x, t), v(x, t))$ in $[0, 1]$. From

Theorem 1 with the assumption that $u_0, v_0 \in W_2^2([0, 1])$, we have that $\sup_{0 \leq t < \infty} |u_{xx}(t)|_2 < \infty$, and $\sup_{0 \leq t < \infty} |v_{xx}(t)|_2 < \infty$. Applying the calculus inequality (5.6) in Section 5 to the functions $u(x, t) - \bar{u}$ and $v(x, t) - \bar{v}$, we obtain the convergence $(u(x, t), v(x, t)) \rightarrow (\bar{u}, \bar{v})$ as $t \rightarrow \infty$ in $W_2^1([0, 1])$. By using the Sobolev embedding theorem we show that $(u(x, t), v(x, t))$ converges to (\bar{u}, \bar{v}) uniformly in $[0, 1]$ as $t \rightarrow \infty$. We also obtain that (\bar{u}, \bar{v}) is locally asymptotically stable in $C([0, 1])$ by using the fact that $H(u(t), v(t))$ is decreasing for $t \geq 0$. Thus we conclude that (\bar{u}, \bar{v}) is globally asymptotically stable. \square

THEOREM 5. *For the system (B) in Case(B) suppose that $-\frac{a_1 b_2}{b_1 c_2} < \frac{a_2}{c_2} < \frac{a_1}{c_1}$, and $0 < b_2 < c_1 + 2 \min\{b_1, c_2\}$. Let u_0, v_0 be in $W_2^2([0, 1])$. If $d_1, d_2 \geq 1$ satisfy that*

$$(2.5) \quad (b_2^2 \alpha_{12}^2 \bar{u}^2 + c_1^2 \alpha_{21}^2 \bar{v}^2) M^2 < 4b_2 c_1 \bar{u} \bar{v} d_1 d_2,$$

where M is the positive constant in Theorem 3, then the solution $(u(x, t), v(x, t))$ converges to (\bar{u}, \bar{v}) uniformly in $[0, 1]$ as $t \rightarrow \infty$, and (\bar{u}, \bar{v}) is globally asymptotically stable.

PROOF. By using the functional $H(u, v)$ as in (2.1) in the proof of Theorem 4 we observe the convergence of global solutions of the cross-diffusion prey-predator system (B). We first estimate the time derivative of $H(u(t), v(t))$ for the solution of the system (B).

$$(2.6) \quad \begin{aligned} & \frac{dH(u(t), v(t))}{dt} \\ &= \int_0^1 \{b_2(1 - \frac{\bar{u}}{u})u_t + c_1(1 - \frac{\bar{v}}{v})v_t\} dx \\ &= \int_0^1 \{b_2(1 - \frac{\bar{u}}{u})(d_1 u + \alpha_{11} u^2 + \alpha_{12} uv)_{xx} \\ & \quad + c_1(1 - \frac{\bar{v}}{v})(d_2 v + \alpha_{21} uv + \alpha_{22} v^2)_{xx}\} dx \\ & \quad + \int_0^1 \{b_2(u - \bar{u})f + c_1(v - \bar{v})g\} dx \\ &= - \int_0^1 \{ \frac{b_2 \bar{u}}{u^2} (d_1 + 2\alpha_{11} u + \alpha_{12} v) u_x^2 + (\frac{b_2 \alpha_{12} \bar{u}}{u} + \frac{c_1 \alpha_{21} \bar{v}}{v}) u_x v_x \\ & \quad + \frac{c_1 \bar{v}}{v^2} (d_2 + \alpha_{21} u + 2\alpha_{22} v) v_x^2 \} dx \\ & \quad - \int_0^1 \{b_1 b_2 (u - \bar{u})^2 + c_1 c_2 (v - \bar{v})^2\} dx, \end{aligned}$$

where we denoted $f = a_1 - b_1 u - c_1 v$ and $g = a_2 + b_2 u - c_2 v$ and used the fact that $b_2(u - \bar{u})f + c_1(v - \bar{v})g = -b_1 b_2 (u - \bar{u})^2 - c_1 c_2 (v - \bar{v})^2$ as shown in the proof of Theorem 4.

Now we remind the uniform boundedness result for the solution of the system (B) in the case $d_1, d_2 \geq 1$, and $\underline{d} \leq \frac{d_2}{d_1} \leq \bar{d}$ as in Theorem 3 that there exist positive constants t_0 and $M = M(\underline{d}, \bar{d}, \alpha_{ij}, a_i, b_i, c_i, i = 1, 2)$ such that

$$(2.7) \quad 0 \leq u(x, t), v(x, t) \leq M \quad \text{for every } (x, t) \in [0, 1] \times (t_0, \infty).$$

Using (2.7) and condition (2.5) in the hypothesis of the present theorem (Theorem 5) for every constant γ such that

$$0 < \gamma < \frac{4b_2c_1\bar{u}\bar{v}d_1d_2 - (b_2^2\alpha_{12}^2\bar{u}^2 + c_1^2\alpha_{21}^2\bar{v}^2)M^2}{4M^2[b_2\bar{u}(d_1 + (2\alpha_{11} + \alpha_{12})M) + c_1\bar{v}(d_2 + (\alpha_{21} + 2\alpha_{22})M)]}$$

we have the following inequality :

$$(2.8) \quad \begin{aligned} & \frac{b_2\bar{u}}{u^2}(d_1 + 2\alpha_{11}u + \alpha_{12}v)u_x^2 \\ & + (\frac{b_2\alpha_{12}\bar{u}}{u} + \frac{c_1\alpha_{21}\bar{v}}{v})u_xv_x + \frac{c_1\bar{v}}{v^2}(d_2 + \alpha_{21}u + 2\alpha_{22}v)v_x^2 \\ & \geq \gamma\{u_x^2 + v_x^2\}, \end{aligned}$$

since

$$\begin{aligned} & (\frac{b_2\alpha_{12}\bar{u}}{u} + \frac{c_1\alpha_{21}\bar{v}}{v})^2 - 4\{\frac{b_2\bar{u}}{u^2}(d_1 + 2\alpha_{11}u + \alpha_{12}v) - \gamma\} \\ & \times \{\frac{c_1\bar{v}}{v^2}(d_2 + \alpha_{21}u + 2\alpha_{22}v) - \gamma\} \\ \leq & \frac{b_2^2\alpha_{12}^2\bar{u}^2}{u^2} + \frac{c_1^2\alpha_{21}^2\bar{v}^2}{v^2} - \frac{4b_2c_1\bar{u}\bar{v}d_1d_2}{u^2v^2} \\ & + 4\gamma\{\frac{b_2\bar{u}}{u^2}(d_1 + (2\alpha_{11} + \alpha_{12})M) + \frac{c_1\bar{v}}{v^2}(d_2 + (\alpha_{21} + 2\alpha_{22})M)\} \\ \leq & \frac{1}{u^2v^2}[(b_2^2\alpha_{12}^2\bar{u}^2 + c_1^2\alpha_{21}^2\bar{v}^2)M^2 - 4b_2c_1\bar{u}\bar{v}d_1d_2 \\ & + 4\gamma M^2\{b_2\bar{u}(d_1 + (2\alpha_{11} + \alpha_{12})M) + c_1\bar{v}(d_2 + (\alpha_{21} + 2\alpha_{22})M)\}] \\ < & 0. \end{aligned}$$

From (2.6) and (2.8) we have

$$\begin{aligned} & \frac{dH(u(t),v(t))}{dt} \\ & \leq -\gamma \int_0^1 \{u_x^2 + v_x^2\} dx - \int_0^1 \{b_1b_2(u - \bar{u})^2 + c_1c_2(v - \bar{v})^2\} dx \leq 0. \end{aligned}$$

Now, by using the same arguments as the proof of Theorem 4 we show that $(u(x, t), v(x, t))$ converges to (\bar{u}, \bar{v}) uniformly in $[0, 1]$ as $t \rightarrow \infty$. We also obtain that (\bar{u}, \bar{v}) is locally asymptotically stable in $C([0, 1])$ from the fact that $H(u(t), v(t))$ is decreasing for $t \geq 0$. Thus we conclude that (\bar{u}, \bar{v}) is globally asymptotically stable. \square

3. Convergence in Case (ii)

In case (ii) the convergence results for system (A) and (B) are stated and proved in Theorems 6 and 7, respectively.

THEOREM 6. *Suppose that $\frac{a_1}{c_1} < \frac{a_2}{c_2}$, $0 < b_2 < c_1 + 2 \min\{b_1, c_2\}$, and $0 < b_2 < 4(a_2 - \frac{a_1c_2}{c_1})$ for the system (A). Let u_0, v_0 be in $W_2^2([0, 1])$. If $d \geq 1$ satisfies that*

$$(3.1) \quad b_2^2c_2^2\alpha_{12}^2M^4 + a_2^2c_1^2\alpha_{21}^2(1 + M)^2 < 4a_2b_2c_1c_2d^2,$$

where M is the positive constant in Theorem 2 (independent of $d \geq 1$), then the solution $(u(x, t), v(x, t))$ converges to $(0, \frac{a_2}{c_2})$ uniformly in $[0, 1]$ as $t \rightarrow \infty$, and $(0, \frac{a_2}{c_2})$ is globally asymptotically stable.

PROOF. Using the functional $E^v(u, v)$ defined below we observe the convergence of global solutions of the cross-diffusion prey-predator system (A) in case (ii) :

$$(3.2) \quad E^v(u, v) = \int_0^1 \left\{ b_2 (u - \log(1 + u)) + c_1 \left(v - \frac{a_2}{c_2} - \frac{a_2}{c_2} \log \frac{v}{a_2/c_2} \right) \right\} dx.$$

$E^v(u, v)$ is always nonnegative and is zero only if $u \equiv 0$ and $v \equiv \frac{a_2}{c_2}$. In order to prove the convergence of the solution first we observe the time derivative of $E^v(u(t), v(t))$ for the system (A) :

$$(3.3) \quad \begin{aligned} & \frac{dE^v(u(t), v(t))}{dt} \\ &= \int_0^1 \left\{ b_2 \left(\frac{u}{1+u} \right) u_t + c_1 \left(1 - \frac{a_2/c_2}{v} \right) v_t \right\} dx \\ &= \int_0^1 \left\{ b_2 \left(\frac{u}{1+u} \right) (d u + \alpha_{12} u v)_{xx} + c_1 \left(1 - \frac{a_2/c_2}{v} \right) (d v + \alpha_{21} u v)_{xx} \right\} dx \\ & \quad + \int_0^1 \left\{ b_2 \left(\frac{u^2}{1+u} \right) f + c_1 \left(v - \frac{a_2}{c_2} \right) g \right\} dx \\ &= - \int_0^1 \left\{ \frac{b_2}{(1+u)^2} (d + \alpha_{12} v) u_x^2 + \left(\frac{b_2 \alpha_{12} u}{(1+u)^2} + \frac{c_1 \alpha_2 \alpha_{21}}{c_2 v} \right) u_x v_x \right. \\ & \quad \left. + \frac{c_1 a_2}{c_2 v^2} (d + \alpha_{21} u) v_x^2 \right\} dx \\ & \quad + \int_0^1 \frac{1}{1+u} \left\{ b_2 u^2 f + c_1 (1+u) \left(v - \frac{a_2}{c_2} \right) g \right\} dx, \end{aligned}$$

where $f = a_1 - b_1 u - c_1 v$ and $g = a_2 + b_2 u - c_2 v$.

First we estimate the terms of the integral in the last line of (3.3) :

$$\begin{aligned} & b_2 u^2 f + c_1 (1+u) \left(v - \frac{a_2}{c_2} \right) g \\ &= b_2 u^2 (a_1 - b_1 u - c_1 v) + c_1 (1+u) \left(v - \frac{a_2}{c_2} \right) (a_2 + b_2 u - c_2 v) \\ &= b_2 u^2 (a_1 - b_1 u - c_1 \left(v - \frac{a_2}{c_2} \right) - \frac{a_2 c_1}{c_2}) \\ & \quad + c_1 (1+u) \left(v - \frac{a_2}{c_2} \right) (b_2 u - c_2 \left(v - \frac{a_2}{c_2} \right)) \\ &= b_2 u^2 (a_1 - \frac{a_2}{c_2} - b_1 u) - b_2 c_1 u^2 \left(v - \frac{a_2 c_1}{c_2} \right) \\ & \quad + b_2 c_1 u (1+u) \left(v - \frac{a_2}{c_2} \right) - c_1 c_2 (1+u) \left(v - \frac{a_2}{c_2} \right)^2 \\ &= b_2 u^2 (a_1 - \frac{a_2}{c_2} - b_1 u) + b_2 c_1 u \left(v - \frac{a_2}{c_2} \right) - c_1 c_2 (1+u) \left(v - \frac{a_2}{c_2} \right)^2, \end{aligned}$$

and, regarding the form in the last line above as a quadratic function of $(v - \frac{a_2}{c_2})$, we observe its determinant :

$$\begin{aligned} & (b_2 c_1 u)^2 + 4 b_2 c_1 c_2 u^2 (1+u) \left(a_1 - \frac{a_2 c_1}{c_2} - b_1 u \right) \\ &= b_2 c_1 u^2 \left\{ b_2 c_1 + 4 c_2 (1+u) \left(a_1 - \frac{a_2 c_1}{c_2} - b_1 u \right) \right\} \\ &\leq b_2 c_1 u^2 \left\{ b_2 c_1 + 4 c_2 \left(a_1 - \frac{a_2 c_1}{c_2} \right) \right\} \\ &\leq 0, \text{ and the equality holds only when } u = 0, \end{aligned}$$

since $b_2c_1 < 4(a_2c_1 - a_1c_2)$ from the condition $0 < b_2 < 4(a_2 - \frac{a_1c_2}{c_1})$ in the hypothesis of the present theorem. Thus we obtain that

$$(3.4) \quad \begin{aligned} & b_2u^2f + c_1(1+u)(v - \frac{a_2}{c_2})g \\ & \leq 0, \text{ and the equality holds only if } (u, v) \equiv (0, \frac{a_2}{c_2}). \end{aligned}$$

Now we estimate the terms with spatial derivatives in (3.3). For the solution of the system (A) we take the uniform bound M satisfying (2.3) in the proof of Theorem 4. From the condition (3.1) in the hypothesis of the present theorem (Theorem 6) for every constant γ such that $0 < \gamma < \frac{4a_2b_2c_1c_2d^2 - b_2^2c_2^2\alpha_{12}^2M^4 - a_2^2c_1^2\alpha_{21}^2(1+M)^2}{4(b_2c_2^2M^2(d+\alpha_{12}M) + a_2c_1c_2(1+M)^2(d+\alpha_{21}M))}$ we have the following inequality :

$$(3.5) \quad \begin{aligned} & \frac{b_2}{(1+u)^2}(d + \alpha_{12}v)u_x^2 + (\frac{b_2\alpha_{12}u}{(1+u)^2} + \frac{c_1a_2\alpha_{21}}{c_2v})u_xv_x + \frac{c_1a_2}{c_2v^2}(d + \alpha_{21}u)v_x^2 \\ & \geq \gamma\{u_x^2 + v_x^2\}, \end{aligned}$$

since

$$\begin{aligned} & \left(\frac{b_2\alpha_{12}u}{(1+u)^2} + \frac{c_1a_2\alpha_{21}}{c_2v}\right)^2 \\ & - 4\left\{\frac{b_2}{(1+u)^2}(d + \alpha_{12}v) - \gamma\right\}\left\{\frac{c_1a_2}{c_2v^2}(d + \alpha_{21}u) - \gamma\right\} \\ \leq & \frac{b_2^2\alpha_{12}^2u^2}{(1+u)^4} + \frac{c_1^2a_2^2\alpha_{21}^2}{c_2^2v^2} - \frac{4a_2b_2c_1d^2}{c_2(1+u)^2v^2} \\ & + 4\gamma\left\{\frac{b_2}{(1+u)^2}(d + \alpha_{12}v) + \frac{c_1a_2}{c_2v^2}(d + \alpha_{21}u)\right\} \\ \leq & \frac{1}{c_2^2(1+u)^2v^2}\left[\frac{b_2^2\alpha_{12}^2u^2v^2}{(1+u)^2} + c_1^2a_2^2\alpha_{21}^2(1+u)^2 - 4a_2b_2c_1c_2d^2\right. \\ & \left.+ 4\gamma\{b_2c_2^2v^2(d + \alpha_{12}v) + a_2c_1c_2(1+u)^2(d + \alpha_{21}u)\}\right] \\ \leq & \frac{1}{c_2^2(1+u)^2v^2}\left[b_2^2\alpha_{12}^2M^4 + c_1^2a_2^2\alpha_{21}^2(1+M)^2 - 4a_2b_2c_1c_2d^2\right. \\ & \left.+ 4\gamma\{b_2c_2^2M^2(d + \alpha_{12}M) + a_2c_1c_2(1+M)^2(d + \alpha_{21}M)\}\right] \\ < & 0. \end{aligned}$$

From (3.4) and (3.5) we have for $t \geq 0$ that $\frac{dE^v(u(t),v(t))}{dt} \leq 0$ and $\frac{dE^v(u(t),v(t))}{dt} = 0$ only if $u(x, t) \equiv 0$ and $v(x, t) \equiv \frac{a_2}{c_2}$.

Thus it is shown that $E^v(u(t), v(t)) \searrow 0$ as $t \rightarrow \infty$. And we obtain the L_2 convergences, $|u(t)|_2 \rightarrow 0$, $|v(t) - \frac{a_2}{c_2}|_2 \rightarrow 0$ as $t \rightarrow \infty$ by using the uniform boundedness of $(u(x, t), v(x, t))$ in $[0, 1]$. From Theorem 1, $\sup_{0 \leq t < \infty} |u_{xx}(t)|_2 < \infty$, and $\sup_{0 \leq t < \infty} |v_{xx}(t)|_2 < \infty$. Applying the calculus inequality (5.6) in Section 5 to the functions $u(x, t)$ and $v(x, t) - \frac{a_2}{c_2}$, we obtain the convergence $(u(x, t), v(x, t)) \rightarrow (0, \frac{a_2}{c_2})$ as $t \rightarrow \infty$ in $W_2^1([0, 1])$. By using the Sobolev embedding theorem we show that $(u(x, t), v(x, t))$ converges to $(0, \frac{a_2}{c_2})$ uniformly in $[0, 1]$ as $t \rightarrow \infty$. We also obtain that $(0, \frac{a_2}{c_2})$ is locally asymptotically stable in $C([0, 1])$ by using the fact that

$E^v(u(t), v(t))$ is decreasing for $t \geq 0$. Thus we conclude that $(0, \frac{a_2}{c_2})$ is globally asymptotically stable. \square

THEOREM 7. For the system (B) in Case(B) suppose that $\frac{a_1}{c_1} < \frac{a_2}{c_2}$, $0 < b_2 < c_1 + 2 \min\{b_1, c_2\}$, and $0 < b_2 < 4(a_2 - \frac{a_1 c_2}{c_1})$. Let u_0, v_0 be in $W_2^2([0, 1])$. If $d_1, d_2 \geq 1$ satisfy that

$$(3.6) \quad b_2^2 c_2^2 \alpha_{12}^2 M^4 + a_2^2 c_1^2 \alpha_{21}^2 (1 + M)^2 < 4a_2 b_2 c_1 c_2 d_1 d_2,$$

where M is the positive constant in Theorem 3, then the solution $(u(x, t), v(x, t))$ converges to $(0, \frac{a_2}{c_2})$ uniformly in $[0, 1]$ as $t \rightarrow \infty$, and $(0, \frac{a_2}{c_2})$ is globally asymptotically stable.

PROOF. Using the functional $E^v(u, v)$ defined as in (3.2) in the proof of Theorem 6 we observe the convergence of global solutions of the cross-diffusion prey-predator system (B). We first estimate the time derivative of $E^v(u(t), v(t))$ for the solution of the system (B).

$$(3.7) \quad \begin{aligned} & \frac{dE^v(u(t), v(t))}{dt} \\ &= \int_0^1 \{b_2(\frac{u}{1+u})u_t + c_1(1 - \frac{a_2/c_2}{v})v_t\} dx \\ &= \int_0^1 \{b_2(\frac{u}{1+u})(d_1 u + \alpha_{11}u^2 + \alpha_{12}uv)_{xx} \\ & \quad + c_1(1 - \frac{a_2/c_2}{v})(d_2 v + \alpha_{21}uv + \alpha_{22}v^2)_{xx}\} dx \\ & \quad + \int_0^1 \{b_2(\frac{u^2}{1+u})f + c_1(v - \frac{a_2}{c_2})g\} dx \\ &= - \int_0^1 \{ \frac{b_2}{(1+u)^2} (d_1 + 2\alpha_{11}u + \alpha_{12}v)u_x^2 \\ & \quad + (\frac{b_2 \alpha_{12}u}{(1+u)^2} + \frac{c_1 a_2 \alpha_{21}}{c_2 v})u_x v_x \\ & \quad + \frac{c_1 a_2}{c_2 v^2} (d_2 + \alpha_{21}u + 2\alpha_{22}v)v_x^2 \} dx \\ & \quad + \int_0^1 \frac{1}{1+u} \{b_2 u^2 f + c_1(1+u)(v - \frac{a_2}{c_2})g\} dx, \end{aligned}$$

where $f = a_1 - b_1 u - c_1 v$ and $g = a_2 + b_2 u - c_2 v$.

We have the same estimates for the terms of the integral in the last line of (3.7) as shown in the proof of Theorem 4 :

$$(3.8) \quad \begin{aligned} & b_2 u^2 f + c_1(1+u)(v - \frac{a_2}{c_2})g \\ & \leq 0, \text{ and the equality holds only if } (u, v) \equiv (0, \frac{a_2}{c_2}). \end{aligned}$$

Now we estimate the terms with spatial derivatives in (3.7). For the solution of the system (B) we take the uniform bound M satisfying (2.7) in the proof of Theorem 5. From the condition (3.6) in the hypothesis of the present theorem (Theorem 7) for every constant γ such that $0 < \gamma < \frac{4a_2 b_2 c_1 c_2 d_1 d_2 - b_2^2 c_2^2 \alpha_{12}^2 M^4 - a_2^2 c_1^2 \alpha_{21}^2 (1+M)^2}{4(b_2 c_2^2 M^2 (d_1 + (2\alpha_{11} + \alpha_{12})M) + a_2 c_1 c_2 (1+M)^2 (d_2 + (\alpha_{21} + 2\alpha_{22})M))}$ we have the

following inequality :

$$(3.9) \quad \frac{b_2}{(1+u)^2}(d_1 + 2\alpha_{11}u + \alpha_{12}v)u_x^2 + \left(\frac{b_2\alpha_{12}u}{(1+u)^2} + \frac{c_1\alpha_2\alpha_{21}}{c_2v}\right)u_xv_x + \frac{c_1\alpha_2}{c_2v^2}(d_2 + \alpha_{21}u + 2\alpha_{22}v)v_x^2 \geq \gamma\{u_x^2 + v_x^2\},$$

since

$$\begin{aligned} & \left(\frac{b_2\alpha_{12}u}{(1+u)^2} + \frac{c_1\alpha_2\alpha_{21}}{c_2v}\right)^2 - 4\left\{\frac{b_2}{(1+u)^2}(d_1 + 2\alpha_{11}u + \alpha_{12}v) - \gamma\right\} \\ & \times \left\{\frac{c_1\alpha_2}{c_2v^2}(d_2 + \alpha_{21}u + 2\alpha_{22}v) - \gamma\right\} \\ \leq & \frac{b_2^2\alpha_{12}^2u^2}{(1+u)^4} + \frac{c_1^2\alpha_2^2\alpha_{21}^2}{c_2^2v^2} - \frac{4a_2b_2c_1d^2}{c_2(1+u)^2v^2} \\ & + 4\gamma\left\{\frac{b_2}{(1+u)^2}(d_1 + 2\alpha_{11}u + \alpha_{12}v) + \frac{c_1\alpha_2}{c_2v^2}(d_2 + \alpha_{21}u + 2\alpha_{22}v)\right\} \\ \leq & \frac{1}{c_2^2(1+u)^2v^2}\left[\frac{b_2^2c_2^2\alpha_{12}^2u^2v^2}{(1+u)^2} + c_1^2\alpha_2^2\alpha_{21}^2(1+u)^2 - 4a_2b_2c_1c_2d_1d_2\right. \\ & \left.+ 4\gamma\{b_2c_2^2v^2(d_1 + 2\alpha_{11}u + \alpha_{12}v) + a_2c_1c_2(1+u)^2(d_2 + \alpha_{21}u + 2\alpha_{22}v)\}\right] \\ \leq & \frac{1}{c_2^2(1+u)^2v^2}\left[b_2^2c_2^2\alpha_{12}^2M^4 + a_2^2c_1^2\alpha_{21}^2(1+M)^2 - 4a_2b_2c_1c_2d_1d_2\right. \\ & \left.+ 4\gamma\{b_2c_2^2M^2(d_1 + (2\alpha_{11} + \alpha_{12})M) + a_2c_1c_2(1+M)^2(d_2 + (\alpha_{21} + 2\alpha_{22})M)\}\right] \\ < & 0. \end{aligned}$$

From (3.8) and (3.9) we have for $t \geq 0$ that $\frac{dE^v(u(t),v(t))}{dt} \leq 0$ and $\frac{dE^v(u(t),v(t))}{dt} = 0$ only if $u(x, t) \equiv 0$ and $v(x, t) \equiv \frac{\alpha_2}{c_2}$.

By using the same arguments as the proof of Theorem 6 we show that $(u(x, t), v(x, t))$ converges to $(0, \frac{\alpha_2}{c_2})$ uniformly in $[0, 1]$ as $t \rightarrow \infty$. We also obtain that $(0, \frac{\alpha_2}{c_2})$ is locally asymptotically stable in $C([0, 1])$ from the fact that $E^v(u(t), v(t))$ is decreasing for $t \geq 0$. Thus we conclude that $(0, \frac{\alpha_2}{c_2})$ is globally asymptotically stable. \square

4. Convergence in Case (iii)

In case (iii) the convergence results for system (A) and (B) are stated and proved in Theorems 8 and 9, respectively.

THEOREM 8. *Suppose for the system (A) that $\frac{\alpha_2}{c_2} < -\frac{a_1b_2}{b_1c_2}$, $0 < b_2 < c_1 + 2\min\{b_1, c_2\}$, and $0 < c_1 < -4(\frac{\alpha_2b_1}{b_2} + a_1)$. Let u_0, v_0 be in $W_2^2([0, 1])$. If $d \geq 1$ satisfies that*

$$(4.1) \quad a_1^2b_1^2\alpha_{12}^2(1+M)^2 + b_1^2c_1^2\alpha_{21}^2M^4 < 4a_1b_1b_2c_1d^2,$$

where M is the positive constant in Theorem 2 (independent of $d \geq 1$), then the solution $(u(x, t), v(x, t))$ converges to $(\frac{a_1}{b_1}, 0)$ uniformly in $[0, 1]$ as $t \rightarrow \infty$, and $(\frac{a_1}{b_1}, 0)$ is globally asymptotically stable.

PROOF. Using the functional $E^u(u, v)$ defined below we observe the convergence of global solutions of the cross-diffusion prey-predator system (A) in case (ii) :

$$(4.2) \quad E^u(u, v) = \int_0^1 \left\{ b_2 \left(u - \frac{a_1}{b_1} - \frac{a_1}{b_1} \log \frac{u}{a_1/b_1} \right) + c_1 (v - \log(1 + v)) \right\} dx.$$

$E^u(u, v)$ is always nonnegative and is zero only if $u \equiv \frac{a_1}{b_1}$ and $v \equiv 0$. In order to prove the convergence of the solution first we observe the time derivative of $E^u(u(t), v(t))$ for the system (A) :

$$(4.3) \quad \begin{aligned} & \frac{dE^u(u(t), v(t))}{dt} \\ &= \int_0^1 \{ b_2(1 - \frac{a_1/b_1}{u})u_t + c_1(\frac{v}{1+v})v_t \} dx \\ &= \int_0^1 \{ b_2(1 - \frac{a_1/b_1}{u})(du + \alpha_{12}uv)_{xx} \\ & \quad + c_1(\frac{v}{1+v})(dv + \alpha_{21}uv)_{xx} \} dx \\ & \quad + \int_0^1 \{ b_2(u - \frac{a_1}{b_1})f + c_1(\frac{v^2}{1+v})g \} dx \\ &= - \int_0^1 \{ \frac{a_1 b_2}{b_1 u^2} (d + \alpha_{12}v)u_x^2 \\ & \quad + (\frac{a_1 b_2 \alpha_{12}}{b_1 u} + \frac{c_1 \alpha_{21} v}{(1+v)^2})u_x v_x + \frac{c_1}{(1+v)^2} (d + \alpha_{21}u)v_x^2 \} dx \\ & \quad + \int_0^1 \frac{1}{1+v} \{ b_2(u - \frac{a_1}{b_1})(1+v)f + c_1 v^2 g \} dx, \end{aligned}$$

where $f = a_1 - b_1 u - c_1 v$ and $g = a_2 + b_2 u - c_2 v$.

First we estimate the terms of the integral in the last line of (4.3) :

$$\begin{aligned} & b_2(u - \frac{a_1}{b_1})(1+v)f + c_1 v^2 g \\ &= b_2(u - \frac{a_1}{b_1})(1+v)(a_1 - b_1 u - c_1 v) + c_1 v^2 (a_2 + b_2 u - c_2 v) \\ &= b_2(u - \frac{a_1}{b_1})(1+v)(-b_1(u - \frac{a_1}{b_1}) - c_1 v) \\ & \quad + c_1 v^2 (a_2 + b_2(u - \frac{a_1}{b_1}) - c_2 v + \frac{a_1 b_2}{b_1}) \\ &= -b_1 b_2 (1+v)(u - \frac{a_1}{b_1})^2 - b_2 c_1 (u - \frac{a_1}{b_1})(1+v)v \\ & \quad + b_2 c_1 v^2 (u - \frac{a_1}{b_1}) + c_1 v^2 (a_2 - c_2 v + \frac{a_1 b_2}{b_1}) \\ &= -b_1 b_2 (1+v)(u - \frac{a_1}{b_1})^2 - b_2 c_1 (u - \frac{a_1}{b_1})v + c_1 v^2 (a_2 + \frac{a_1 b_2}{b_1} - c_2 v), \end{aligned}$$

and, regarding the form in the last line above as a quadratic function of $(u - \frac{a_1}{b_1})$, we observe its determinant :

$$\begin{aligned} & (b_2 c_1 v)^2 + 4b_1 b_2 c_1 v^2 (1+v)(a_2 + \frac{a_1 b_2}{b_1} - c_2 v) \\ &= b_2 c_1 v^2 \{ b_2 c_1 + 4b_1 (1+v)(a_2 + \frac{a_1 b_2}{b_1} - c_2 v) \} \\ &\leq b_2 c_1 v^2 \{ b_2 c_1 + 4b_1 (a_2 + \frac{a_1 b_2}{b_1}) \} \\ &\leq 0, \quad \text{and the equality holds only when } u = 0, \end{aligned}$$

since $b_2c_1 < -4(a_2b_1 + a_1b_2)$ from the condition $0 < c_1 < -4(\frac{a_2b_1}{b_2} + a_1)$ in the hypothesis of the present theorem. Also notice that $a_2 < 0$ from the condition (iii) $\frac{a_2}{c_2} < -\frac{a_1b_2}{b_1c_2}$. Thus we obtain that

$$(4.4) \quad \begin{aligned} & b_2(u - \frac{a_1}{b_1})(1 + v)f + c_1v^2g \\ & \leq 0, \text{ and the equality holds only if } (u, v) \equiv (\frac{a_1}{b_1}, 0). \end{aligned}$$

Now we estimate the terms with spatial derivatives in (4.3). For the solution of the system (A) we take the uniform bound M satisfying (2.3) in the proof of Theorem 4. From the condition (4.1) in the hypothesis of the present theorem (Theorem 8) for every constant γ such that $0 < \gamma < \frac{4a_1b_1b_2c_1d^2 - a_1^2b_1^2\alpha_{12}^2(1+M)^2 - b_1^2c_1^2\alpha_{21}^2M^4}{4(a_1b_1b_2(1+M)^2(d+\alpha_{12}M) + b_1^2c_1M^2(d\alpha_{21}+M))}$ we have the following inequality :

$$(4.5) \quad \begin{aligned} & \frac{a_1b_2}{b_1u^{\frac{3}{2}}}(d + \alpha_{12}v)u_x^2 + (\frac{a_1b_2\alpha_{12}}{b_1u} + \frac{c_1\alpha_{21}v}{(1+v)^2})u_xv_x + \frac{c_1}{(1+v)^2}(d + \alpha_{21}u)v_x^2 \\ & \geq \gamma\{u_x^2 + v_x^2\}, \end{aligned}$$

since

$$\begin{aligned} & \left(\frac{a_1b_2\alpha_{12}}{b_1u} + \frac{c_1\alpha_{21}v}{(1+v)^2}\right)^2 - 4\left\{\frac{a_1b_2}{b_1u^{\frac{3}{2}}}(d + \alpha_{12}v) - \gamma\right\} \\ & \times \left\{\frac{c_1}{(1+v)^2}(d + \alpha_{21}u) - \gamma\right\} \\ & \leq \frac{a_1^2b_2^2\alpha_{12}^2}{b_1^2u^2} + \frac{c_1^2\alpha_{21}^2v^2}{(1+v)^4} - \frac{4a_1b_1b_2d^2}{b_1u^2(1+v)} \\ & \quad + 4\gamma\left\{\frac{a_1b_2}{b_1u^{\frac{3}{2}}}(d + \alpha_{12}v) + \frac{c_1}{(1+v)^2}(d + \alpha_{21}u)\right\} \\ & \leq \frac{1}{b_1^2u^2(1+v)^2}\left[a_1^2b_1^2\alpha_{12}^2(1+v)^2 + \frac{b_1^2c_1^2\alpha_{21}^2u^2v^2}{(1+v)^2} - 4a_1b_1b_2c_1d^2\right. \\ & \quad \left.+ 4\gamma\{a_1b_1b_2(1+v)^2(d + \alpha_{12}v) + b_1^2c_1u^2(d + \alpha_{21}u)\}\right] \\ & \leq \frac{1}{c_2^2(1+u)^2v^2}\left[a_1^2b_1^2\alpha_{12}^2(1+M)^2 + b_1^2c_1^2\alpha_{21}^2M^4 - 4a_2b_2c_1c_2d^2\right. \\ & \quad \left.+ 4\gamma\{a_1b_1b_2(1+M)^2(d + \alpha_{12}M) + b_1^2c_1M^2(d + \alpha_{21}M)\}\right] \\ & < 0. \end{aligned}$$

From (4.4) and (4.5) we have for $t \geq 0$ that $\frac{dE^u(u(t),v(t))}{dt} \leq 0$ and $\frac{dE^u(u(t),v(t))}{dt} = 0$ only if $u(x, t) \equiv \frac{a_1}{b_1}$ and $v(x, t) \equiv 0$.

Thus it is shown that $E^u(u(t), v(t)) \searrow 0$ as $t \rightarrow \infty$. And we obtain the L_2 convergences, $|u(t) - \frac{a_1}{b_1}|_2 \rightarrow 0, |v(t)|_2 \rightarrow 0$ as $t \rightarrow \infty$ by using the uniform boundedness of $(u(x, t), v(x, t))$ in $[0, 1]$. From Theorem 1, $\sup_{0 \leq t < \infty} |u_{xx}(t)|_2 < \infty$, and $\sup_{0 \leq t < \infty} |v_{xx}(t)|_2 < \infty$. Applying the calculus inequality (5.6) in Section 5 to the functions $u(x, t) - \frac{a_1}{b_1}$ and $v(x, t)$, we obtain the convergence $(u(x, t), v(x, t)) \rightarrow (\frac{a_1}{b_1}, 0)$ as $t \rightarrow \infty$ in $W_2^1([0, 1])$. By using the Sobolev embedding theorem we show that $(u(x, t), v(x, t))$ converges to $(\frac{a_1}{b_1}, 0)$ uniformly in $[0, 1]$ as $t \rightarrow \infty$. We also obtain that

$(\frac{a_1}{b_1}, 0)$ is locally asymptotically stable in $C([0, 1])$ by using the fact that $E^u(u(t), v(t))$ is decreasing for $t \geq 0$. Thus we conclude that $(\frac{a_1}{b_1}, 0)$ is globally asymptotically stable. \square

THEOREM 9. For the system (B) in Case(B) suppose that $\frac{a_2}{c_2} < -\frac{a_1 b_2}{b_1 c_2}$, $0 < b_2 < c_1 + 2 \min\{b_1, c_2\}$, and $0 < c_1 < -4(\frac{a_2 b_1}{b_2} + a_1)$. Let u_0, v_0 be in $W_2^2([0, 1])$. If $d_1, d_2 \geq 1$ satisfy that

$$(4.6) \quad a_1^2 b_1^2 \alpha_{12}^2 (1 + M)^2 + b_1^2 c_1^2 \alpha_{21}^2 M^4 < 4a_1 b_1 b_2 c_1 d_1 d_2,$$

where M is the positive constant in Theorem 3, then the solution $(u(x, t), v(x, t))$ converges to $(\frac{a_1}{b_1}, 0)$ uniformly in $[0, 1]$ as $t \rightarrow \infty$, and $(\frac{a_1}{b_1}, 0)$ is globally asymptotically stable.

PROOF. Using the functional $E^u(u, v)$ defined as in (4.2) in the proof of Theorem 8 we observe the convergence of global solutions of the cross-diffusion prey-predator system (B). We first estimate the time derivative of $E^u(u(t), v(t))$ for the solution of the system (B).

$$(4.7) \quad \begin{aligned} & \frac{dE^u(u(t), v(t))}{dt} \\ &= \int_0^1 \{b_2(1 - \frac{a_1/b_1}{u})u_t + c_1(\frac{v}{1+v})v_t\} dx \\ &= \int_0^1 \{b_2(1 - \frac{a_1/b_1}{u})(d_1 u + \alpha_{11}u^2 + \alpha_{12}uv)_{xx} \\ & \quad + c_1(\frac{v}{1+v})(d_2 v + \alpha_{21}uv + \alpha_{22}v^2)_{xx}\} dx \\ & \quad + \int_0^1 \{b_2(u - \frac{a_1}{b_1})f + c_1(\frac{v^2}{1+v})g\} dx \\ &= - \int_0^1 \{ \frac{a_1 b_2}{b_1 u^2} (d_1 + 2\alpha_{11}u + \alpha_{12}v)u_x^2 \\ & \quad + (\frac{a_1 b_2 \alpha_{12}}{b_1 u} + \frac{c_1 \alpha_{21} v}{(1+v)^2})u_x v_x + \frac{c_1}{(1+v)^2} (d_2 + \alpha_{21}u + 2\alpha_{22}v)v_x^2 \} dx \\ & \quad + \int_0^1 \frac{1}{1+v} \{b_2(u - \frac{a_1}{b_1})(1+v)f + c_1 v^2 g\} dx, \end{aligned}$$

where $f = a_1 - b_1 u - c_1 v$ and $g = a_2 + b_2 u - c_2 v$.

We have the same estimates for the terms of the integral in the last line of (4.7) as shown in the proof of Theorem 8 :

$$(4.8) \quad \begin{aligned} & b_2(u - \frac{a_1}{b_1})(1+v)f + c_1 v^2 g \\ & \leq 0, \text{ and the equality holds only if } (u, v) \equiv (\frac{a_1}{b_1}, 0). \end{aligned}$$

Now we estimate the terms with spatial derivatives in (4.7). For the solution of the system (B) we take the uniform bound M satisfying (2.7) in the proof of Theorem 5. From the condition (4.6) in the hypothesis of the present theorem (Theorem 9) for every constant γ such that $0 <$

$\gamma < \frac{4a_1 b_1 b_2 c_1 d_1 d_2 - a_1^2 b_1^2 \alpha_{12}^2 (1+M)^2 - b_1^2 c_1^2 \alpha_{21}^2 M^4}{4(a_1 b_1 b_2 (1+M)^2 (d_1 + (2\alpha_{11} + \alpha_{12})M) + b_1^2 c_1 M^2 (d_2 + (\alpha_{21} + 2\alpha_{22})M))}$ we have the

following inequality :

$$(4.9) \quad \begin{aligned} & \frac{a_1 b_2}{b_1 u^2} (d_1 + 2\alpha_{11}u + \alpha_{12}v) u_x^2 \\ & + \left(\frac{a_1 b_2 \alpha_{12}}{b_1 u} + \frac{c_1 \alpha_{21} v}{(1+v)^2} \right) u_x v_x + \frac{c_1}{(1+v)^2} (d_2 + \alpha_{21}u + 2\alpha_{22}v) v_x^2 \\ & \geq \gamma \{u_x^2 + v_x^2\}, \end{aligned}$$

since

$$\begin{aligned} & \left(\frac{a_1 b_2 \alpha_{12}}{b_1 u} + \frac{c_1 \alpha_{21} v}{(1+v)^2} \right)^2 \\ & - 4 \left\{ \frac{a_1 b_2}{b_1 u^2} (d_1 + 2\alpha_{11}u + \alpha_{12}v) - \gamma \right\} \left\{ \frac{c_1}{(1+v)^2} (d_2 + \alpha_{21}u + 2\alpha_{22}v) - \gamma \right\} \\ \leq & \frac{a_1^2 b_2^2 \alpha_{12}^2}{b_1^2 u^2} + \frac{c_1^2 \alpha_{21}^2 v^2}{(1+v)^4} - \frac{4a_1 b_1 b_2 d_1 d_2}{b_1 u^2 (1+v)} \\ & + 4\gamma \left\{ \frac{a_1 b_2}{b_1 u^2} (d_1 + 2\alpha_{11}u + \alpha_{12}v) + \frac{c_1}{(1+v)^2} (d_2 + \alpha_{21}u + 2\alpha_{22}v) \right\} \\ \leq & \frac{1}{b_1^2 u^2 (1+v)^2} [a_1^2 b_1^2 \alpha_{12}^2 (1+v)^2 + \frac{b_1^2 c_1^2 \alpha_{21}^2 u^2 v^2}{(1+v)^2} - 4a_1 b_1 b_2 c_1 d_1 d_2 \\ & + 4\gamma \{a_1 b_1 b_2 (1+v)^2 (d_1 + 2\alpha_{11}u + \alpha_{12}v) \\ & + b_1^2 c_1 u^2 (d_2 + \alpha_{21}u + 2\alpha_{22}v)\}] \\ \leq & \frac{1}{c_2^2 (1+u)^2 v^2} [a_1^2 b_1^2 \alpha_{12}^2 (1+M)^2 + b_1^2 c_1^2 \alpha_{21}^2 M^4 - 4a_2 b_2 c_1 c_2 d_1 d_2 \\ & + 4\gamma \{a_1 b_1 b_2 (1+M)^2 (d_1 + (2\alpha_{11} + \alpha_{12})M) \\ & + b_1^2 c_1 M^2 (d_2 + (\alpha_{21} + 2\alpha_{22})M)\}] \\ < & 0. \end{aligned}$$

From (4.8) and (4.9) we have for $t \geq 0$ that $\frac{dE^u(u(t),v(t))}{dt} \leq 0$ and $\frac{dE^u(u(t),v(t))}{dt} = 0$ only if $u(x, t) \equiv \frac{a_1}{b_1}$ and $v(x, t) \equiv 0$.

By using the same arguments as the proof of Theorem 8 we show that $(u(x, t), v(x, t))$ converges to $(\frac{a_1}{b_1}, 0)$ uniformly in $[0, 1]$ as $t \rightarrow \infty$. We also obtain that $(\frac{a_1}{b_1}, 0)$ is locally asymptotically stable in $C([0, 1])$ from the fact that $E^u(u(t), v(t))$ is decreasing for $t \geq 0$. Thus we conclude that $(\frac{a_1}{b_1}, 0)$ is globally asymptotically stable. □

5. Calculus inequalities

THEOREM 10. *Let $\Omega \in \mathbb{R}^n$ be a bounded domain with $\partial\Omega$ in C^m . For every function u in $W^{m,r}(\Omega)$, $1 \leq q, r \leq \infty$, the derivative $D^j u$, $0 \leq j < m$, satisfies the inequality*

$$(5.1) \quad |D^j u|_p \leq C(|D^m u|_r^a |u|_q^{1-a} + |u|_q),$$

where $\frac{1}{p} = \frac{j}{n} + a \left(\frac{1}{r} - \frac{m}{n} \right) + (1-a) \frac{1}{q}$, for all a in the interval $\frac{j}{m} \leq a < 1$, provided one of the following three conditions :

- (i) $r \leq q$,

(ii) $0 < \frac{n(r-q)}{mrq} < 1$, or

(iii) $\frac{n(r-q)}{mrq} = 1$ and $m - \frac{n}{q}$ is not a nonnegative integer.

(The positive constant C depends only on n, m, j, q, r, a .)

PROOF. We refer the reader to A. Friedman [7] or L. Nirenberg [14] for the proof of this well-known calculus inequality. \square

COROLLARY 11. There exist positive constants C, \tilde{C} and \hat{C} such that for every function u in $W_2^1([0, 1])$

$$(5.2) \quad |u|_2 \leq C(|u_x|_2^{\frac{1}{3}}|u|_1^{\frac{2}{3}} + |u|_1),$$

$$(5.3) \quad |u|_{\frac{5}{2}} \leq \tilde{C}(|u_x|_2^{\frac{2}{5}}|u|_1^{\frac{3}{5}} + |u|_1),$$

$$(5.4) \quad |u|_3 \leq \hat{C}(|u_x|_2^{\frac{4}{9}}|u|_1^{\frac{5}{9}} + |u|_1).$$

PROOF. $m = 1, r = 2, q = 1$ satisfy the condition (ii) in Theorem 10. \square

COROLLARY 12. For every function u in $W_2^2([0, 1])$

$$(5.5) \quad |u_x|_2 \leq C(|u_{xx}|_2^{\frac{3}{5}}|u|_1^{\frac{2}{5}} + |u|_1).$$

PROOF. $m = 2, r = 2, q = 1$ satisfy the condition (ii) in Theorem 10. \square

LEMMA 13. For every function u in $W_2^2([0, 1])$ with $u_x(0) = u_x(1) = 0$

$$(5.6) \quad |u_x|_2 \leq |u_{xx}|_2^{\frac{1}{2}}|u|_2^{\frac{1}{2}}.$$

LEMMA 14. If a function f is in the space $W_2^1([0, 1])$ then there exists a constant $C > 0$ such that

$$(5.7) \quad |f^2|_\infty \leq C\left(1 + \frac{1}{\epsilon}\right)|f|_2^2 + \epsilon|f_x|_2^2,$$

for every $0 < \epsilon < 1$.

PROOF. Suppose first $f \in C^1[0, 1]$. By Lemma 5.2 of [7] there exists a function F in $C_0^1(\mathbb{R}^1)$ such that $F = f$ in the interval $[0, 1]$ and $\|F\|_{W_2^j(\mathbb{R}^1)} \leq C\|f\|_{j,2}$, $j = 0, 1$. For the function F we have the inequalities

$$\begin{aligned} |F^2|_{L_\infty(\mathbb{R}^1)} &\leq \int_{\mathbb{R}^1} |(F^2)_x| dx = 2 \int_{\mathbb{R}^1} |FF_x| dx \\ &\leq \int_{\mathbb{R}^1} (\epsilon|F_x|^2 + \frac{1}{\epsilon}|F|^2) dx \\ &= \epsilon|F_x|_{L_2(\mathbb{R}^1)}^2 + \frac{1}{\epsilon}|F|_{L_2(\mathbb{R}^1)}^2. \end{aligned}$$

Thus now for f we have

$$(5.8) \quad |f^2|_\infty \leq |F^2|_{L^\infty(\mathbb{R}^1)} \leq \epsilon |F_x|_{L_2(\mathbb{R}^1)}^2 + \frac{1}{\epsilon} |F|_{L_2(\mathbb{R}^1)}^2 \leq \epsilon C \|f\|_{1,2}^2 + \frac{C}{\epsilon} |f|_2^2$$

for every $\epsilon > 0$.

Suppose now that $f \in W_2^1([0, 1])$. There exists a sequence $\{f_i\}$ in $C^1[0, 1]$ such that $\|f_i - f\|_{1,2} \rightarrow 0$, $\|f_i - f\|_{0,2} \rightarrow 0$, $|f_i - f|_\infty \rightarrow 0$ as $i \rightarrow \infty$. Hence by passing limits in the inequality (5.8) for f_i we obtain the inequality (5.8) for $f \in W_2^1([0, 1])$ and thus the inequality (5.7) for every $0 < \epsilon < 1$. □

6. Uniform boundedness for the system (A)

PROOF OF Theorem 2.

Step 1. Taking integration of the first equation in the system (A) over the domain $[0, 1]$ we have

$$\begin{aligned} \frac{d}{dt} \int_0^1 u(t) \, dx &= \int_0^1 (a_1 u - b_1 u^2 - c_1 uv) \, dx \\ &\leq a_1 \int_0^1 u \, dx - b_1 \int_0^1 u^2 \, dx \\ &\leq a_1 \int_0^1 u \, dx - b_1 \left(\int_0^1 u \, dx\right)^2 \\ &= b_1 \left(\frac{a_1}{b_1} - \int_0^1 u \, dx\right) \int_0^1 u \, dx. \end{aligned}$$

In the case that $\int_0^1 u_0 \, dx < \frac{a_1}{b_1}$ we have that $\int_0^1 u(x, t) \, dx \leq \frac{a_1}{b_1}$ for all $t > 0$. In the case that $\int_0^1 u_0 \, dx \geq \frac{a_1}{b_1}$ there exist positive constants δ and τ_0 such that $\int_0^1 u(x, t) \, dx < \delta + \frac{a_1}{b_1}$ for all $t \in (\tau_0, \infty)$. Hence we obtain the L_1 -bound of u for all time.

Now, taking integration of the second equation in the system (A) over the domain $[0, 1]$ we have

$$\frac{d}{dt} \int_0^1 v(t) \, dx = \int_0^1 (a_2 v + b_2 uv - c_2 v^2) \, dx,$$

where a_2 can be any real number, positive, zero, and negative as well in prey-predator type reactions. Here, let us observe the addition of the equations of integrations of u and v together as :

$$\frac{d}{dt} \int_0^1 (u + v) \, dx = \int_0^1 \{(a_1 u + a_2 v) + (b_2 - c_1)uv - (b_1 u^2 + c_2 v^2)\} \, dx.$$

By using the condition $0 < b_2 < c_1 + 2 \min\{b_1, c_2\}$, given in the assumption of the present theorem, we find a constant $\delta = \delta(b_1, b_2, c_1, c_2)$ such that

$$b_1 u^2 + c_2 v^2 - (b_2 - c_1)uv \geq \delta(u^2 + v^2).$$

Thus we have that

$$\begin{aligned} \frac{d}{dt} \int_0^1 (u+v) dx &\leq \int_0^1 (a_1 u + a_2 v) dx - \delta \int_0^1 (u^2 + v^2) dx \\ &\leq \max\{a_1, a_2\} \int_0^1 (u+v) dx - \delta \int_0^1 (u+v)^2 dx \\ &\leq \max\{a_1, a_2\} \int_0^1 (u+v) dx - \delta \left(\int_0^1 (u+v) dx\right)^2. \end{aligned}$$

This gives that $\int_0^1 (u+v) dx$ is bounded for all time.

Hence we conclude that there exist positive constants τ_0 and $M_0 = M_0(a_i, b_i, c_i, i = 1, 2)$ such that

$$\int_0^1 u(t) dx < M_0, \quad \int_0^1 v(t) dx < M_0 \quad \text{for all } t \in (\tau_0, \infty).$$

Now, for Step 2 and 3 we reduce the system (A) into the following system by using the scaling $u(x, \frac{\tau}{d}) = \frac{d}{\alpha_{21}} \tilde{u}(x, \tau)$, $v(x, \frac{\tau}{d}) = \frac{d}{\alpha_{12}} \tilde{v}(x, \tau)$, $t = \frac{\tau}{d}$ and then use u, v and t instead of \tilde{u}, \tilde{v} and τ , respectively :

$$(Ar) \quad \begin{cases} u_t = (u + uv)_{xx} + u\left(\frac{a_1}{d} - \frac{b_1}{\alpha_{21}}u - \frac{c_1}{\alpha_{12}}v\right) & \text{in } [0, 1] \times (0, \infty), \\ v_t = (v + uv)_{xx} + v\left(\frac{a_2}{d} + \frac{b_2}{\alpha_{21}}u - \frac{c_2}{\alpha_{12}}v\right) & \text{in } [0, 1] \times (0, \infty), \\ u_x(x, t) = v_x(x, t) = 0 & \text{at } x = 0, 1, \\ u(x, 0) = \tilde{u}_0(x), \quad v(x, 0) = \tilde{v}_0(x) & \text{in } [0, 1]. \end{cases}$$

Then the result in Step 1 is restated as follows :

There exist positive constants τ_0 and $M_0 = M_0(\alpha_{12}, \alpha_{21}, a_i, b_i, c_i, i = 1, 2)$ such that

$$\int_0^1 du(t) dx < M_0, \quad \int_0^1 dv(t) dx < M_0 \quad \text{for all } t \in (\tau_0, \infty).$$

Step 2. We use $\zeta = v - u$ as an auxiliary function to obtain necessary estimates and then the system (Ar) is rewritten as

$$(6.1) \quad u_t = (u + u^2 + u\zeta)_{xx} + u\tilde{f},$$

$$(6.2) \quad v_t = (v + v^2 - v\zeta)_{xx} + v\tilde{g},$$

$$(6.3) \quad \zeta_t = \zeta_{xx} + G,$$

where $\tilde{f} = \frac{a_1}{d} - \frac{b_1}{\alpha_{21}}u - \frac{c_1}{\alpha_{12}}v$, $\tilde{g} = \frac{a_2}{d} + \frac{b_2}{\alpha_{21}}u - \frac{c_2}{\alpha_{12}}v$, and $G = v\tilde{g} - u\tilde{f}$.

Multiplying (6.1), (6.2), (6.3) by u , v , $-\zeta_{xx}$, respectively and integrating over $[0, 1]$ we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_0^1 u^2 dx &= \int_0^1 u(uu^2 + u\zeta)_{xx} dx + \int_0^1 u^2 \tilde{f} dx \\ &= -\int_0^1 u_x(u_x + 2uu_x + u_x\zeta + u\zeta_x) dx + \int_0^1 u^2 \tilde{f} dx \\ &= -\int_0^1 (u_x^2 + 2uu_x^2 + u_x^2\zeta) dx - \int_0^1 uu_x\zeta_x dx + \int_0^1 u^2 \tilde{f} dx \\ &= -\int_0^1 (u_x^2 + uu_x^2 + vv_x^2) dx + \frac{1}{2} \int_0^1 u^2 \zeta_{xx} dx + \int_0^1 u^2 \tilde{f} dx \\ &\leq -\int_0^1 (1+u)u_x^2 dx + \frac{1}{2} \int_0^1 u^2 \zeta_{xx} dx + \int_0^1 \frac{a_1}{d} u^2 dx \\ &\quad - \int_0^1 \frac{c_1}{\alpha_{12}} u^2 v dx, \end{aligned}$$

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_0^1 v^2 dx &= \int_0^1 v(v + v^2 - v\zeta)_{xx} dx + \int_0^1 v^2 \tilde{g} dx \\ &= -\int_0^1 v_x(v_x + 2vv_x - v_x\zeta - v\zeta_x) dx + \int_0^1 v^2 \tilde{g} dx \\ &= -\int_0^1 (v_x^2 + 2vv_x^2 - v_x^2\zeta) dx + \int_0^1 vv_x\zeta_x dx \\ &\quad + \int_0^1 v^2 \tilde{g} dx \\ &= -\int_0^1 (v_x^2 + vv_x^2 + uv_x^2) dx - \frac{1}{2} \int_0^1 v^2 \zeta_{xx} dx \\ &\quad + \int_0^1 v^2 \tilde{g} dx \\ &\leq -\int_0^1 (1+v)v_x^2 dx - \frac{1}{2} \int_0^1 v^2 \zeta_{xx} dx + \int_0^1 \frac{a_2}{d} v^2 dx \\ &\quad + \int_0^1 \frac{b_2}{\alpha_{21}} uv^2 dx, \end{aligned}$$

$$\frac{1}{2} \frac{d}{dt} \int_0^1 \zeta_x^2 dx = -\int_0^1 (\zeta_{xx})^2 dx - \int_0^1 \zeta_{xx} G dx$$

from which it follows that

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int_0^1 (u^2 + v^2 + \zeta_x^2) dx \\ (6.4) \quad &\leq -\int_0^1 (1+u)u_x^2 dx - \int_0^1 (1+v)v_x^2 dx \\ &\quad - \int_0^1 (\zeta_{xx})^2 dx + \frac{1}{2} \int_0^1 \zeta_{xx}(u^2 - v^2 - 2G) dx \\ &\quad + \frac{C_{1,1}}{d} \int_0^1 (u^2 + v^2) dx - \frac{c_1}{\alpha_{12}} \int_0^1 u^2 v dx \\ &\quad + \frac{b_2}{\alpha_{21}} \int_0^1 uv^2 dx, \end{aligned}$$

where $C_{1,1} = \max\{a_1, a_2\}$. Using Young's inequality we notice that

$$\begin{aligned} \frac{b_2}{\alpha_{21}} \int_0^1 uv^2 dx &\leq \frac{\epsilon b_2}{2\alpha_{21}} \int_0^1 u^2 v dx + \frac{b_2}{2\epsilon\alpha_{21}} \int_0^1 v^3 dx \\ &= \frac{c_1}{\alpha_{12}} \int_0^1 u^2 v dx + \frac{b_2^2}{4c_1\alpha_{21}} \int_0^1 v^3 dx, \end{aligned}$$

where $\epsilon = \frac{2c_1\alpha_{21}}{b_2\alpha_{12}}$. Hence we can reduce (6.4) to :

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int_0^1 (u^2 + v^2 + \zeta_x^2) dx \\ (6.5) \quad &\leq -\int_0^1 (1+u)u_x^2 dx - \int_0^1 (1+v)v_x^2 dx \\ &\quad - \int_0^1 (\zeta_{xx})^2 dx + \frac{1}{2} \int_0^1 \zeta_{xx}(u^2 - v^2 - 2G) dx \\ &\quad + \frac{C_{1,1}}{d} \int_0^1 (u^2 + v^2) dx + \frac{b_2^2}{4c_1\alpha_{21}} \int_0^1 v^3 dx, \end{aligned}$$

Applying the inequality (5.4) to the functions v and using the uniform boundedness of $|v|_1$ from *Step 1* we have

$$|v|_3 \leq C(|v_x|_2^{\frac{4}{9}}|v|_1^{\frac{5}{9}} + |v|_1) \leq C_1 d^{-\frac{5}{9}}(|v_x|_2^{\frac{4}{9}} + d^{-\frac{4}{9}}),$$

and thus

$$-\int_0^1 v_x^2 dx \leq 2d^{-2} - C_2 d^{5/2} \left(\int_0^1 v^3 dx \right)^{3/2},$$

where C_1 and C_2 are positive constants depending only on α_{12} , α_{21} , a_i , b_i , c_i , $i = 1, 2$. We observe that the term with $\int_0^1 v^3 dx$ in the last line of (6.5) has strictly lower order than the term with $\int_0^1 v_x^2 dx$. Therefore for the rest estimates in *Step 2* we can follow that part in [22] to obtain the L_2 -bound of u and v for all time :

There exist positive constants τ_1 and $M_1 = M_1(d, \alpha_{12}, \alpha_{21}, a_i, b_i, c_i, i = 1, 2)$ such that

$$(6.6) \quad \int_0^1 (d u(t))^2 dx < M_1, \quad \int_0^1 (d v(t))^2 dx < M_1 \quad \text{for all } t \in (\tau_1, \infty).$$

For $d \geq 1$ we have

$$(6.7) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \int_0^1 d^2(u^2 + v^2 + \zeta_x^2) dx &\leq C_{1,12} + C_{1,1} \int_0^1 d^2(u^2 + v^2 + \zeta_x^2) dx \\ &\quad - C_{1,11} \left(\int_0^1 d^2(u^2 + v^2 + \zeta_x^2) dx \right)^{\frac{5}{3}}, \end{aligned}$$

where $C_{1,12}$ is a positive constant depending only on α_{12} , α_{21} , a_i , b_i , c_i , $i = 1, 2$. Thus for $d \geq 1$ the positive constant M_1 in (6.6) is independent of $d \geq 1$, that is, $M_1 = M_1(\alpha_{12}, \alpha_{21}, a_i, b_i, c_i, i = 1, 2)$.

Step 3. Multiplying (6.1), (6.2) by $-u_{xx}$, $-v_{xx}$, respectively and integrating over $[0, 1]$ we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_0^1 u_x^2 dx &= -\int_0^1 u_{xx}(u + u^2 + u\zeta)_{xx} dx - \int_0^1 u_{xx} u \tilde{f} dx \\ &= -\int_0^1 u_{xx}(u_{xx} 2u_x^2 + 2uu_{xx} + \zeta u_{xx} + 2u_x \zeta_x + u \zeta_{xx}) dx \\ &\quad - \frac{a_1}{d} \int_0^1 uu_{xx} dx - \frac{b_1}{\alpha_{21}} \int_0^1 u^2 u_{xx} dx - \frac{c_1}{\alpha_{12}} \int_0^1 uvu_{xx} dx \\ &= -\int_0^1 (u_{xx})^2 dx - \int_0^1 (u + v)(u_{xx})^2 dx \\ &\quad - \int_0^1 (u \zeta_{xx} + 2u_x \zeta_x) u_{xx} dx - 2 \int_0^1 u_x^2 u_{xx} dx \\ &\quad + \frac{a_1}{d} \int_0^1 u_x^2 dx + \frac{b_1}{\alpha_{21}} \int_0^1 u^2 u_{xx} dx + \frac{c_1}{\alpha_{12}} \int_0^1 uvu_{xx} dx, \end{aligned}$$

and

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_0^1 v_x^2 dx &= -\int_0^1 v_{xx}(v + v^2 - v\zeta)_{xx} dx - \int_0^1 v_{xx} v \tilde{g} dx \\ &= -\int_0^1 (v_{xx})^2 dx - \int_0^1 (u + v)(v_{xx})^2 dx \\ &\quad + \int_0^1 (v \zeta_{xx} + 2v_x \zeta_x) v_{xx} dx - 2 \int_0^1 v_x^2 v_{xx} dx \\ &\quad + \frac{a_2}{d} \int_0^1 v_x^2 dx - \frac{b_2}{\alpha_{21}} \int_0^1 uvv_{xx} dx + \frac{c_2}{\alpha_{12}} \int_0^1 v^2 v_{xx} dx. \end{aligned}$$

Notice that $\int_0^1 u_x^2 u_{xx} \, dx = \int_0^1 v_x^2 v_{xx} \, dx = 0$ by using the Neumann boundary conditions. Thus we have

$$(6.8) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \int_0^1 u_x^2 \, dx &\leq - \int_0^1 (u_{xx})^2 \, dx - \int_0^1 (u \zeta_{xx} + 2u_x \zeta_x) u_{xx} \, dx \\ &\quad + \frac{b_1}{\alpha_{21}} \int_0^1 u^2 |u_{xx}| \, dx + \frac{c_1}{\alpha_{12}} \int_0^1 uv |u_{xx}| \, dx \\ &\quad + \frac{a_1}{d} \int_0^1 u_x^2 \, dx, \end{aligned}$$

and

$$(6.9) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \int_0^1 v_x^2 \, dx &\leq - \int_0^1 (v_{xx})^2 \, dx + \int_0^1 (v \zeta_{xx} + 2v_x \zeta_x) v_{xx} \, dx \\ &\quad + \frac{c_2}{\alpha_{12}} \int_0^1 v^2 |v_{xx}| \, dx + \frac{b_2}{\alpha_{21}} \int_0^1 uv |v_{xx}| \, dx \\ &\quad + \frac{a_2}{d} \int_0^1 v_x^2 \, dx. \end{aligned}$$

Using (6.8) and (6.9) we can follow [22] for the rest estimates in Step 3 to obtain the L_2 -bound of u_x and v_x for all time :

There exist positive constants τ_2 and $M_2 = M_2(d, \alpha_{12}, \alpha_{21}, a_i, b_i, c_i, i = 1, 2)$ such that

$$(6.10) \quad \int_0^1 (d u_x(t))^2 \, dx < M_2, \quad \int_0^1 (d v_x(t))^2 \, dx < M_2 \quad \text{for all } t \in (\tau_2, \infty).$$

For $d \geq 1$ we have

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int_0^1 d^2 (u_x^2 + v_x^2 + (\zeta_{xx})^2) \, dx \\ &\leq C_{2,11} + C_{2,12} \int_0^1 d^2 (u_x^2 + v_x^2 + (\zeta_{xx})^2) \, dx \\ &\quad - C_{2,13} \left\{ \int_0^1 d^2 (u_x^2 + v_x^2 + (\zeta_{xx})^2) \, dx \right\}^{\frac{3}{2}}, \end{aligned}$$

where $C_{2,11}, C_{2,12}, C_{2,13}$ are positive constants depending only on $\alpha_{12}, \alpha_{21}, a_i, b_i, c_i, i = 1, 2$. Thus for $d \geq 1$ the positive constant M_2 in (6.10) is independent of $d \geq 1$, that is, $M_2 = M_2(\alpha_{12}, \alpha_{21}, a_i, b_i, c_i, i = 1, 2)$.

From the results of Step 1, Step 2 and Step 3 we have positive constants \tilde{t}_0 and $\tilde{M} = \tilde{M}(d, \alpha_{12}, \alpha_{21}, a_i, b_i, c_i, i = 1, 2)$ such that

$$(6.11) \quad \max\{\|d u(\cdot, t)\|_{1,2}, \|d v(\cdot, t)\|_{1,2} : t \in (\tilde{t}_0, \tilde{T})\} \leq \tilde{M}$$

for the maximal solution (u, v) of the system (Ar). By scaling back and using the Sobolev embedding inequalities we obtain the desired estimate for the system (A) as the following :

We have positive constants $t_0, M' = M'(d, \alpha_{12}, \alpha_{21}, a_i, b_i, c_i, i = 1, 2)$, and $M = M(d, \alpha_{12}, \alpha_{21}, a_i, b_i, c_i, i = 1, 2)$ such that

$$(6.12) \quad \begin{aligned} &\max\{\|u(\cdot, t)\|_{1,2}, \|v(\cdot, t)\|_{1,2} : t \in (t_0, T)\} \leq M', \\ &\max\{u(x, t), v(x, t) : (x, t) \in [0, 1] \times (t_0, T)\} \leq M \end{aligned}$$

for the maximal solution (u, v) of the system (A). We also conclude that $T = +\infty$ from Theorem 1. For $d \geq 1$ the positive constants M' and M in

(6.12) are independent of $d \geq 1$, that is, $M' = M'(\alpha_{12}, \alpha_{21}, a_i, b_i, c_i, i = 1, 2)$, $M = M(\alpha_{12}, \alpha_{21}, a_i, b_i, c_i, i = 1, 2)$. □

7. Uniform boundedness for the system (B)

PROOF OF Theorem 3.

Step 1. By using the condition $0 < b_2 < c_1 + 2 \min\{b_1, c_2\}$, given in the assumption of the present theorem, we obtain the same L_1 -bounds for u and v as in Step 1 of the proof of Theorem 2 for the system (A) in Section 6. Thus we have that :

There exist positive constants τ_0 and $M_0 = M_0(a_i, b_i, c_i, i = 1, 2)$ such that

$$\int_0^1 u(t) \, dx < M_0, \quad \int_0^1 v(t) \, dx < M_0 \quad \text{for all } t \in (\tau_0, \infty).$$

Step 2. In this step we use the scaling $\tilde{u}(x, t) = \alpha_{21}u(x, t)$, $\tilde{v}(x, t) = \alpha_{12}v(x, t)$, and then use u, v instead of \tilde{u}, \tilde{v} , respectively to reduce the system (B) to

$$(Br1) \quad \begin{cases} u_t = (d_1u + \frac{\alpha_{11}}{\alpha_{21}}u^2 + uv)_{xx} + u(a_1 - \frac{b_1}{\alpha_{21}}u - \frac{c_1}{\alpha_{12}}v) \\ \quad \text{in } [0, 1] \times (0, \infty), \\ v_t = (d_2v + uv + \frac{\alpha_{22}}{\alpha_{12}}v^2)_{xx} + v(a_2 + \frac{b_2}{\alpha_{21}}u - \frac{c_2}{\alpha_{12}}v) \\ \quad \text{in } [0, 1] \times (0, \infty), \\ u_x(x, t) = v_x(x, t) = 0 \quad \text{at } x = 0, 1, \\ u(x, 0) = \tilde{u}_0(x), \quad v(x, 0) = \tilde{v}_0(x) \quad \text{in } [0, 1]. \end{cases}$$

In this step we will observe the solution of (Br1) to prove the uniform $L_2[0, 1]$ -boundedness of its solution. Multiplying the first equation in (Br1) by u and integrating it over the domain $[0, 1]$, we have

$$(7.1) \quad \frac{1}{2} \frac{d}{dt} \int_0^1 u(t)^2 \, dx \leq -d_1 \int_0^1 u_x^2 \, dx - \int_0^1 ((2\frac{\alpha_{11}}{\alpha_{21}}u + v)u_x^2 + uu_xv_x) \, dx + a_1 \int_0^1 u^2 \, dx - \frac{c_1}{\alpha_{12}} \int_0^1 u^2 v \, dx.$$

From the second equation in (Br1) we obtain similar inequalities for v :

$$(7.2) \quad \frac{1}{2} \frac{d}{dt} \int_0^1 v(t)^2 \, dx \leq -d_2 \int_0^1 v_x^2 \, dx - \int_0^1 ((u + 2\frac{\alpha_{22}}{\alpha_{12}}v)v_x^2 + vv_xv_x) \, dx + a_2 \int_0^1 v^2 \, dx + \frac{b_2}{\alpha_{21}} \int_0^1 uv^2 \, dx.$$

Adding up the inequalities (7.1) and (7.2) we have

$$\begin{aligned}
 (7.3) \quad & \frac{1}{2} \frac{d}{dt} \int_0^1 (u(t)^2 + v(t)^2) dx \\
 & \leq -d_1 \int_0^1 u_x^2 dx - d_2 \int_0^1 v_x^2 dx + a_1 \int_0^1 u^2 dx \\
 & \quad + a_2 \int_0^1 v^2 dx - \int_0^1 ((2\frac{\alpha_{11}}{\alpha_{21}}u + v)u_x^2 + (u + v)u_x v_x \\
 & \quad + (u + 2\frac{\alpha_{22}}{\alpha_{12}}v)v_x^2) dx - \frac{c_1}{\alpha_{12}} \int_0^1 u^2 v dx \\
 & \quad + \frac{b_2}{\alpha_{21}} \int_0^1 uv^2 dx.
 \end{aligned}$$

Using Young’s inequality we notice that

$$\begin{aligned}
 \frac{b_2}{\alpha_{21}} \int_0^1 uv^2 dx & \leq \frac{\epsilon b_2}{2\alpha_{21}} \int_0^1 u^2 v dx + \frac{b_2}{2\epsilon\alpha_{21}} \int_0^1 v^3 dx \\
 & = \frac{c_1}{\alpha_{12}} \int_0^1 u^2 v dx + \frac{b_2^2}{4c_1\alpha_{21}} \int_0^1 v^3 dx,
 \end{aligned}$$

where $\epsilon = \frac{2c_1\alpha_{21}}{b_2\alpha_{12}}$. Thus we can reduce (7.3) to :

$$\begin{aligned}
 (7.4) \quad & \frac{1}{2} \frac{d}{dt} \int_0^1 (u(t)^2 + v(t)^2) dx \\
 & \leq -d_1 \int_0^1 u_x^2 dx - d_2 \int_0^1 v_x^2 dx + a_1 \int_0^1 u^2 dx + a_2 \int_0^1 v^2 dx \\
 & \quad - \int_0^1 ((2\frac{\alpha_{11}}{\alpha_{21}}u + v)u_x^2 + (u + v)u_x v_x + (u + 2\frac{\alpha_{22}}{\alpha_{12}}v)v_x^2) dx \\
 & \quad + \frac{b_2^2}{4c_1\alpha_{21}} \int_0^1 v^3 dx.
 \end{aligned}$$

We observe that the term with $\int_0^1 v^3 dx$ in the last line of (7.4) has strictly lower order than the term with $\int_0^1 v_x^2 dx$ by the same derivation in Step 2 of the proof of Theorem 2 for the system (A) in Section 6. Hence for the rest estimates in Step 2 we can follow that part in [23] to obtain the L_2 -bound of u and v for all time :

There exist positive constants τ_1 and $M_1 = M_1(d_i, \alpha_{ij}, a_i, b_i, c_i, i, j = 1, 2)$ such that

$$(7.5) \quad \int_0^1 (u(t))^2 dx < M_1, \quad \int_0^1 (v(t))^2 dx < M_1 \quad \text{for all } t \in (\tau_1, \infty)$$

for the solution $(u(t), v(t))$ of the system (B). Especially for $d_1, d_2 \geq 1$ the positive constant M_1 in (7.5) is independent of $d_1, d_2 \geq 1$, that is, $M_1 = M_1(\alpha_{ij}, a_i, b_i, c_i, i, j = 1, 2)$.

Step 3. We use in this section another way of scaling that $\tilde{u}(x, \tau) = \frac{\alpha_{21}}{d_2}u(x, t)$, $\tilde{v}(x, \tau) = \frac{\alpha_{12}}{d_1}v(x, t)$, $\tau = d_1 t$, and then use u, v, t instead of $\tilde{u}, \tilde{v}, \tau$, respectively to reduce the system (B) to

$$(Br2) \quad \begin{cases} u_t = (u + \frac{\alpha_{11}\xi}{\alpha_{21}}u^2 + uv)_{xx} + u(\frac{a_1}{d_1} - \frac{b_1\xi}{\alpha_{21}}u - \frac{c_1}{\alpha_{12}}v) & \text{in } [0, 1] \times (0, \infty), \\ v_t = \xi(v + uv + \frac{\alpha_{22}}{\alpha_{12}\xi}v^2)_{xx} + v(\frac{a_2}{d_1} + \frac{b_2\xi}{\alpha_{21}}u - \frac{c_2}{\alpha_{12}}v) & \text{in } [0, 1] \times (0, \infty), \\ u_x(x, t) = v_x(x, t) = 0 & \text{at } x = 0, 1, \\ u(x, 0) = \tilde{u}_0(x), \quad v(x, 0) = \tilde{v}_0(x) & \text{in } [0, 1], \end{cases}$$

where $\xi = \frac{d_2}{d_1}$. We will observe the solution of the system (Br2) to obtain a uniform bounded of $|u_x|_2$ and $|v_x|_2$. We denote that

$$P = u + \frac{\alpha_{11}\xi}{\alpha_{21}}u^2 + uv, \quad Q = v + uv + \frac{\alpha_{22}}{\alpha_{12}\xi}v^2, \\ \tilde{f} = \frac{a_1}{d_1} - \frac{b_1\xi}{\alpha_{21}}u - \frac{c_1}{\alpha_{12}}v, \quad \tilde{g} = \frac{a_2}{d_1} + \frac{b_2\xi}{\alpha_{21}}u - \frac{c_2}{\alpha_{12}}v.$$

In order to estimate $\int u_x dx$ and $\int v_x dx$ we start with multiplying the first equation in the system (Br2) by P_t and the second equation by Q_t .

$$\int_0^1 (P_u u_t^2 + P_v v_t v_t) dx = \int_0^1 P_t P_{xx} dx + \int_0^1 P_t u \tilde{f} dx \\ = -\frac{1}{2} \frac{d}{dt} \int_0^1 P_x^2 dx + \int_0^1 (P_u u_t + P_v v_t) u \tilde{f} dx, \\ \int_0^1 (Q_u u_t v_t + Q_v v_t^2) dx = \xi \int_0^1 Q_t Q_{xx} dx + \int_0^1 Q_t v \tilde{g} dx \\ = -\frac{\xi}{2} \frac{d}{dt} \int_0^1 Q_x^2 dx + \int_0^1 (Q_u u_t + Q_v v_t) v \tilde{g} dx.$$

Hence

$$(7.6) \quad \frac{1}{2d_1} \frac{d}{dt} \int_0^1 P_x^2 dx = -\frac{1}{d_1} \int_0^1 \left\{ \left(1 + \frac{2\alpha_{11}\xi}{\alpha_{21}}u + v\right) u_t^2 + u u_t v_t \right\} dx \\ + \frac{1}{d_1} \int_0^1 \left(1 + \frac{2\alpha_{11}\xi}{\alpha_{21}}u + v\right) u \tilde{f} u_t dx + \frac{1}{d_1} \int_0^1 u^2 \tilde{f} v_t dx,$$

$$(7.7) \quad \frac{1}{2d_1} \frac{d}{dt} \int_0^1 Q_x^2 dx = -\frac{1}{d_2} \int_0^1 \left\{ v u_t v_t + \left(1 + u + \frac{2\alpha_{22}}{\alpha_{12}\xi}v\right) v_t^2 \right\} dx \\ + \frac{1}{d_2} \int_0^1 v^2 \tilde{g} u_t dx + \frac{1}{d_2} \int_0^1 \left(1 + u + \frac{2\alpha_{22}}{\alpha_{12}\xi}v\right) v \tilde{g} v_t dx.$$

We note from Theorem 1 that $P, Q \in C([0, T], W_2^1([0, 1])) \cap C^\infty([0, 1] \times (0, T))$ for $0 \leq t < T$. Adding up the equations (7.6) and (7.7)

$$(7.8) \quad \frac{1}{2d_1} \frac{d}{dt} \int_0^1 (P_x^2 + Q_x^2) dx \\ = -\frac{1}{d_1} \int_0^1 u_t^2 dx - \frac{1}{d_2} \int_0^1 v_t^2 dx - \frac{1}{d_1} \int_0^1 \left(\frac{2\alpha_{11}\xi}{\alpha_{21}}u + v\right) u_t^2 dx \\ - \int_0^1 \left(\frac{1}{d_1}u + \frac{1}{d_2}v\right) u_t v_t dx - \frac{1}{d_2} \int_0^1 \left(u + \frac{2\alpha_{22}}{\alpha_{12}\xi}v\right) v_t^2 dx \\ + \frac{1}{d_1} \int_0^1 \left(1 + \frac{2\alpha_{11}\xi}{\alpha_{21}}u + v\right) u \tilde{f} u_t dx + \frac{1}{d_1} \int_0^1 u^2 \tilde{f} v_t dx \\ + \frac{1}{d_2} \int_0^1 v^2 \tilde{g} u_t dx + \frac{1}{d_2} \int_0^1 \left(1 + u + \frac{2\alpha_{22}}{\alpha_{12}\xi}v\right) v \tilde{g} v_t dx.$$

In Step 3 of proof of Theorem 1.2 in [23] for competition type reactions, all the terms in the reaction functions \tilde{f} and \tilde{g} are estimated in absolute values. Hence, once we obtained the inequalities in (7.8), the rest of estimates in Step 2 can follow that part of [23] to obtain the L_2 -bound of u_x and v_x for all time :

There exist positive constants τ_2 and $M_2 = M_2(d_i, \alpha_{ij}, a_i, b_i, c_i, i, j = 1, 2)$ such that

$$\int_0^1 u_x^2 dx < M_2, \quad \int_0^1 v_x^2 dx < M_2 \quad \text{for all } t \in (\tau_2, \infty)$$

for the solution $(u(t), v(t))$ of the system (B). And for $d_1, d_2 \geq 1$, $\underline{d} \leq \xi \leq \bar{d}$, the positive constant M_2 is independent of $d_1, d_2 \geq 1$, that is, $M_2 = M_2(\underline{d}, \bar{d}, \alpha_{ij}, a_i, b_i, c_i, i = 1, 2)$.

From the results of Step 1, Step 2 and Step 3 and the Sobolev embedding inequality we have positive constants $t_0, M' = M'(d_i, \alpha_{ij}, a_i, b_i, c_i, i = 1, 2)$, and $M = M(d_i, \alpha_{ij}, a_i, b_i, c_i, i = 1, 2)$ such that

$$(7.9) \quad \begin{aligned} \max\{\|u(\cdot, t)\|_{1,2}, \|v(\cdot, t)\|_{1,2} : t \in (t_0, T)\} &\leq M', \\ \max\{u(x, t), v(x, t) : (x, t) \in [0, 1] \times (t_0, T)\} &\leq M \end{aligned}$$

for the maximal solution (u, v) of the system (B). We also conclude that $T = +\infty$ from Theorem 1.

For $d_1, d_2 \geq 1$, $\underline{d} \leq \xi \leq \bar{d}$, the positive constants M' and M in (7.9) are independent of $d_1, d_2 \geq 1$, that is, $M' = M'(\underline{d}, \bar{d}, \alpha_{ij}, a_i, b_i, c_i, i = 1, 2)$, $M = M(\underline{d}, \bar{d}, \alpha_{ij}, a_i, b_i, c_i, i = 1, 2)$. □

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