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A CHARACTERIZATION OF BLOCH FUNCTIONS

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Abstract. On the unit disk of the complex plane, a characterization of Bloch function is expressed extending known result.

1. Introduction

Let $D = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc in the complex plane $\mathbb{C}$ and let $dA(z) = dxdy$ denote the Lebesgue area measure of $\mathbb{C}$. For $a \in D$, the Möbius transformation φ_a is defined by

$$\varphi_a(z) = \frac{a - z}{1 - \overline{a}z}, \quad z \in D.$$

The group of automorphisms of D will be denoted by $\mathcal{M}$, which consists of rotations(around the origin) of φ_a, $a \in D$. The Bloch space $\mathcal{B}$ is the set of all holomorphic functions f on D for which

$$\|f\|_{\mathcal{B}} = \sup_{a \in D} |(f \circ \varphi_a)'(0)| < \infty.$$

The quantity $\|f\|_{\mathcal{B}}$ is easily seen to be $\mathcal{M}$ invariant, so is the space $\mathcal{B}$. $\mathcal{B}$ is known to be maximal among all $\mathcal{M}$ invariant Banach spaces of holomorphic functions on D [5]. There are a lot of characterizations for the membership $f \in \mathcal{B}$, and among them is the following.

Theorem A [6]. *Let $0 < p < \infty$ and let f be an holomorphic function on D. Then $f \in \mathcal{B}$ if and only if*

$$\sup_{a \in D} \iint_D |f'(z)|^p (1 - |z|^2)^{p-2} (1 - |\varphi_a(z)|^2)^2 \, dxdy < \infty. \tag{1.1}$$

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The left integral of (1.1) *is equivalent to* $\|f\|_{\mathcal{B}}^p$.

(1.1) says that $|f'(z)|^p(1-|z|^2)^{p-1}dxdy$ is a Bergman Carleson measure and the case $p=2$ of (1.1) says that $\mathcal{B}$ is Bergman $BMOA$ (see [2] for $BMOA$).

We extend Theorem A to the following :

THEOREM 1. *Let* $0<p<\infty$ *and* f *be holomorphic on* D. *If either* $1<\delta<\infty$ *and* $0\le\eta<\infty$ *or if* $\delta=1$ *and* $1<\eta<\infty$, *then*

$$\|f\|_{\mathcal{B}}^p \approx \sup_{w\in D}\int_D |f'(z)|^p(1-|z|^2)^{p-2}(1-|\varphi_w(z)|^2)^\delta \times \left(\log\frac{1}{1-|\varphi_w(z)|^2}\right)^{-\eta} dA(z).$$

Throughout " $\approx$ " means the equivalence of two quantities, that is, the quotient of the left side and the right side lies between two positive constants unless both are zero. We give another equivalent condition for the membership $f\in\mathcal{B}$ that is similar to (1.1):

THEOREM 2. *Suppose that* $0<p<\infty$ *and* f *is holomorphic on* D. *Then*

$$(1.2)\quad \|f\|_{\mathcal{B}}^p \approx \sup_{w\in D}\int_D |f(z)-f(w)|^{p-2}|f'(z)|^2(1-|\varphi_w(z)|^2)^2\, dA(z).$$

(1.2) says that $f\in\mathcal{B}$ if and only if the right side of (1.2) is finite. For $p\ge 2$ this fact is a special case of Theorem 1 in [4] and the restriction $p\ge 2$ was removed in [3, Theorem 2], where the following characterization is formulated : an analytic function f on D is a Bloch function if and only if

$$\sup_{a\in D}\int_D |f(z)-f(a)|^{p-2}\,|f'(z)|^2\,(1-|\varphi_a(z)|^2)\log\frac{1}{|\varphi_a(z)|}\,dxdy<\infty.$$

This is equivalent to the finiteness of the right side quantity of (1.2), since

$$\log\frac{1}{|\varphi_a(z)|}\approx 1-|\varphi_a(z)|^2$$

whenever $|\varphi_a(z)|$ is bounded away from 0. We further give an elementary proof for the equivalence of the quantities in (1.2).

2. Proof of Theorem 1

Let $D(w,r)=\{z\in D: |\varphi_w(z)|<r\}$. It is known [7] that

$$\|f\|_{\mathcal{B}}^p \approx \sup_{w\in D}\int_{D(w,\ r)} |f'(z)|^p(1-|z|^2)^{p-2}\, dA(z),$$

where $r: 0<r<1$ is fixed. Since the constant

$$C_r := (1-r^2)^\delta\left(\log\frac{1}{1-r^2}\right)^{-\eta}$$

depending on r is finite, we have

(1.3)

$$\|f\|_{\mathcal{B}}^p \approx \sup_{w\in D}\int_{D(w,\ r)} |f'(z)|^p(1-|z|^2)^{p-2}\,(1-r^2)^\delta\left(\log\frac{1}{1-r^2}\right)^{-\eta} dA(z).$$

Since

$$z\in D(w,r) \iff |\varphi_w(z)|<r \iff 1-|\varphi_w(z)|^2>1-r^2,$$

the right side of (1.3) is bounded by

$$\sup_{w\in D}\int_D |f'(z)|^p(1-|z|^2)^{p-2}(1-|\varphi_w(z)|^2)^\delta\left(\log\frac{1}{1-|\varphi_w(z)|^2}\right)^{-\eta} dA(z).$$

Hence

$$\begin{aligned}\|f\|_{\mathcal{B}}^p \lesssim\ & \sup_{w\in D}\int_D |f'(z)|^p(1-|z|^2)^{p-2}(1-|\varphi_w(z)|^2)^\delta \\ & \times\left(\log\frac{1}{1-|\varphi_w(z)|^2}\right)^{-\eta} dA(z).\end{aligned}$$

Conversely, noting that

$$(1-|z|^2)|\varphi_w'(z)| = 1-|\varphi_w(z)|^2,$$

we have

$$\begin{aligned}& \sup_{w\in D}\int_D |f'(z)|^p(1-|z|^2)^{p-2}(1-|\varphi_w(z)|^2)^\delta \\ & \times\left(\log\frac{1}{1-|\varphi_w(z)|^2}\right)^{-\eta} dA(z) \\ \lesssim\ & \|f\|_{\mathcal{B}}^p\cdot\sup_{w\in D}\int_D |\varphi_w'(z)|^2\frac{\left(\log\frac{1}{1-|\varphi_w(z)|^2}\right)^{-\eta}}{(1-|\varphi_w(z)|^2)^{2-\delta}}\, dA(z).\end{aligned}$$

By the change of the variables for $u = \varphi_w(z)$,

$$\begin{aligned}
&\int_D |\varphi_w'(z)|^2 \frac{\left(\log \frac{1}{1-|\varphi_w(z)|^2}\right)^{-\eta}}{(1-|\varphi_w(z)|^2)^{2-\delta}}\, dA(z) \\
&= \int_D \frac{\left(\log \frac{1}{1-|u|^2}\right)^{-\eta}}{(1-|u|^2)^{2-\delta}}\, dA(u) \\
&\approx \int_0^1 \frac{\left(\log \frac{1}{1-t}\right)^{-\eta}}{(1-t)^{2-\delta}} dt,
\end{aligned}$$

which is finite if either $1 < \delta < \infty$ and $0 \leq \eta < \infty$ or if $\delta = 1$ and $1 < \eta < \infty$. Hence

$$\begin{aligned}
\|f\|_{\mathcal{B}}^p \gtrsim \sup_{w\in D} \int_{D(w,\, r)} &|f'(z)|^p (1-|z|)^{p-2} \left(1-|\varphi_w(z)|^2\right) \\
&\times \left(\log \frac{1}{1-|\varphi_w(z)|^2}\right)^{-\eta} dA(z).
\end{aligned}$$

The proof is complete.

3. Proof of Theorem 2

It is known [1, 6] that

$$\|f\|_{\mathcal{B}} \approx \sup_{a\in D} \|f\circ\varphi_a - f(a)\|_{L^p(dA)}.$$

We fix $a \in D$ for a moment and consider

$$\|f\circ\varphi_a - f(a)\|_{L^p(dA)}^p = \int_D |F|^p\, dA,$$

where $F = f\circ\varphi_a - f(a)$. By the Hardy-Stein identity,

$$r\frac{d}{dr}\int_0^{2\pi} |F(re^{i\theta})|^p d\theta = p^2 \int_{|z|<r} |F(z)|^{p-2} |F'(z)|^2 dA(z)\ .$$

Noting that $F(0)=0$ we have

$$\begin{aligned}\int_0^{2\pi}|F(\rho e^{i\theta})|^p d\theta &= p^2\int_0^{\rho}\frac{dr}{r}\int_{|z|<r}|F(z)|^{p-2}|F'(z)|^2 dA(z)\\ &= p^2\int_{|z|<\rho}|F(z)|^{p-2}|F'(z)|^2\left[\int_0^{\rho}\frac{1}{r}\mathcal{X}_{|z|<r}\,dr\right]dA(z)\\ &= p^2\int_{|z|<\rho}|F(z)|^{p-2}|F'(z)|^2\log\frac{\rho}{|z|}\,dA(z)\ .\end{aligned}$$

Now taking the integration $\int_0^1 \rho\, d\rho$ on both sides,

$$\begin{aligned}&\int_D |F(w)|^p dA(w)\\ \approx\ &\int_0^1 \rho\, d\rho\int_{|z|<\rho}|F(z)|^{p-2}|F'(z)|^2\log\frac{\rho}{|z|}\,dA(z)\\ =\ &\int_D |F(z)|^{p-2}|F'(z)|^2\left[\int_0^1\rho\log\frac{\rho}{|z|}\mathcal{X}_{|z|<\rho}\,d\rho\right]dA(z)\ .\end{aligned}$$

We see that

$$\begin{aligned}\int_0^1\rho\log\frac{\rho}{|z|}\mathcal{X}_{|z|<\rho}\,d\rho\ &=\ \int_{|z|}^1\rho\log\frac{\rho}{|z|}\,d\rho\\ =\frac{1}{4}\log\frac{1}{|z|^2}-\frac{1-|z|^2}{4}\ &=\ \frac{1}{4}\left\{\sum_{k=2}^{\infty}\frac{(1-|z|^2)^k}{k}\right\}.\end{aligned}$$

And it is not difficult to see that

$$\sum_{k=2}^{\infty}\frac{(1-|z|^2)^{k-2}}{k}\ \approx\ 1+\log\frac{1}{|z|^2}.$$

Thus, we obtain

$$\begin{aligned}&\int_D |F(w)|^p\,dA(w)\\ \approx\ &\int_D |F(z)|^{p-2}|F'(z)|^2(1-|z|^2)^2\left(1+\log\frac{1}{|z|^2}\right)\,dA(z).\end{aligned}$$

Since the right integrand is integrable on a neighborhood of $z = 0$ in D, it is simple to see that

$$\|f\|_{\mathcal{B}}^p \approx \int_D |F(z)|^{p-2}|F'(z)|^2(1-|z|^2)^2\, dA(z).$$

Therefore we conclude

$$\|f\|_{\mathcal{B}}^p \approx \sup_{a\in D}\int_D |f\circ\varphi_a(z) - f(a)|^{p-2}|(f\circ\varphi_a)'(z)|^2(1-|z|^2)^2\, dA(z).$$

By the change of variable formula the last integral equals

$$\sup_{a\in D}\int_D |f(w) - f(a)|^{p-2}|f'(w)|^2\left(1-|\varphi_a(w)|^2\right)^2\, dA(w).$$

The proof is complete. □

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