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ITERATIVE APPROXIMATIONS OF ZEROES FOR ACCRETIVE OPERATORS IN BANACH SPACES

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Abstract. In this paper, we introduce and study a new iterative algorithm for approximating zeroes of accretive operators in Banach spaces.

1. Introduction

Let E be a real Banach space, $A \subset E \times E$ be an m-accretive operator and $J_r = (I + rA)^{-1}$ be the resolvent of A for all $r > 0$. A well-known method is the following:

$$(1.1) \qquad \begin{cases} x_0 = x \in E, \\ x_{n+1} = J_{r_n} x_n, \quad n \geq 0, \end{cases}$$

where $\{r_n\}$ is a sequence of positive real numbers. The convergence of (1.1) has been studied by Brézis and Lions [1], Bruck and Passty [3], Bruck and Reich [4], Jung and Takahashi [6], Lions [8], Nevanlinna and Reich [11], Pazy [13], Reich [14]-[16], Rockafellar [17], etc.

On the other hand, Halpern [5] and Mann [10] introduced the following iterative schemes for approximating fixed points of nonexpansive mappings T of E into itself:

$$(1.2) \qquad \begin{cases} x \in E, \\ x_{n+1} = \alpha_n x + (1 - \alpha_n) T x_n, \quad n \geq 0, \end{cases}$$

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and

$$(1.3)\qquad \begin{cases} x_0 \in E, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \quad n \geq 0, \end{cases}$$

respectively, where $\{\alpha_n\}$ is a sequence in [0,1]. The iterative schemes (1.2) and (1.3) have been studied extensively by Takahashi [19], [20] and others (see the references therein).

Motivated by (1.1), (1.2) and (1.3), Kamimura, Khan and Takahashi [7] studied two iterative schemes to solve the relation $0 \in Av$, where A is an accretive operator satisfying the range condition, that is,

$$\overline{D(A)} \subset \cap_{r>0} R(I + rA).$$

Let C be a nonempty closed convex subset of E such that $\overline{D(A)} \subset C \subset \cap_{r>0} R(I+rA)$. Then the correspondence to (1.2) and (1.3) are the following, respectively:

$$x_{n+1} = P(\alpha_n x + (1-\alpha_n)J_{r_n}x_n + f_n), \quad n \geq 0,$$

and

$$x_{n+1} = P(\alpha_n x_n + (1-\alpha_n)J_{r_n}x_n + f_n), \quad n \geq 0,$$

where P is a nonexpansive retraction of E onto C and f_n is the term showing a computational error.

In this paper, we introduce a new iterative scheme

$$(1.4)\qquad x_{n+1} = \alpha_n u + \beta_n J_{r_n} x_n + \gamma_n P e_n, \quad n \geq 0,$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in $[0,1]$, $\{e_n\}$ is a sequence in E and $\{r_n\}$ is a sequence in $(0,\infty)$ satisfying the following:

(i) $\alpha_n + \beta_n + \gamma_n = 1$,
(ii) $\sum_{n=0}^{\infty} \alpha_n = +\infty$ and $\sum_{n=0}^{\infty} \gamma_n < +\infty$,
(iii) $r_n \to \infty$ as $n \to \infty$,
(iv) $\{e_n\}$ is bounded

and show that, if $A^{-1}0 \neq \emptyset$, then the sequence $\{x_n\}$ defined by (1.4) converges strongly to a zero of A. Also we give some weak convergence theorems of the sequence $\{x_n\}$ defined by (1.4).

2. Preliminaries

Throughout this paper, let E be a real Banach space with norm $\|\cdot\|$ and let E^* denote the dual space of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^* \rangle$. When $\{x_n\}$ is a sequence in E, we denote strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and weak convergence by $x_n \to x$ weakly. The *modulus of convexity* of E is defined by

$$\delta(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| \leq 1,\ \|y\| \leq 1,\ \|x-y\| \geq \epsilon\right\}$$

for any ϵ with $0 \leq \epsilon \leq 2$. A Banach space E is said to be *uniformly convex* if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. If E is uniformly convex, then δ satisfies

$$\left\|\frac{x+y}{x}\right\| \leq r\left(1 - \delta\left(\frac{\epsilon}{r}\right)\right)$$

for any $x, y \in E$ with $\|x\| \leq r$, $\|y\| \leq r$ and $\|x - y\| \geq \epsilon$. Let $U = \{x \in E : \|x\| = 1\}$. The *normalized duality mapping* J from E into 2^{E^*} is defined by

$$Jx = \{f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2\}$$

for any $x \in E$. The norm of E is said to be *uniformly Gâteaux differentiable* if, for each $y \in U$, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

is attained uniformly for $x \in U$. It is also said to be *Fréchet differentiable* if, for each $x \in U$, the limit (2.1) is attained uniformly for any $y \in U$. It is known that, if the norm of E is uniformly Gâteaux differentiable, then the normalized duality mapping J is single valued and uniformly norm to weak* continuous on each bounded subset of E. A Banach space E is said to satisfy *Opial's condition* ([12]) if, for any sequence $\{x_n\}$ in E, $x_n \to y$ weakly implies

$$\liminf_{n\to\infty} \|x_n - y\| < \liminf_{n\to\infty} \|x_n - z\|$$

for any $z \in E$ with $z \neq y$.

Let C be a closed convex subset of E. A mapping $T : C \to C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$. We denote the set of all fixed points of T by $F(T)$. A closed convex subset C of E is said to have the *fixed point property* for nonexpansive mappings if every nonexpansive mapping of a bounded closed convex subset D of C into itself has a fixed point in D. Let D be a subset of C. We denote the closure of the convex hull of D by $\overline{co}D$.

A mapping P of D into itself is said to be a *retraction* if $P^2 = P$. A subset D of C is said to be a *nonexpansive retract* of C if there exists a nonexpansive retraction of C onto D.

Let I denote the identity operator on E. An operator $A \subset E \times E$ with domain $D(A) = \{z \in E : Az \neq \emptyset\}$ and range $R(A) = \cup\{Az : z \in D(A)\}$ is said to be *accretive* if, for each $x_i \in D(A)$ and $y_i \in Ax_i$, $i = 1, 2$, there exists $j \in J(x_1 - x_2)$ such that

$$\langle y_1 - y_2, j(x_1 - x_2)\rangle \geq 0.$$

If A is accretive, then we have

$$\|x_1 - x_2\| \leq \|x_1 - x_2 + r(y_1 - y_2)\|$$

for any $x_i \in D(A)$, $y_i \in Ax_i$, $i = 1, 2$, and $r > 0$.

An accretive operator A is said to satisfy the *range condition* if

$$\overline{D(A)} \subset \cap_{r>0} R(I + rA).$$

If A is accretive, then we can define a nonexpansive single valued mapping $J_r : R(I + rA) \to D(A)$ by

$$J_r = (I + rA)^{-1}$$

for each $r > 0$, which is called the *resolvent* of A. We also define the *Yosida approximation* A_r by

$$A_r = \frac{1}{r}(I - J_r).$$

We know that $A_r x \in AJ_r x$ for all $x \in R(I + rA)$ and $\|A_r x\| \leq \inf\{\|y\| : y \in Ax\}$ for all $x \in D(A) \cap R(I + rA)$. Also it follows that, for an accretive operator A satisfying the range condition, $A^{-1}0 = F(J_r)$ for all $r > 0$.

An accretive operator A is said to be *m-accretive* if $R(I + rA) = E$ for any $r > 0$.

In the sequel, unless stated otherwise, we assume that $A \subset E \times E$ is an accretive operator satisfying the range condition and that J_r is the resolvent of A for $r > 0$.

Now, we need the following lemmas for our main results.

LEMMA 2.1. ([9]) *Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences of positive real numbers satisfying*

$$a_{n+1} \leq (1 - t_n)a_n + b_n + c_n, \quad n \geq 0,$$

where $\{t_n\} \subset [0,1]$ with $t_n \to 0$ as $n \to \infty$ and $\sum_{n=0}^{\infty} t_n = \infty$, $b_n = o(t_n)$ and $\sum_{n=0}^{\infty} c_n < +\infty$. Then $a_n \to 0$ as $n \to \infty$.

LEMMA 2.2. ([22]) *Let E be a real reflexive Banach space whose norm is uniformly Gâteanx differentiable and $A \subset E \times E$ be an accretive operator. Suppose that every weakly compact convex subset of E has the fixed point property for nonexpansive mappings. Let C be a nonempty closed convex subset of E such that $\overline{D(A)} \subset C \subset \cap_{r>0} R(I + rA)$. If $A^{-1}0 \neq \emptyset$, then the strong limit $\lim_{t \to \infty} J_t x$ exists and belongs to $A^{-1}0$ for all $x \in C$, where $J_t = (I + tA)^{-1}$ is the resolvent of A for all $t > 0$.*

3. Strong convergence theorems

First, we give some strong convergence theorems of a new iterative sequence $\{x_n\}$ defined by (1.4) to a zero of the given mapping A in Banach spaces.

THEOREM 3.1. *Let E be a real reflexive Banach space with a uniformly Gâteaux differentiable norm and $A \subset E \times E$ be an accretive operator. Let C be a nonempty closed convex nonexpansive retract of E onto C. Suppose that every weakly compact convex subset of E has the fixed point property for nonexpansive mappings. Let $\{x_n\}$ be a sequence generated by*

$$\text{(3.1)} \qquad \begin{cases} u,\, x_0 \in C, \\ x_{n+1} = \alpha_n u + \beta_n J_{r_n} x_n + \gamma_n P e_n, \quad n \geq 0, \end{cases}$$

where the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset [0,1]$, $\{e_n\} \subset E$ and $\{r_n\} \subset (0, \infty)$ satisfy the following:

(i) $\alpha_n + \beta_n + \gamma_n = 1$,

(ii) $\sum_{n=0}^{\infty} \alpha_n = +\infty$ *and* $\sum_{n=0}^{\infty} \gamma_n < +\infty$,
(iii) $r_n \to \infty$ *as* $n \to \infty$,
(iv) $\{e_n\}$ *is bounded.*

If $A^{-1}0 \neq \emptyset$, *then the sequence* $\{x_n\}$ *converges strongly to a zero of* A.

PROOF. We finish the proof by three steps.

Step 1. $\{x_n\}$ is bounded.

To prove this, take a fixed element $p \in A^{-1}0$ and set

$$M = \max\{\|x_0 - p\|, \|u - p\|, \sup_{n\geq 0} \|e_n - p\|\}.$$

Then we have $\|x_0 - p\| \leq M$ and $\|u - p\| \leq M$. Assume that $\|x_n - p\| \leq M$ for some positive integer n. Then, by using (3.1), we have

$$\begin{aligned}\|x_{n+1} - p\| &\leq \alpha_n\|u - p\| + \beta_n\|J_{r_n}x_n - p\| + \gamma_n\|Pe_n - p\| \\ &\leq \alpha_n M + \beta_n\|x_n - p\| + \gamma_n\|e_n - p\| \\ &\leq (\alpha_n + \beta_n + \gamma_n)M \\ &= M.\end{aligned}$$

Thus, by induction, we assert that $\|x_n - p\| \leq M$ for all $n \geq 0$ and hence $\{x_n\}$ is bounded and, further, $\{J_{r_n}x_n\}$ is also bounded since J_{r_n} is nonexpansive.

Step 2. $\limsup_{n\to\infty}\langle u - z, J(x_{n+1} - z\rangle \leq 0$, where $z = \lim_{t\to\infty} J_t u$, which is guaranteed by Lemma 2.2.

We first prove that, for all $t > 0$,

$$\limsup_{n\to\infty}\langle u - J_t u, J(J_{r_n}x_n - J_t u)\rangle \leq 0. \tag{3.2}$$

Indeed, since $\frac{(u-J_t u)}{t} \in AJ_t u$, $A_{r_n}x_n \in AJ_{r_n}x_n$ and A is accretive, we have

$$\langle A_{r_n}x_n - \frac{1}{t}(u - J_t u), J(J_{r_n}x_n - J_t u)\rangle \geq 0$$

and so

$$\langle u - J_t u, J(J_{r_n}x_n - J_t u)\rangle \leq t\langle A_{r_n}x_n, J(J_{r_n}x_n - J_t u)\rangle$$

for all $n \geq 0$ and $t > 0$. Then, by using the fact that $A_{r_n}x_n = \frac{(x_n - y_n)}{r_n} \to 0$ as $n \to \infty$, we obtain

$$\limsup_{n\to\infty}\langle u - J_t u, J(J_{r_n}x_n - J_t u\rangle \leq 0$$

for all $t > 0$, which proves our assertion. Since the norm of E is uniformly Gâteaux differentiable and $J_t u \to z \in A^{-1}0$ as $t \to \infty$, we can prove that

$$\limsup_{n\to\infty}\langle u - z, J(J_{r_n}x_n - z)\rangle \le 0. \tag{3.3}$$

Noting that

$$x_{n+1} - J_{r_n}x_n = \alpha_n u + (\beta_n - 1)J_{r_n}x_n + \gamma_n Pe_n \to 0$$

as $n \to \infty$ and using the uniformly Gâteaux differentiability of the norm again, we have

$$\lim_{n\to\infty} |\langle u - z, J(x_{n+1} - z)\rangle) - \langle u - z, J(J_{r_n}x_n - z)\rangle| = 0$$

and hence

$$\limsup_{n\to\infty}\langle u - z, J(x_{n+1} - z)\rangle \le 0,$$

which finish the proof of Step 2.

Step 3. $x_n \to z \in A^{-1}0$ as $n \to \infty$.

From

$$\begin{aligned} & x_{n+1} - z \\ &= \alpha_n(u - z) + \beta_n(J_{r_n}x_n - z) + \gamma_n(Pe_n - z) \\ &= (1 - \alpha_n)(J_{r_n}x_n - z) + \alpha_n(u - z) + \gamma_n(Pe_n - J_{r_n}x_n), \end{aligned}$$

it follows that

$$\begin{aligned} & \|x_{n+1} - z\|^2 \\ &\le (1 - \alpha_n)^2\|J_{r_n}x_n - z\|^2 + 2\alpha_n\langle u - z, J(x_{n+1} - z)\rangle \\ &\quad + 2r_n\langle Pe_n - J_{r_n}x_n, J(x_{n+1} - z)\rangle \\ &\le (1 - \alpha_n)\|x_n - z\|^2 + 2\alpha_n\langle u - z, J(x_{n+1} + z)\rangle \\ &\quad + 2r_n\|e_n - J_{r_n}x_n\|\|x_{n+1} - z\| \\ &\le (1 - \alpha_n)\|x_n - z\|^2 + 2\alpha_n\langle u - z, J(x_{n+1} - z)\rangle + K\gamma_n \end{aligned} \tag{3.4}$$

for some constant $K > 0$. Define $\sigma_n = \max\{\langle u - z, J(x_{n+1} - z)\rangle, 0\}$. Then, by virtue of Step 2, we can prove that $\sigma_n \to 0$ as $n \to \infty$. It follows from (3.4) that

$$\|x_{n+1} - z\|^2 \le (1 - \alpha_n)\|x_n - z\|^2 + 2\alpha_n\sigma_n + K\gamma_n. \tag{3.5}$$

By setting $a_n = \|x_n - z\|^2$, $t_n = \alpha_n$, $b_n = 2\alpha_n\sigma_n$ and $c_n = K\gamma_n$, then (3.5) reduces to the following:

$$a_{n+1} \leq (1 - t_n)a_n + b_n + c_n, \quad n \geq 0,$$

with $t_n \to 0$, $\sum_{n=0}^{\infty} t_n = \infty$, $b_n = o(t_n)$ and $\sum_{n=0}^{\infty} c_n < +\infty$. Now Lemma 2.1 is applicable to get the desired conclusion. This completes the proof. □

From Theorem 3.1, we have the following:

COROLLARY 3.2. *Let C be a nonempty closed convex nonexpansive retract of a reflexive Banach space E whose norm is uniformly Gâteaux differentiable, P be a nonexpansive retraction of E onto C and $T : C \to C$ be a continuous pseudo-contractive mapping. Suppose that every weakly compact convex subset of E has the fixed point property. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} u,\, x_0 \in C, \\ y_n = \frac{1}{1+r_n}x_n + \frac{r_n}{1+r_n}Ty_n, \quad n \geq 0, \\ xn + 1 = \alpha_n u + \beta_n y_n + \gamma_n Pe_n, \quad n \geq 0, \end{cases}$$

where the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset [0,1]$, $\{e_n\} \subset E$ and $\{r_n\} \subset (0,\infty)$ satisfy the following:

(i) $\alpha_n + \beta_n + \gamma_n = 1$,
(ii) $\alpha_n \to 0$ *as* $n \to \infty$,
(iii) $\Sigma_{n=0}^{\infty}\alpha_n = \infty$ *and* $\Sigma_{n=0}^{\infty}\gamma_n < +\infty$,
(iv) $r_n \to \infty$ *as* $n \to \infty$,
(v) $\{e_n\}$ *is bounded.*

If $F(T) \neq \emptyset$, then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

PROOF. Set $A = I - T$. Then $A : C \to E$ is continuous accretive, satisfies the range condition:

$$\overline{D(A)} = C \subset \cap_{r>0}(I + rA)$$

and $A^{-1}0 = F(T)$ (see [18]). Now, applying Theorem 3.1, we can obtain the desired conclusion of the theorem. □

4. Weak convergence theorems

Now, we give some weak convergence theorems of the sequence defined by (1.4). We need the following lemmas for our main results in this section.

LEMMA 4.1. ([2]) *Let C be a closed bounded convex subset of a uniformly convex Banach space E and $T : C \to C$ be a nonexpansive mapping. If a sequence $\{x_n\}$ in C converges weakly to $z \in C$ and $\{x_n - Tx_n\}$ converges strongly to 0 as $n \to \infty$, then $Tz = z$.*

REMARK. Lemma 4. 1 is still true without assuming the condition that C is bounded.

LEMMA 4.2. ([15], [21]) *Let E be a uniformly convex Banach space whose norm is Fréchet differentiable, C be a nonempty closed convex subset of E and $\{T_n\}_{n=0}^{\infty}$ be a sequence of nonexpansive mappings of C into itself such that $\cap_{n=0}^{\infty} F(T_n)$ is nonempty. Let $x \in C$ and $S_n = T_n T_{n-1} \cdots T_0$ for all $n \geq 0$. Then the set $\cap_{n=0}^{\infty} \overline{co}\{S_m x : m \geq n\} \cap U$ consists of at most one point, where $U = \cap_{n=0}^{\infty} F(Tn)$.*

THEOREM 4.3. *Let E be a uniformly convex Banach space whose norm is Fréchet differentiable or which satisfies Opial's condition and $A \subset E \times E$ be an accretive operator. Let C be a nonempty closed convex nonexpansive retract of E such that*

$$\overline{D(A)} \subset C \subset \cap_{r>0} R(I + rA)$$

and P be a nonexpansive retraction of E onto C. Let $\{x_n\}$ be a sequence generated by

$$\begin{cases} x_0 = x \in C, \\ x_{n+1a} = \alpha_n x_n + \beta_n J_{r_n} x_n + \gamma_n P e_n, \quad n \geq 0, \end{cases}$$

where the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset [0, 1]$, $\{r_n\} \subset (0, \infty)$ and $\{e_n\} \subset E$ satisfy the following:

(i) $\alpha_n + \beta_n + \gamma_n = 1$,
(ii) $\limsup_{n \to \infty} \alpha_n \leq 1$,
(iii) $\Sigma_{n=0}^{\infty} \gamma_n < +\infty$,
(iv) $\liminf_{n \to \infty} r_n > 0$,
(v) $\{e_n\}$ *is bounded.*

If $A^{-1}0 \neq \emptyset$, then the sequence $\{x_n\}$ converges weakly to a zero of A.

PROOF. Since $A^{-1}0 \neq \emptyset$, we can fix an element $p \in A^{-1}0$ and set

$$M = \max\{\|x_0 - p\|, \sup_{n \geq 0}\{\|e_n - p\|\}\}.$$

Then we can prove that $\{x_n\}$ is bounded. Indeed, $\|x_0 - p\| \leq M$. Assume that $\|x_n - p\| \leq M$ for some $n \geq 0$. Then we have

$$\begin{aligned}\|x_{n+1} - p\| &\leq \alpha_n\|x_n - p\| + \beta_n\|J_{r_n}x_n - p\| + \gamma_n\|Pe_n - p\| \\ &\leq \alpha_n M + \beta_n\|x_n - p\| + \gamma_n\|e_n - p\| \\ &\leq (\alpha_n + \beta_n + \gamma_n)M \\ &= M.\end{aligned}$$

By induction, we assert that $\{x_n\}$ is bounded.

Next, we prove that $\lim_{n\to\infty}\|x_n - u\|$ exists for any $u \in A^{-1}0$. In fact,

$$\begin{aligned}\|x_{n+1} - u\| &\leq (\alpha_n + \beta_n)\|x_n - u\| + \gamma_n M \\ &\leq \|x_n - u\| + \gamma_n M,\end{aligned}$$

which implies that $\lim_{n\to\infty}\|x_n - u\|$ exists since $\Sigma_{n=0}^{\infty}\gamma_n < +\infty$. Write $d = \lim_{n\to\infty}\|x_n - u\|$ for any $u \in A^{-1}0$. Without loss of generality, we assume that $d > 0$. Since A is accretive and E is uniformly convex, we have

$$\begin{aligned}\|J_{r_n}x_n - u\| &\leq \left\|J_{r_n}x_n - u + \frac{r_n}{2}(A_{r_n}x_n - 0)\right\| \\ &= \left\|J_{r_n}x_n - u + \frac{1}{2}(x_n - J_{r_n}x_n)\right\| \\ &= \left\|\frac{x_n - J_{r_n}x_n}{2} - u\right\| \\ &\leq \|x_n - u\|\Big[1 - \delta\Big(\frac{\|x_n - J_{r_n}x_n\|}{\|x_n - u\|}\Big)\Big]\end{aligned}$$

and hence

$$\begin{aligned}&(1-\alpha_n)\|x_n - u\|\delta\Big(\frac{\|x_n - J_{r_n}x_n\|}{\|x_n - u\|}\Big) \\ &\leq (1-\alpha_n)\{\|x_n - u\| - \|J_{r_n}x_n - u\|\} \\ &= \|x_n - u\| - \alpha_n\|x_n - u\| - (1-\alpha_n)\|J_{r_n}x_n - u\| \\ &= \|x_n - u\| - \alpha_n\|x_n - u\| - (1-\alpha_n)\|J_{r_n}x_n - u\| - \gamma_n M + \gamma_n M \\ &\leq \|x_n - u\| - \|x_{n+1} - u\| + \gamma_n M\end{aligned}$$

for all $n \geq 0$. Using assumptions $\lim_{n\to\infty} \alpha_n < 1$ and $\lim_{n\to\infty} \|x_n - u\| = d > 0$, we obtain

$$\delta\Big(\frac{\|x_n J_{r_n} x_n\|}{\|x_n - u\|}\Big) \to 0 \quad (n \to \infty),$$

which implies that $x_n - J_{r_n} x_n \to 0$ as $n \to \infty$. Let $v \in E$ be a weak subsequential limit of $\{x_n\}$ such that $x_{n_i} \to v$ as $i \to \infty$. Then it follows that $J_{r_{n_i}} x_{n_i} \to v$ as $i \to \infty$. From

$$\begin{aligned}\|J_{r_n} x_n - J_1 J_{r_n} x_n\| &= \|(I - J_1) J_{r_n} x_n\| \\ &= \|A_1 J_{r_n} x_n\| \\ &\leq \inf\{\|z\| : z \in A J_{r_n} x_n\} \\ &\leq \|A_{r_n} x_n\| = \Big\|\frac{x_n J_{r_n} x_n}{r_n}\Big\|\end{aligned}$$

and $\lim_{n\to\infty} r_n > 0$, we obtain $J_{r_n} x_n - J_1 J_{r_n} x_n \to 0$ as $n \to \infty$. Thus it follows from Lemma 4.1 that $v \in F(J_1) = A^{-1}0$.

Now, we only prove Theorem 4.3 for the case that E has a Fréchet differentiable norm. Define a mapping $T_n : C \to C$ by $T_n x = \alpha_n x + \beta_n J_{r_n} x + \gamma_n P x$ for all $n \geq 0$. Then $\{T_n\}$ is a family of nonexpansive mappings of C into itself and $\cap_{n=0}^{\infty} F(T_n) = A^{-1}0$. It is clear that

$$x_{n+1} = T_n x_n + \gamma_n (P e_n - x_n), \quad n \geq 0. \tag{4.1}$$

Now, we consider the following two cases:

Case 1. $\gamma_n = 0$ for all $n \geq 0$.

In the case, (4.1) reduces to $x_{n+1} = T_n x_n$ for all $n \geq 0$. Putting $S_n = T_n T_{n-1} \cdots T_0$, then $x_{n+1} = S_n x$ for all $n \geq 0$. It follows from Lemma 4.2 that $x_n \to v \in A^{-1}0$ weakly as $n \to \infty$.

Case 2. $\gamma_n \neq 0$ for some $n \geq 0$.

Our discussion follows an idea of Brézis and Lions [1, Remarque 14]. Let $U_n z = T_n z + \gamma_n (P e_n - z)$ for all $z \in C$ and $n \geq 0$. Then $x_{n+1} = U_n x_n$. Now, for any fixed $m \geq 0$, define a sequence $\{z_n(m)\}$ by

$$z_0(m) = x_m, \; z_{n+1}(m) = T_{n+m} z_n(m), \quad n \geq 0.$$

From the case 1, we know that $\{z_n(m)\}$ converges weakly to some $z(m) \in A^{-1}0$ and hence

$$\|z(m+1) - z(m)\| \leq \liminf_{n\to\infty} \|z_n(m+1) - z_{n+1}(m)\|. \tag{4.2}$$

Observe that

$$\begin{aligned}
&\|z_n(m+1) - z_{n+1}(m)\| \\
&= \|T_{n+m}T_{n+m-1}\cdots T_{m+1}x_{m+1} - T_{n+m}T_{n+m-1}\cdots T_m x_m\| \\
&\leq \|x_{m+1} - T_m x_m\| \\
&= \gamma_n \|Pe_m - x_m\| \\
&\leq M\gamma_m, \quad n, m \geq 0,
\end{aligned}$$

where $M = \sup\{\|Pe_m - x_m\| : m \geq 0\}$. This together with (4.2) implies that

$$\|z(m+1) - z(m)\| \leq M\gamma_m, \quad m \geq 0,$$

and hence

$$\sum_{m=0}^{\infty} \|z(m+1) - z(m)\| \leq M \sum_{m=0}^{\infty} \gamma_m < +\infty,$$

which implies that $\{z(m)\}$ is a Cauchy sequence and so $z(m) \to a \in A^{-1}0$ as $m \to \infty$ since $A^{-1}0$ is closed. Since

$$\begin{aligned}
&\|x_{n+m+1} - z_{n+1}(m)\| \\
&= \|U_{n+m}U_{n+m-1}\cdots U_m x_m - T_{n+m}T_{n+m-1}\cdots T_m x_m\| \\
&\leq M \sum_{l=m}^{n+m} \gamma_l,
\end{aligned}$$

we have

$$\begin{aligned}
&|\langle x_{n+m+1} - a, h\rangle| \\
&= |\langle x_{n+m+1} - z_{n+1}(m), h\rangle + \langle z_{n+1}(m) - z(m), h\rangle + \langle z(m) - a, h\rangle| \\
&\leq \Big(M \sum_{l=m}^{n+m} \gamma_l + \|z(m) - a\|\Big)\|h\| + |\langle z_{n+1}(m) - z(m), h\rangle|
\end{aligned}$$

for all $h \in E^*$ and $n, m \geq 0$. Fixing $m \geq 0$ and letting $n \to \infty$, then we have

$$\begin{aligned}
\limsup_{n\to\infty} |\langle x_n - a, h\rangle| &= \limsup_{n\to\infty} |\langle x_{n+m+1} - a, h\rangle| \\
&\leq \Big(M \sum_{l=m}^{\infty} \gamma_l + \|z(m) - a\|\Big)\|h\|
\end{aligned} \tag{4.3}$$

for all $h \in E^*$ and $m \geq 0$. Letting $m \to \infty$ in (4.3) yields that

$$\limsup_{n\to\infty} |\langle x_n - a, h\rangle| \leq 0,$$

which implies that the sequence $\{x_n\}$ converges weakly to $a \in A^{-1}0$. This completes the proof. □

From Theorem 4.3, we have the following:

COROLLARY 4.4. *Let C be a nonempty closed convex nonexpansive retract of a uniformly convex Banach space E whose norm is Fréchet differentiable or which satisfies Opial's condition, P be a nonexpansive retraction of E onto C and T be a continuous pseudo contraction of C into itself. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0,\, x \in C, \\ y_n = \frac{1}{1+r_n} x_n + \frac{r_n}{1+r_n} T y_n, \quad n \geq 0, \\ x_{n+1} = \alpha_n x_n + \beta_n y_n + \gamma_n P e_n, \quad n \geq 0, \end{cases}$$

where the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset [0,1]$, $\{r_n\} \subset (0,\infty)$ and $\{e_n\} \subset E$ satisfy the following:

(i) $\alpha_n + \beta_n + \gamma_n = 1,$
(ii) $\limsup_{n\to\infty} \alpha_n < 1,$
(iii) $\sum_{n=0}^{\infty} \gamma_n < +\infty,$
(iv) $\liminf_{n\to\infty} r_n > 0,$
(v) $\{e_n\}$ *is bounded.*

If $F(T) \neq \emptyset$, then the sequence $\{x_n\}$ converges weakly to a fixed point of T.

References

[1] H. Brézis and P. L. Lions, *Produits infinis de resolvants*, Israel J. Math. **29** (1978), 329–345.
[2] F. E. Browder, *Semicontractive and semiaccretive nonlinear mappings in Banach spaces*, Bull. Amer. Math. Soc. **74** (1968), 660–665.
[3] R. E. Bruck and G. B. Passty, *Almost convergence of the infinite product of resolvents in Banach spaces*, Nonlinear Anal. **3** (1979), 279–282.
[4] R. E. Bruck and S. Reich, *Nonexpansive projections and resolvents of accretive operators in Banach spaces*, Houston J. Math. **3** (1977), 459–470.
[5] B. Halpern, *Fixed points of nonexpansive maps*, Bull. Amer. Math. Soc. **73** (1967), 957–961.

[6] J. S. Jung and W. Takahashi, *Dual convergence theorems for the infinite products of resolvents in Banach spaces*, Kodai Math. J. **14** (1991), 358–364.

[7] S. Kamimura, S. H. Khan and W. Takahashi, *Iterative schemes for approximating solutions of relations involving accretive operators in Banach spaces*, Fixed Point Theory and Applications, Vol. 5, Edited by Y. J. Cho, J. K. Kim and S. M. Kang, Nova Science Publishers, Inc., 2003, 41–52.

[8] P. L. Lions, *Une methode iterative de resolution d'une inequation variationnelle*, Israel J. Math. **31** (1978), 204–208.

[9] L. S. Liu, *Ishikawa and Mann iterative processes with errors for nonlinear strongly accretive mappings in Banach spaces*, J. Math. Anal. Appl. **194** (1995), 114–125.

[10] W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4** (1953), 506–510.

[11] O. Nevanlinna and S. Reich, *Strong convergence of contraction semigroups and of iterative methods for accretive operators in Banach spaces*, Israel J. Math. **32** (1979), 44–58.

[12] Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc. **73** (1967), 591–597.

[13] A. Pazy, *Remarks on nonlinear ergodic theory in Hilbert spaces*, Nonlinear Anal. **6** (1979), 863–871.

[14] S. Reich, *On infinite products of resolvents*, Atti Acad. Naz. Lincei **63** (1977), 338–340.

[15] ———, *Weak convergence theorems for resolvents of accretive operators in Banach spaces*, J. Math. Anal. Appl. **67** (1979), 274–276.

[16] ———, *Strong convergence theorems for resolvents of accretive operators in Banach spaces*, J. Math. Anal. Appl. **75** (1980), 287–292.

[17] R. T. Rockafellar, *Monotone operators and the proximal point algorithm*, SIAM J. Control and Optim. **14** (1976), 877–898.

[18] W. Takahashi, *Nonlinear Functional Analysis*, Kindai-Kagaku-Sha, Tokyo, 1988.

[19] ———, *Fixed point theorems and nonlinear ergodic theorems for nonlinear semigroups and their applications*, Nonlinear Anal. **30** (1997), 1283–1293.

[20] ———, *Nonlinear Analysis and Convex Analysis*, Edited by W. Takahashi and T. Tanaka, World Scientific Publishing Company, 1999, p. 87–94.

[21] W. Takahashi and G. E. Kim, *Approximating fixed points of nonexpansive mappings in Banach spaces*, Math. Japon. **48** (1998), 1–9.

[22] W. Takahashi and Y. Ueda, *On Reich's strong convergence theorems for resolve of accretive operators*, J. Math. Anal. Appl. **104** (1984), 546–553.

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