

DERIVATIONS OF A RESTRICTED WEYL TYPE ALGEBRA ON A LAURENT EXTENSION

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ABSTRACT. Several authors find all the derivations of an algebra [1], [3], [7]. A Weyl type non-associative algebra and its subalgebra are defined in the paper [2], [3], [10]. All the derivations of the non-associative algebra $\overline{WN}_{0,0,s_1}$ is found in this paper [4]. We find all the derivations of the non-associative algebra $\overline{WN}_{0,s,0_1}$ in this paper [5].

1. Introduction

Generally, there is an infinite dimensional simple algebra with outer derivation. Thus it is an interesting problem to find all the derivations of an infinite dimensional (non-)associative algebra [2]. The Weyl type non-associative algebras are defined in the papers [3], [12]. All the derivations of the restricted Weyl type non-associative algebras $\overline{WN}_{0,0,1_1}$ and $\overline{WN}_{0,0,2_1}$ are found in the paper [1] and [2]. All the derivations of the non-associative algebra $\overline{WN}_{0,0,s_1}$ is found in this paper [4]. In this paper, we find all the derivations of the restricted Weyl type non-associative algebras $\overline{WN}_{0,s,0_1}$.

2. Preliminaries

Let \mathbf{F} be a field of characteristic zero (not necessarily algebraically closed). Throughout this paper, \mathbf{N} and \mathbf{Z} will denote the non-negative

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integers and the integers, respectively. Let $\mathbf{F}[x_1, \dots, x_{m+s}]$ be the polynomial ring with the variables x_1, \dots, x_{m+s} . Let g_1, \dots, g_n be given polynomials in $\mathbf{F}[x_1, \dots, x_{m+s}]$. For $n, m, s \in \mathbf{N}$, let us define the commutative, associative \mathbf{F} -algebra $F_{g_n, m, s} = \mathbf{F}[e^{\pm g_1}, \dots, e^{\pm g_n}, x_1^{\pm 1}, \dots, x_m^{\pm 1}, x_{m+1}, \dots, x_{m+s}]$ in the formal power series ring $\mathbf{F}[[x_1, \dots, x_{m+s}]]$ which is called a stable algebra in the paper [7] with the standard basis

$$(1) \quad \mathbf{B} = \{e^{a_1 g_1} \dots e^{a_n g_n} x_1^{i_1} \dots x_{m+s}^{i_{m+s}} \mid a_1, \dots, a_n, i_1, \dots, i_m \in \mathbf{Z}, \\ i_{m+1}, \dots, i_{m+s} \in \mathbf{N}\}$$

and with the obvious addition and the multiplication [3], [7], [11], [13], [14] where we take g_1, \dots, g_n so that \mathbf{B} is the standard basis of $F_{g_n, m, s}$. ∂_w , $1 \leq w \leq m+s$, denotes the usual partial derivative with respect to x_w on $F_{g_n, m, s}$. For partial derivatives $\partial_u, \dots, \partial_v$ of $F_{g_n, m, s}$, the composition $\partial_u^{j_u} \circ \dots \circ \partial_v^{j_v}$ of them is denoted $\partial_u^{j_u} \dots \partial_v^{j_v}$ where $j_u, \dots, j_v \in \mathbf{N}$. Let us define the vector space $WN(g_n, m, s)$ over \mathbf{F} which is spanned by the standard basis

$$(2) \quad \{e^{a_1 g_1} \dots e^{a_n g_n} x_1^{i_1} \dots x_{m+s}^{i_{m+s}} \partial_u^{j_u} \dots \partial_v^{j_v} \mid e^{a_1 g_1} \dots e^{a_n g_n} x_1^{i_1} \dots x_{m+s}^{i_{m+s}} \in \mathbf{B}, \\ j_u, \dots, j_v \in \mathbf{N}, 1 \leq u, \dots, v \leq m+s\}.$$

Thus we can define the multiplication $*$ on $WN(g_n, m, s)$ as follows:

$$(3) \quad \begin{aligned} & e^{a_{11} g_1} \dots e^{a_{1n} g_n} x_1^{i_{11}} \dots x_{m+s}^{i_{1, m+s}} \partial_u^{j_u} \dots \partial_v^{j_v} \\ * & e^{a_{21} g_1} \dots e^{a_{2n} g_n} x_1^{i_{21}} \dots x_{m+s}^{i_{2, m+s}} \partial_h^{j_h} \dots \partial_w^{j_w} \\ = & e^{a_{11} g_1} \dots e^{a_{1n} g_n} x_1^{i_{11}} \dots x_{m+s}^{i_{1, m+s}} \partial_u^{j_u} \dots \partial_v^{j_v} \\ & (e^{a_{21} g_1} \dots e^{a_{2n} g_n} x_1^{i_{21}} \dots x_{m+s}^{i_{2, m+s}}) \partial_h^{j_h} \dots \partial_w^{j_w} \end{aligned}$$

for any basis elements $e^{a_{11} g_1} \dots e^{a_{1n} g_n} x_1^{i_{11}} \dots x_{m+s}^{i_{1, m+s}} \partial_u^{j_u} \dots \partial_v^{j_v}$ and $e^{a_{21} g_1} \dots e^{a_{2n} g_n} x_1^{i_{21}} \dots x_{m+s}^{i_{2, m+s}} \partial_h^{j_h} \dots \partial_w^{j_w} \in WN(g_n, m, s)$. Thus we can define the Weyl-type non-associative algebra $\overline{WN}_{g_n, m, s}$ with the multiplication $*$ in (3) and with the set $WN(g_n, m, s)$ [10], [11], [12]. For $r \in \mathbf{N}$, let us define the restricted Weyl type non-associative subalgebra $\overline{WN}_{g_n, m, s_r}$ of the non-associative algebra $\overline{WN}_{g_n, m, s}$ spanned by

$$(4) \quad \{e^{a_1 g_1} \dots e^{a_n g_n} x_1^{i_1} \dots x_{m+s}^{i_{m+s}} \partial_u^{j_u} \dots \partial_v^{j_v} \mid e^{a_1 g_1} \dots e^{a_n g_n} x_1^{i_1} \dots x_s^{i_s} \in \mathbf{B}, \\ j_u, \dots, j_v \in \mathbf{N}, j_u + \dots + j_v \leq r, 1 \leq u, \dots, v \leq m+s\}.$$

For any basis element $e^{a_1 g_1} \dots e^{a_n g_n} x_1^{i_1} \dots x_s^{i_s} \partial_u^{j_u} \dots \partial_v^{j_v}$ of $\overline{WN}_{g_n, m, s}$, let us define the degree $\deg_r(e^{a_1 g_1} \dots e^{a_n g_n} x_1^{i_1} \dots x_s^{i_s} \partial_u^{j_u} \dots \partial_v^{j_v})$ of x_r as i_r and the total degree $\deg_{tot}(e^{a_1 g_1} \dots e^{a_n g_n} x_1^{i_1} \dots x_s^{i_s} \partial_u^{j_u} \dots \partial_v^{j_v})$ of $e^{a_1 g_1} \dots$

$e^{a_n g_n} x_1^{i_1} \cdots x_s^{i_s} \partial_u^{j_u} \cdots \partial_v^{j_v}$ as $i_1 + \cdots + i_s$. Thus for any element l of $\overline{WN}_{g_n, m, s}$, we define $deg_r(l)$ of x_r as the highest degree of x_r in the basis terms of l and $deg_{tot}(l)$ as the highest total degree of basis terms of l . It is well known that the non-associative algebra $\overline{WN}_{g_n, m, s}$ is simple, even though it has the right annihilator [7], [8]. For any element of l of $\overline{WN}_{g_n, m, s}$, an element l_1 (resp. l_2) is right (resp. left) identity of l if $l * l_1 = l$ (resp. $l_2 * l = l$).

3. Derivations of $\overline{WN}_{0, s, 0_1}$

LEMMA 1. For any derivation D of the non-associative algebra $\overline{WN}_{0, 3, 0_1}$, then we have that

$$D(x_1^i \partial_1) = (1 - i)a_{1,0,0}x_1^i \partial_1 - ib_{1,0,0}x_1^{i-1}x_2 \partial_1 - ic_{1,0,0}x_1^{i-1}x_3 \partial_1 + id_{1,0,0}x_1^{i-1} \partial_1 + a_{2,0,0}x_1^i \partial_2 + a_{3,0,0}x_1^i \partial_3$$

and $D(x_u^j \partial_u)$ has the similar formula as $D(x_1^i \partial_1)$ where $a_{1,0,0}, \dots, a_{3,0,0} \in \mathbf{F}$, $2 \leq u \leq 3$, and $i, j \in \mathbf{Z}$.

PROOF. The proof of the lemma is similar to the proof of Lemma by considering \mathbf{Z} instead of \mathbf{N} in the paper [4]. Let us omit it. \square

LEMMA 2. For any derivation D of the non-associative algebra $\overline{WN}_{0, 3, 0_1}$,

$$D(x_1 \partial_2) = -a_{1,0,0}x_1 \partial_2 + b_{2,0,0}x_1 \partial_2 - b_{1,0,0}x_2 \partial_2 - c_{1,0,0}x_3 \partial_2 + d_{1,0,0} \partial_2 + b_{1,0,0}x_1 \partial_1 + b_{3,0,0}x_1 \partial_3$$

hold with appropriate scalars. $D(x_u \partial_v)$ has the similar formula as $D(x_1 \partial_2)$ where $1 \leq u \neq v \leq 3$.

PROOF. The proof of the lemma comes from the similar comments of the proof of Lemma 1. Let us omit it. \square

LEMMA 3. For any derivation D of the non-associative algebra $\overline{WN}_{0, 3, 0_1}$

$$D(x_1 x_2 \partial_1) = -a_{2,0,0}x_1^2 \partial_1 - b_{1,0,0}x_2^2 \partial_1 - b_{2,0,0}x_1 x_2 \partial_1 - c_{1,0,0}x_2 x_3 \partial_1 - c_{2,0,0}x_1 x_3 \partial_1 + d_{1,0,0}x_2 \partial_1 + h_{2,0,0}x_1 \partial_1 + a_{2,0,0}x_1 x_2 \partial_2 + a_{3,0,0}x_1 x_2 \partial_3$$

$$D(x_1 x_3 \partial_1) = -a_{3,0,0}x_1^2 \partial_1 - b_{1,0,0}x_2 x_3 \partial_1 - b_{3,0,0}x_1 x_2 \partial_1 - c_{1,0,0}x_3^2 \partial_1$$

$$\begin{aligned}
& -c_{3,0,0}x_1x_3\partial_1 + d_{1,0,0}x_3\partial_1 + g_{3,0,0}x_1\partial_1 + a_{2,0,0}x_1x_3\partial_2 \\
& + a_{3,0,0}x_1x_3\partial_3 \\
D(x_2x_3\partial_1) & = a_{1,0,0}x_2x_3\partial_1 - a_{2,0,0}x_1x_3\partial_1 - a_{3,0,0}x_1x_2\partial_1 - b_{2,0,0}x_2x_3\partial_1 \\
& - b_{3,0,0}x_2^2\partial_1 - c_{2,0,0}x_3^2\partial_1 - c_{3,0,0}x_2x_3\partial_1 + h_{2,0,0}x_3\partial_1 \\
(5) \quad & + g_{3,0,0}x_2\partial_1 + a_{2,0,0}x_2x_3\partial_2 + a_{3,0,0}x_2x_3\partial_3
\end{aligned}$$

hold with appropriate scalars. We have the similar formulas of $D(x_u x_v \partial_w)$ as (5) where $1 \leq u, v \leq 3$ and $2 \leq w \leq 3$.

PROOF. The proof of the lemma comes from the similar comments of the proof of Lemma 1. Let us omit it. \square

The proof of the following lemma is similar to the proof of Lemma 4 in [4], but let us put it for the readers.

LEMMA 4. For any derivation D of the non-associative algebra $\overline{WN}_{0,3,0_1}$ and for any basis element $x_1^n x_2^m x_3^k \partial_u$, $1 \leq u \leq 3$, of $\overline{WN}_{0,3,0_1}$, then we have that

$$\begin{aligned}
D(x_1^n x_2^m x_3^k \partial_1) & = (1-n)a_{1,0,0}x_1^n x_2^m x_3^k \partial_1 - mb_{2,0,0}x_1^n x_2^m x_3^k \partial_1 \\
& - kc_{3,0,0}x_1^n x_2^m x_3^k \partial_1 + nd_{1,0,0}x_1^{n-1}x_2^m x_3^k \partial_1 \\
& + mh_{2,0,0}x_1^n x_2^{m-1}x_3^k \partial_1 + kg_{3,0,0}x_1^n x_2^m x_3^{k-1} \partial_1 \\
& - ma_{2,0,0}x_1^{n+1}x_2^{m-1}x_3^k \partial_1 - ka_{3,0,0}x_1^{n+1}x_2^m x_3^{k-1} \partial_1 \\
& - nb_{1,0,0}x_1^{n-1}x_2^{m+1}x_3^k \partial_1 - kb_{3,0,0}x_1^n x_2^{m+1}x_3^{k-1} \partial_1 \\
& - nc_{1,0,0}x_1^{n-1}x_2^m x_3^{k+1} \partial_1 - mc_{2,0,0}x_1^n x_2^{m-1}x_3^{k+1} \partial_1 \\
(6) \quad & + a_{2,0,0}x_1^n x_2^m x_3^k \partial_2 + a_{3,0,0}x_1^n x_2^m x_3^k \partial_3,
\end{aligned}$$

where $a_{i,0,0}, b_{i,0,0}, c_{i,0,0} \in \mathbf{F}$, $1 \leq i \leq 3$, and $d_{1,0,0}, h_{2,0,0}, g_{3,0,0} \in \mathbf{F}$. We have the similar formulas of $D(x_1^n x_2^m x_3^k \partial_2)$ and $D(x_1^n x_2^m x_3^k \partial_3)$ as (6).

PROOF. Let D be the derivation in the lemma. By Lemma 1 and $D(\partial_1 * x_1 x_2 x_3 \partial_1) = D(x_2 x_3 \partial_1)$, we have that $\partial_1 * D(x_1 x_2 x_3 \partial_1) = -2a_{2,0,0}x_1x_3\partial_1 - c_{2,0,0}x_3^2\partial_1 - 2a_{3,0,0}x_1x_2\partial_1 - b_{2,0,0}x_2x_3\partial_1 - b_{3,0,0}x_2^2\partial_1 - c_{3,0,0}x_2x_3\partial_1 + h_{2,0,0}x_3\partial_1 + g_{3,0,0}x_2\partial_1 + a_{2,0,0}x_2x_3\partial_2 + a_{3,0,0}x_2x_3\partial_3$. This implies that

$$\begin{aligned}
D(x_1x_2x_3\partial_1) & = -a_{2,0,0}x_1^2x_3\partial_1 - a_{3,0,0}x_1^2x_2\partial_1 - b_{2,0,0}x_1x_2x_3\partial_1 \\
& - b_{3,0,0}x_1x_2^2\partial_1 - c_{2,0,0}x_1x_3^2\partial_1 - c_{3,0,0}x_1x_2x_3\partial_1
\end{aligned}$$

$$\begin{aligned}
 & +h_{2,0,0}x_1x_3\partial_1 + g_{3,0,0}x_1x_2\partial_1 + a_{2,0,0}x_1x_2x_3\partial_2 \\
 & +a_{3,0,0}x_1x_2x_3\partial_3 + \sum_{i,j \geq 0} s_{1,i,j}x_2^i x_3^j \partial_1 \\
 (7) \quad & + \sum_{i,j \geq 0} s_{2,i,j}x_2^i x_3^j \partial_2 + \sum_{i,j \geq 0} s_{3,i,j}x_2^i x_3^j \partial_3,
 \end{aligned}$$

where $s_{1,i,j}, s_{2,i,j}, s_{3,i,j} \in \mathbf{F}$, $i, j \geq 0$. By $D(\partial_2 * x_1x_2x_3\partial_1) = D(x_1x_3\partial_1)$, we have that

$$\begin{aligned}
 & \sum_{i \geq 1, j \geq 0} i s_{1,i,j} x_2^{i-1} x_3^j \partial_1 + \sum_{i \geq 1, j \geq 0} i s_{2,i,j} x_2^{i-1} x_3^j \partial_2 \\
 & + \sum_{i \geq 1, j \geq 0} i s_{3,i,j} x_2^{i-1} x_3^j \partial_3 \\
 = & -2b_{1,0,0}x_2x_3\partial_1 - c_{1,0,0}x_3^2\partial_1 + d_{1,0,0}x_3\partial_1.
 \end{aligned}$$

So we have that

$$\begin{aligned}
 D(x_1x_2x_3\partial_1) = & -a_{2,0,0}x_1^2x_3\partial_1 - a_{3,0,0}x_1^2x_2\partial_1 - b_{1,0,0}x_2^2x_3\partial_1 \\
 & -b_{2,0,0}x_1x_2x_3\partial_1 - b_{3,0,0}x_1x_2^2\partial_1 - c_{1,0,0}x_2x_3^2\partial_1 \\
 & -c_{2,0,0}x_1x_3^2\partial_1 - c_{3,0,0}x_1x_2x_3\partial_1 + d_{1,0,0}x_2x_3\partial_1 \\
 & +h_{2,0,0}x_1x_3\partial_1 + g_{3,0,0}x_1x_2\partial_1 + a_{2,0,0}x_1x_2x_3\partial_2 \\
 & +a_{3,0,0}x_1x_2x_3\partial_3 + \sum_{j \geq 0} s_{1,0,j}x_3^j\partial_1 + \sum_{j \geq 0} s_{2,0,j}x_3^j\partial_2 \\
 (8) \quad & + \sum_{j \geq 0} s_{3,0,j}x_3^j\partial_3.
 \end{aligned}$$

By $D(\partial_3 * x_1x_2x_3\partial_1) = D(x_1x_2\partial_1)$ and (8), we have that

$$\begin{aligned}
 D(x_1x_2x_3\partial_1) = & -a_{2,0,0}x_1^2x_3\partial_1 - a_{3,0,0}x_1^2x_2\partial_1 - b_{1,0,0}x_2^2x_3\partial_1 \\
 & -b_{2,0,0}x_1x_2x_3\partial_1 - b_{3,0,0}x_1x_2^2\partial_1 - c_{1,0,0}x_2x_3^2\partial_1 \\
 & -c_{2,0,0}x_1x_3^2\partial_1 - c_{3,0,0}x_1x_2x_3\partial_1 + d_{1,0,0}x_2x_3\partial_1 \\
 & +h_{2,0,0}x_1x_3\partial_1 + g_{3,0,0}x_1x_2\partial_1 + a_{2,0,0}x_1x_2x_3\partial_2 \\
 (9) \quad & +a_{3,0,0}x_1x_2x_3\partial_3 + s_{1,0,0}\partial_1 + s_{2,0,0}\partial_2 + s_{3,0,0}\partial_3.
 \end{aligned}$$

Since $x_1\partial_1$ is a left identity of $x_1x_2x_3\partial_1$, by Lemma 1 and (9), we have that

$$D(x_1x_2x_3\partial_1) = -a_{2,0,0}x_1^2x_3\partial_1 - a_{3,0,0}x_1^2x_2\partial_1 - b_{1,0,0}x_2^2x_3\partial_1$$

$$\begin{aligned}
& -b_{2,0,0}x_1x_2x_3\partial_1 - b_{3,0,0}x_1x_2^2\partial_1 - c_{1,0,0}x_2x_3^2\partial_1 \\
& -c_{2,0,0}x_1x_3^2\partial_1 - c_{3,0,0}x_1x_2x_3\partial_1 + d_{1,0,0}x_2x_3\partial_1 \\
& +h_{2,0,0}x_1x_3\partial_1 + g_{3,0,0}x_1x_2\partial_1 + a_{2,0,0}x_1x_2x_3\partial_2 \\
& +a_{3,0,0}x_1x_2x_3\partial_3.
\end{aligned}$$

By $D(x_1^n\partial_1 * x_1x_2x_3\partial_1) = D(x_1^n x_2x_3\partial_1)$, Lemma 1, and (10), we have that

$$\begin{aligned}
D(x_1^n x_2x_3\partial_1) &= (1-n)a_{1,0,0}x_1^n x_2x_3\partial_1 - a_{2,0,0}x_1^{n+1}x_3\partial_1 \\
& -a_{3,0,0}x_1^{n+1}x_2\partial_1 - nb_{1,0,0}x_1^{n-1}x_2^2x_3\partial_1 \\
& -b_{2,0,0}x_1^n x_2x_3\partial_1 - b_{3,0,0}x_1^n x_2^2\partial_1 \\
& -nc_{1,0,0}x_1^{n-1}x_2x_3^2\partial_1 - c_{2,0,0}x_1^n x_3^2\partial_1 \\
& -c_{3,0,0}x_1^n x_2x_3\partial_1 + nd_{1,0,0}x_1^{n-1}x_2x_3\partial_1 \\
& +h_{2,0,0}x_1^n x_3\partial_1 + g_{3,0,0}x_1^n x_2\partial_1 \\
(10) \quad & +a_{2,0,0}x_1^n x_2x_3\partial_2 + a_{3,0,0}x_1^n x_2x_3\partial_3.
\end{aligned}$$

By $D(x_2^m\partial_2 * x_1^n x_2x_3\partial_1) = D(x_1^n x_2^m x_3\partial_1)$, Lemma 1, and (10), we have that

$$\begin{aligned}
D(x_1^n x_2^m x_3\partial_1) &= (1-n)a_{1,0,0}x_1^n x_2^m x_3\partial_1 - ma_{2,0,0}x_1^{n+1}x_2^{m-1}x_3\partial_1 \\
& -a_{3,0,0}x_1^{n+1}x_2^m\partial_1 - nb_{1,0,0}x_1^{n-1}x_2^{m+1}x_3\partial_1 \\
& -mb_{2,0,0}x_1^n x_2^m x_3\partial_1 - b_{3,0,0}x_1^n x_2^{m+1}\partial_1 \\
& -nc_{1,0,0}x_1^{n-1}x_2^m x_3^2\partial_1 - mc_{2,0,0}x_1^n x_2^{m-1}x_3^2\partial_1 \\
& -c_{3,0,0}x_1^n x_2^m x_3\partial_1 + nd_{1,0,0}x_1^{n-1}x_2^m x_3\partial_1 \\
& +mh_{2,0,0}x_1^n x_2^{m-1}x_3\partial_1 + g_{3,0,0}x_1^n x_2^m\partial_1 \\
(11) \quad & +a_{2,0,0}x_1^n x_2^m x_3\partial_2 + a_{3,0,0}x_1^n x_2^m x_3\partial_3.
\end{aligned}$$

By $D(x_3^k\partial_3 * x_1^n x_2^m x_3\partial_1) = D(x_1^n x_2^m x_3^k\partial_1)$, Lemma 1, and (11), we have that

$$\begin{aligned}
D(x_1^n x_2^m x_3^k\partial_1) &= (1-n)a_{1,0,0}x_1^n x_2^m x_3^k\partial_1 - mb_{2,0,0}x_1^n x_2^m x_3^k\partial_1 \\
& -kc_{3,0,0}x_1^n x_2^m x_3^k\partial_1 + nd_{1,0,0}x_1^{n-1}x_2^m x_3^k\partial_1 \\
& +mh_{2,0,0}x_1^n x_2^{m-1}x_3^k\partial_1 + kg_{3,0,0}x_1^n x_2^m x_3^{k-1}\partial_1 \\
& -ma_{2,0,0}x_1^{n+1}x_2^{m-1}x_3^k\partial_1 - ka_{3,0,0}x_1^{n+1}x_2^m x_3^{k-1}\partial_1 \\
& -nb_{1,0,0}x_1^{n-1}x_2^{m+1}x_3^k\partial_1 - kb_{3,0,0}x_1^n x_2^{m+1}x_3^{k-1}\partial_1 \\
& -nc_{1,0,0}x_1^{n-1}x_2^m x_3^{k+1}\partial_1 - mc_{2,0,0}x_1^n x_2^{m-1}x_3^{k+1}\partial_1 \\
(12) \quad & +a_{2,0,0}x_1^n x_2^m x_3^k\partial_2 + a_{3,0,0}x_1^n x_2^m x_3^k\partial_3.
\end{aligned}$$

Similarly, we can prove that

$$\begin{aligned}
 D(x_1^n x_2^m x_3^k \partial_2) &= -na_{1,0,0} x_1^n x_2^m x_3^k \partial_2 + (1-m)b_{2,0,0} x_1^n x_2^m x_3^k \partial_2 \\
 &\quad -kc_{3,0,0} x_1^n x_2^m x_3^k \partial_2 + nd_{1,0,0} x_1^{n-1} x_2^m x_3^k \partial_2 \\
 &\quad +mh_{2,0,0} x_1^n x_2^{m-1} x_3^k \partial_2 + kg_{3,0,0} x_1^n x_2^m x_3^{k-1} \partial_2 \\
 &\quad -ma_{2,0,0} x_1^{n+1} x_2^{m-1} x_3^k \partial_2 - ka_{3,0,0} x_1^{n+1} x_2^m x_3^{k-1} \partial_2 \\
 &\quad -kb_{3,0,0} x_1^n x_2^{m+1} x_3^{k-1} \partial_2 - nc_{1,0,0} x_1^{n-1} x_2^m x_3^{k+1} \partial_2 \\
 &\quad -mc_{2,0,0} x_1^n x_2^{m-1} x_3^{k+1} \partial_2 - nb_{1,0,0} x_1^{n-1} x_2^{m+1} x_3^k \partial_2 \\
 &\quad +b_{1,0,0} x_1^n x_2^m x_3^k \partial_1 + b_{3,0,0} x_1^n x_2^m x_3^k \partial_3
 \end{aligned}$$

$$\begin{aligned}
 D(x_1^n x_2^m x_3^k \partial_3) &= -na_{1,0,0} x_1^n x_2^m x_3^k \partial_3 - mb_{2,0,0} x_1^n x_2^m x_3^k \partial_3 \\
 &\quad +(1-k)c_{3,0,0} x_1^n x_2^m x_3^k \partial_3 + nd_{1,0,0} x_1^{n-1} x_2^m x_3^k \partial_3 \\
 &\quad +mh_{2,0,0} x_1^n x_2^{m-1} x_3^k \partial_3 + kg_{3,0,0} x_1^n x_2^m x_3^{k-1} \partial_3 \\
 &\quad -ma_{2,0,0} x_1^{n+1} x_2^{m-1} x_3^k \partial_3 - ka_{3,0,0} x_1^{n+1} x_2^m x_3^{k-1} \partial_3 \\
 &\quad -nb_{1,0,0} x_1^{n-1} x_2^{m+1} x_3^k \partial_3 - kb_{3,0,0} x_1^n x_2^{m+1} x_3^{k-1} \partial_3 \\
 &\quad -nc_{1,0,0} x_1^{n-1} x_2^m x_3^{k+1} \partial_3 - mc_{2,0,0} x_1^n x_2^{m-1} x_3^{k+1} \partial_3 \\
 &\quad +c_{1,0,0} x_1^n x_2^m x_3^k \partial_1 + c_{2,0,0} x_1^n x_2^m x_3^k \partial_2
 \end{aligned}$$

as (12). This completes the proof of the lemma. □

Note 1. For any basis element $x_1^{n_1} x_2^{n_2} x_3^{n_3} \partial_u$, $1 \leq u \leq 3$, of $\overline{WN_{0,3,0_1}}$, we define **F**-maps D_v , $1 \leq v \leq 3$, $D_{v,w}$, $1 \leq v \neq w \leq 3$, and D_{9+v} , $1 \leq v \leq 3$, of $\overline{WN_{0,3,0_1}}$ as follows:

$$\begin{aligned}
 D_v(x_1^{n_1} x_2^{n_2} x_3^{n_3} \partial_u) &= (\delta_{u,v} - n_v) x_1^{n_1} x_2^{n_2} x_3^{n_3} \partial_u \quad \text{for } 1 \leq v \leq 3, \\
 D_{v,w}(x_1^{n_1} x_2^{n_2} x_3^{n_3} \partial_u) &= -n_w x_v^{n_v+1} x_w^{n_w-1} x_k^{n_k} \partial_u + \delta_{u,v} x_1^{n_1} x_2^{n_2} x_3^{n_3} \partial_w \\
 &\quad \text{for } 1 \leq v \neq w \leq 3, k \notin \{v, w\}, \\
 D_{9+v}(x_1^{n_1} x_2^{n_2} x_3^{n_3} \partial_u) &= n_v x_v^{n_v-1} x_k^{n_k} x_w^{n_w} \partial_u \\
 &\quad \text{for } 1 \leq v \leq 3 \text{ and } 1 \leq k \neq w \leq 3, v \notin \{k, w\},
 \end{aligned}$$

where $\delta_{u,v}$ is Kronecker delta. The **F**-maps D_v , $1 \leq v \leq 3$, $D_{v,w}$, $1 \leq v \neq w \leq 3$, and D_{9+v} , $1 \leq v \leq 3$, of $\overline{WN_{0,3,0_1}}$ can be linearly extended to non-associative algebra derivations of $\overline{WN_{0,3,0_1}}$.

THEOREM 1. *The additive group $Der(\overline{WN_{0,3,0_1}})$ of the non-associative algebra $\overline{WN_{0,3,0_1}}$ is spanned by D_v , $1 \leq v \leq 3$, $D_{v,w}$, $1 \leq v \neq w \leq 3$, and D_{9+v} , $1 \leq v \leq 3$, which are defined in Note 1, and which are inner [5].*

PROOF. By Lemma 4 and Note 1, any derivation D of $\overline{WN_{0,3,0_1}}$ can be written as linear sum of D_v , $1 \leq v \leq 3$, $D_{v,w}$, $1 \leq v \neq w \leq 3$, and D_{9+v} , $1 \leq v \leq 3$, which are defined in Note 1. This completes the proof of the theorem. \square

Note 2. For any $x_1^{n_1} x_2^{n_2} \cdots x_s^{n_s} \partial_u$, $1 \leq u \leq s$, of $\overline{WN_{0,s,0_1}}$, we define \mathbf{F} -maps D_1, \dots, D_{s^2+s} of $\overline{WN_{0,s,0_1}}$ as follows:

$$\begin{aligned}
 D_v(x_1^{n_1} x_2^{n_2} \cdots x_s^{n_s} \partial_u) &= (\delta_{u,v} - n_v) x_1^{n_1} x_2^{n_2} \cdots x_s^{n_s} \partial_u && \text{for } 1 \leq v \leq s \\
 D_{v,w}(x_1^{n_1} x_2^{n_2} \cdots x_s^{n_s} \partial_u) &= -n_w x_1^{n_1} \cdots x_v^{n_v+1} x_{v+1}^{n_{v+1}} \cdots x_w^{n_w-1} \\
 &\quad \times x_{w+1}^{n_{w+1}} \cdots x_s^{n_s} \partial_u + \delta_{u,v} x_1^{n_1} x_2^{n_2} \cdots x_s^{n_s} \partial_w && \text{for } 1 \leq v \neq w \leq s \\
 D_{s^2+v}(x_1^{n_1} x_2^{n_2} \cdots x_s^{n_s} \partial_u) &= n_v x_1^{n_1} \cdots x_v^{n_v-1} x_{v+1}^{n_{v+1}} \cdots x_s^{n_s} \partial_u && \text{for } 1 \leq v \leq s.
 \end{aligned}
 \tag{13}$$

Then the \mathbf{F} -maps D_1, \dots, D_s , $D_{v,w}$, $1 \leq v \neq w \leq s$, D_{s^2+v} , $1 \leq v \leq s$, can be linearly extended to non-associative algebra derivations of $\overline{WN_{0,s,0_1}}$.

THEOREM 2. *The additive group $\text{Der}(\overline{WN_{0,s,0_1}})$ of the non-associative algebra $\overline{WN_{0,s,0_1}}$ is spanned by D_u , $1 \leq u \leq s$, $D_{v,w}$, $1 \leq v \neq w \leq s$, and D_{s^2+v} , $1 \leq v \leq s$, which are defined in Note 2.*

PROOF. Let D be any derivation $\text{Der}(\overline{WN_{0,s,0_1}})$. For any element $x_q^{n_q} \partial_v$ of $\overline{WN_{0,s,0_1}}$, it is easy to prove that

$$\begin{aligned}
 D(\partial_v) &= \sum_{1 \leq t \leq s} c_t \partial_t \\
 D(x_q^{n_q} \partial_v) &= a_{q,v} (\delta_{q,v} - n_q) x_q^{n_q} \partial_v + b_{q,v} n_q x_q^{n_q-1} \partial_v \\
 &\quad + \sum_{1 \leq w \leq s} (1 - \delta_{w,q}) c_{q,w} n_q x_q^{n_q-1} x_w \partial_v \\
 &\quad + \sum_{1 \leq t \leq s} (1 - \delta_{t,v}) d_{q,t,v} x_q^{n_q} \partial_t.
 \end{aligned}
 \tag{15}$$

Without loss of generality, we may assume that at least one of c_t , $1 \leq t \leq s$, is non-zero with appropriate scalars. For any basis element

$x_1^{n_1} \cdots x_s^{n_s} \partial_v$ of $\overline{WN_{0,s,0_1}}$, let us put the following formula:

$$\begin{aligned}
 D(x_1^{n_1} \cdots x_s^{n_s} \partial_v) &= \sum_{1 \leq t \leq s} a_{t,v} (\delta_{t,v} - n_t) x_1^{n_1} \cdots x_s^{n_s} \partial_v \\
 &+ \sum_{1 \leq t \leq s} b_{t,v} n_t x_1^{n_1} \cdots x_t^{n_t-1} x_{t+1}^{n_{t+1}} \cdots x_s^{n_s} \partial_v \\
 &+ \sum_{1 \leq r \neq t \leq s} c_{r,t,v} (n_t x_1^{n_1} \cdots x_r^{n_r+1} x_{r+1}^{n_{r+1}} \cdots x_t^{n_t-1} x_{t+1}^{n_{t+1}} \cdots x_s^{n_s} \partial_v \\
 (16) \quad &+ \delta_{v,r} x_1^{n_1} \cdots x_s^{n_s} \partial_t)
 \end{aligned}$$

with appropriate scalars. By (14) and (15), we know that D holds (14), (15) and (16) for ∂_u , $1 \leq u \leq s$, and $x_q^{n_q} \partial_v$, $1 \leq q, v \leq s$. Thanks to the similar proofs of Theorem 3.3 in [1], Theorem 1 in [2], and Theorem 1, for any basis element of $\overline{WN_{0,1,0_1}}$, $\overline{WN_{0,2,0_1}}$, and $\overline{WN_{0,3,0_1}}$, D holds (13) appropriately. Thus by induction on s of $\overline{WN_{0,s,0_1}}$, we assume that the theorem holds for $\overline{WN_{0,s-1,0_1}}$. This implies that by (14), (15) and (16), we can assume that the theorem holds for any $x_1^{n_1} \cdots x_s^{n_s} \partial_v$ of $\overline{WN_{0,s,0_1}}$ such that $deg_{tot}(x_1^{n_1} \cdots x_s^{n_s} \partial_v) \in \mathbb{N}$ by induction on the total degree of $x_1^{n_1} \cdots x_s^{n_s} \partial_v$ and a fixed positive integer n_s . We take a basis element $x_1^{n_1} \cdots x_s^{n_s+1} \partial_v$ such that $deg_s(x_1^{n_1} \cdots x_s^{n_s+1} \partial_v) = n_s + 1$. By $D(\partial_s * x_1^{n_1} \cdots x_s^{n_s+1} \partial_v) = (n_s + 1)D(x_1^{n_1} \cdots x_s^{n_s} \partial_v)$, $D(x_1^{n_1} \cdots x_s^{n_s+1} \partial_w * x_w \partial_v) = D(x_1^{n_1} \cdots x_s^{n_s+1} \partial_v)$ and by appropriate inductions, we can prove that $D(x_1^{n_1} \cdots x_s^{n_s+1} \partial_v)$ holds (13) routinely where $1 \leq w \leq s$ and wD is the sum of derivations in Note 2 and let us omit the remaining steps of its proof because of routine calculations. \square

COROLLARY 1. *If $s_1 \neq s_2$, then the non-associative algebras $\overline{WN_{0,s_1,0_1}}$ and $\overline{WN_{0,s_2,0_1}}$ are not isomorphic to each other as non-associative algebras.*

PROOF. By Theorem 2, the dimension of $Der(\overline{WN_{0,s_1,0_1}})$ is $s_1^2 + s_1$ and the dimension of $Der(\overline{WN_{0,s_2,0_1}})$ is $s_2^2 + s_2$. Thus there is no isomorphism between them. This completes the proof of the corollary. \square

COROLLARY 2. *$Der(\overline{WN_{0,s,0_1}})$ is isomorphic to $Der(\overline{WN_{0,0,s_1}})$ as additive groups.*

PROOF. The proof of the corollary is straightforward by Theorem 2 and Theorem 2 in the paper [4]. \square

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