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# Moments and Estimation From Progressively Censored Data of Half Logistic Distribution

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**Abstract.** In this paper, we derive recurrence relations for the single and product moments of progressively Type-II right censored order statistics from half logistic distribution. Next, we derive the maximum likelihood estimators (MLEs) of the location and scale parameters of the half logistic distribution. In addition, we use the setup proposed by Balakrishnan and Aggarwala (2000) to compute the approximate best linear unbiased estimates (ABLUEs) of the location and scale parameters. Finally, we point out a simulation study to compare between the efficiency of the techniques considered for the estimation.

**Key Words :** *Single moments; product moments; approximate best linear unbiased estimate; maximum likelihood estimate and Monte Carlo simulation.*

## 1. INTRODUCTION

Let us consider the following progressive type-II censoring scheme: suppose $n$ units taken from the same population are placed on a life test. At the first failure time of one of the $n$ units, a number $R_1$ of the surviving units is randomly withdrawn

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from the test, at the second failure time, another $R_2$ surviving units are selected at random and taken out of the experiment, and so on. Finally, at the time of the $m$th failure, the remaining $R_m = n - R_1 - R_2 - ... - R_{m-1} - m$ units are removed. In this scheme, $\mathbf{R} = (R_1, R_2, ..., R_m)$ is pre-sanctified. The resulting $m$ ordered failure times, which we will denote by $X_{1:m:n}^{(R_1,R_2,...,R_m)}, X_{2:m:n}^{(R_1,R_2,...,R_m)}, ..., X_{m:m:n}^{(R_1,R_2,...,R_m)}$ are referred to as *progressive Type-II right censored order statistics.* The special case when $R_1 = R_2 = ... = R_{m-1} = 0$ so that $R_m = n - m$ is the case of conventional Type-II right censored sampling. Also, when $R_1 = R_2 = ... = R_m = 0$, so that $m = n$, the progressive Type-II right censoring scheme reduces to the case of no censoring (ordinary order statistics). If the failure times are based on an absolutely continuous distribution function $F$ with probability density function $f$, the joint probability density function of the progressively censored failure times $X_{1:m:n}, X_{2:m:n}, ..., X_{m:m:n}$, is given by [see Balakrishnan and Aggarwala (2000)]

$$f_{X_{1:m:n},X_{2:m:n},...,X_{m:m:n}}(x_1, x_2, ..., x_m) = A_{n;R_1,...,R_{m-1}} \prod_{i=1}^{m} f(x_i)\{1 - F(x_i)\}^{R_i},$$
$$-\infty < x_1 < x_2 < ... < x_m < \infty, \quad (1.1)$$

where $f(.)$ and $F(.)$ are, respectively, the pdf and the cdf of the random sample and

$$A_{n;R_1,...,R_{m-1}} = n(n - R_1 - 1)...(n - R_1 - R_2 - ... - R_{m-1} - m + 1). \quad (1.2)$$

For simplicity, we write $A_{n;R_1,...,R_{m-1}} = A_{n;\tilde{R}_{m-1}}$; $1 \le m \le n$ and $A_{n;\tilde{R}_0} = n$. In this paper, we are concerned with progressive Type-II censored data from the half logistic distribution with probability density function

$$f(x) = \frac{2e^{-x}}{(1 + e^{-x})^2}, \qquad x \ge 0, \quad (1.3)$$

and cumulative distribution function

$$F(x) = \frac{1 - e^{-x}}{1 + e^{-x}}, \qquad x \ge 0. \quad (1.4)$$

Form (1.3) and (1.4), we have the relation

$$f(x) = \{1 - F(x)\} - \frac{1}{2}\{1 - F(x)\}^2, \quad x \ge 0. \quad (1.5)$$

The density function $f(x)$ in (1.3), obtained by folding the logistic density of the form $f(x) = e^{-x}/(1 + e^{-x})^2$, $-\infty < x < \infty$ at $x = 0$, is a monotonic decreasing function of $x$ in the interval $[0, \infty)$ and has an increasing hazard rate. It, therefore, serves as a good failure-time model in life testing problems. Also, it fits the data better than exponential distribution [see Balakrishnan and Puthenpura(1986)]. For the theory methods and applications of progressive censoring, readers are referred to the book by Balakrishnan and Aggarwala (2000).

Since the publication of the book by Balakrishnan and Aggarwala (2000), considerable amount of research work has been carried out on progressive censoring methodology. Balasooriya and Balakrishnan (2000) and Balasooriya, Saw and Gadag (2000) have studied progressively censored reliability sampling plans for Weibull and lognormal distributions, respectively. Ng, Chan and Balakrishnan (2002) have discussed the estimation of parameters from progressively censored data using the EM Algorithm. Viveros and Balakrishnan (1994) and Balasooriya, Saw and Gadag (2000) have developed inferential methods based on progressively Type-II censored samples.

Recently, Ng, Chan and Balakrishnan (2004) have computed the expected Fisher information and asymptotic variance-covariance matrix of the maximum likelihood estimates based on a progressively type II censored sample from Weibull distribution. Also, they used these values to determine the optimal progressive censoring plans.

In this paper, we establish recurrence relations for the single and product moments of progressively Type-II right censored order statistics from half logistic distribution in Section 2. In Section 3, we calculate the MLEs and the approximate BLUEs of the half logistic parameters. To calculate the efficiency of the techniques considered for the estimation, we point out a simulation study In section 4.

## 2. MOMENTS

In this section, we establish some new recurrence relations satisfied by the single and product moments of progressively Type-II right censored order statistics from the half logistic distribution.

### 2.1 Single moments

Let $X_{1:m:n}^{(R_1,R_2,...,R_m)}, X_{2:m:n}^{(R_1,R_2,...,R_m)}, .., X_{m:m:n}^{(R_1,R_2,...,R_m)}$ be a progressive Type-II right censored order statistics with censoring scheme $(R_1, R_2, ..., R_m)$ from half logistic distribution. The single moments of the progressive Type-II censoring can be written from (1.1) as follows

$$\begin{aligned}\mu_{i:m:n}^{(R_1,...,R_m)^{(k)}} &= E[X_{i:m:n}^{(R_1,...,R_m)^{(k)}}]\\ &= A_{n;\tilde{R}_{m-1}} \int\int_{0<x_1<x_2<...<x_m<\infty}...\int x_i^k f(x_1)[1-F(x_1)]^{R_1}\\ &\times \quad f(x_2)[1-F(x_2)]^{R_2}...f(x_m)[1-F(x_m)]^{R_m} dx_1...dx_m, \quad (2.1)\end{aligned}$$

where $A_{n;\tilde{R}_{m-1}}$ is defined in (1.2). When $k = 1$, the superscript in the notation of the mean of the progressively Type-II right censored order statistics may be omitted without any confusion. The single moment of progressive Type-II right censored order statistics given in (2.1) satisfies the following recurrence relations.

**Relation 2.1**

For $2 \le m \le n$ and $k \ge 0$,

$$\begin{aligned}\mu_{1:m:n}^{(R_1,R_2,\ldots,R_m)(k)} &= \frac{(R_1+1)}{(k+1)}\mu_{1:m:n}^{(R_1,R_2,\ldots,R_m)(k+1)} + \frac{(n-R_1-1)}{(k+1)}\mu_{1:m-1:n}^{(R_1+R_2+1,\ldots,R_m)(k+1)} \\ &- \frac{n(n-R_1-1)}{2(k+1)(n+1)}\mu_{1:m-1:n+1}^{(R_1+R_2+2,\ldots,R_m)(k+1)} \\ &- \frac{(R_1+2)}{2(k+1)}\frac{n}{n+1}\mu_{1:m:n+1}^{(R_1+1,R_2,\ldots,R_m)(k+1)}, \qquad (2.2)\end{aligned}$$

for $m = 1$, $n = 1, 2, \ldots$ and $k \ge 0$,

$$\mu_{1:1:n+1}^{(n)(k+1)} = 2\left[\mu_{1:1:n}^{(n-1)(k+1)} - \frac{(k+1)}{n}\mu_{1:1:n}^{(n-1)(k)}\right]. \qquad (2.3)$$

**Proof**

Starting from (2.1), we write

$$\begin{aligned}\mu_{1:m:n}^{(R_1,R_2,\ldots,R_m)(k)} &= A_{n;\tilde{R}_{m-1}} \underset{0<x_2<\ldots<x_m<\infty}{\int\int\cdots\int} I(x_2)f(x_2)[1-F(x_2)]^{R_2}\ldots f(x_m) \\ &\times\ [1-F(x_m)]^{R_m}dx_2\ldots dx_m, \qquad (2.4)\end{aligned}$$

where $A_{n;\tilde{R}_{m-1}}$ is defined in (1.2) and

$$I(x_2) = \int_0^{x_2} x_1^k[1-F(x_1)]^{R_1}f(x_1)dx_1,$$

which upon using (1.5) and integrating by parts gives

$$\begin{aligned}I(x_2) &= \frac{1}{k+1}\left\{x_2^{k+1}[1-F(x_2)]^{R_1+1} - \frac{1}{2}x_2^{k+1}[1-F(x_2)]^{R_1+2}\right\} \\ &+ \frac{(R_1+1)}{(k+1)}\int_0^{x_2} x_1^{k+1}[1-F(x_1)]^{R_1}f(x_1)dx_1 \\ &- \frac{(R_1+2)}{2(k+1)}\int_0^{x_2} x_1^{k+1}[1-F(x_1)]^{R_1+1}f(x_1)dx_1.\end{aligned}$$

By using the above expression of $I(x_2)$ into (2.4), and simplifying in view of definition of the single moment given in (2.1), we obtain (2.2).

Now for $m = 1$, $n = 1, 2, \ldots$, $k \ge 0$, and by using (2.1), we have

$$\begin{aligned}\mu_{1:1:n}^{(R_1)(k)} &= A_{n;\tilde{R}_0}\int_0^\infty x_1^k f(x_1)[1-F(x_1)]^{R_1}dx_1 \\ &= \frac{n(R_1+1)}{k+1}\int_0^\infty x_1^{k+1}f(x_1)[1-F(x_1)]^{R_1}dx_1 \\ &- \frac{n(R_1+2)}{2(k+1)}\int_0^\infty x_1^{k+1}f(x_1)[1-F(x_1)]^{R_1+1}dx_1\end{aligned}$$

$$= \frac{n}{k+1}\mu_{1:1:n}^{(R_1)^{(k+1)}} - \frac{n}{2(k+1)}\mu_{1:1:n+1}^{(R_1+1)^{(k+1)}}, \tag{2.5}$$

and hence (2.3) is proved.

**Relation 2.2**

For $2 \le i \le m-1$, $m \le n$ and $k \ge 0$,

$$\begin{aligned} \mu_{i:m:n+1}^{(R_1,\ldots,R_i+1,\ldots,R_m)^{(k+1)}} &= \frac{2}{(R_i+2)}\frac{A_{n+1;\tilde{R}_{i-1}}}{A_{n;\tilde{R}_{i-1}}}\Big[(R_i+1)\mu_{i:m:n}^{(R_1,\ldots,R_m)^{(k+1)}} \\ &- (k+1)\mu_{i:m:n}^{(R_1,\ldots,R_m)^{(k)}} \\ &+ (n-R_1-\ldots-R_i-i)\mu_{i:m-1:n}^{(R_1,\ldots,R_{i-1},R_i+R_{i+1}+1,R_{i+2},\ldots,R_m)^{(k+1)}} \\ &- \frac{A_{n;\tilde{R}_i}}{2A_{n+1;\tilde{R}_{i-1}}}\mu_{i:m-1:n+1}^{(R_1,\ldots,R_{i-1},R_i+R_{i+1}+2,R_{i+2},\ldots,R_m)^{(k+1)}} \\ &- (n-R_1-\ldots-R_{i-1}-i+1)\mu_{i-1:m-1:n}^{(R_1,\ldots,R_{i-2},R_{i-1}+R_i+1,R_{i+1},\ldots,R_m)^{(k+1)}} \\ &+ \frac{A_{n;\tilde{R}_{i-1}}}{2A_{n+1;\tilde{R}_{i-2}}}\mu_{i-1:m-1:n+1}^{(R_1,\ldots,R_{i-2},R_{i-1}+R_i+2,R_{i+1},\ldots,R_m)^{(k+1)}}\Big]. \end{aligned} \tag{2.6}$$

**Proof**

Again, staring from (2.1), we write

$$\begin{aligned} \mu_{i:m:n}^{(R_1,R_2,\ldots,R_m)^{(k)}} &= A_{n;\tilde{R}_{m-1}} \int_{0<x_1<\ldots<x_{i-1}<x_{i+1}<\ldots<\infty} \cdots \int \cdots \int \left\{ \int_{x_{i-1}}^{x_{i+1}} x_i^k f(x_i)[1-F(x_i)]^{R_i} dx_i \right\} \\ &\times f(x_1)[1-F(x_1)]^{R_1} f(x_{i-1})[1-F(x_{i-1})]^{R_{i-1}} f(x_{i+1})[1-F(x_{i+1})]^{R_{i+1}} \\ &\times \ldots \times f(x_m)[1-F(x_m)]^{R_m} dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_m. \end{aligned} \tag{2.7}$$

By using (1.5) in (2.7) and integrating the innermost integral by parts with some simplifications, we get (2.6).

**Relation 2.3**

For $2 \le m \le n$ and $k \ge 0$,

$$\begin{aligned} \mu_{m:m:n}^{(R_1,\ldots,R_m)^{(k+1)}} &= \frac{1}{(R_m+1)}[(k+1)\mu_{m:m:n}^{(R_1,\ldots,R_m)^{(k)}} \\ &+ (n-R_1-\ldots-R_{m-1}-m+1)\mu_{m-1:m-1:n}^{(R_1,\ldots,R_{m-2},R_{m-1}+R_m+1)^{(k+1)}} \\ &- \frac{A_{n;\tilde{R}_{m-1}}}{2A_{n+1;\tilde{R}_{m-2}}}\mu_{m-1:m-1:n+1}^{(R_1,\ldots,R_{m-2},R_{m-1}+R_m+2)^{(k+1)}} \\ &+ (R_m+2)\frac{A_{n;\tilde{R}_{m-1}}}{2A_{n+1;\tilde{R}_{m-1}}}\mu_{m:m:n+1}^{(R_1,\ldots,R_{m-1},R_m+1)^{(k+1)}}]. \end{aligned} \tag{2.8}$$

**Proof**

Form (2.1), we have

$$\mu_{m:m:n}^{(R_1,R_2,\ldots,R_m)(k)} = A_{n;\tilde{R}_{m-1}} \int\int_{0<x_1<\ldots<x_{m-1}<\infty}\cdots\int \left\{\int_{x_{m-1}}^{\infty} x_m^k[1-F(x_m)]^{R_m} f(x_m)dx_m\right\}$$
$$\times \; f(x_1)[1-F(x_1)]^{R_1}\ldots f(x_{m-1})[1-F(x_{m-1})]^{R_{m-1}}dx_1\ldots dx_{m-1}, \tag{2.9}$$

which upon using (1.5) and integrating the innermost integral by parts, we obtain (2.8).

## 2.2 Product moments

The $(i,j)-th$ product moment of progressively Type-II right censored order statistics from half the logistic distribution can be written from (1.1) as

$$\begin{aligned}\mu_{i,j:m:n}^{(R_1,\ldots,R_m)} &= E[X_{i:m:n}^{(R_1,\ldots,R_m)}X_{j:m:n}^{(R_1,\ldots,R_m)}]\\ &= A_{n;\tilde{R}_{m-1}} \int\int_{0<x_1<x_2<\ldots<x_m<\infty}\cdots\int x_i x_j f(x_1)[1-F(x_1)]^{R_1}\\ &\times \; f(x_2)[1-F(x_2)]^{R_2}\ldots f(x_m)[1-F(x_m)]^{R_m}dx_1\ldots dx_m, \end{aligned}\tag{2.10}$$

where $f(.)$ and $F(.)$ are given respectively, by (1.3) and (1.4) and $A_{n;\tilde{R}_{m-1}}$ is defined in (1.2). The product moment defined in (2.10) satisfies the following recurrence relations.

**Relation 2.4**

For $1 \le i < j \le m-1$ and $m \le n$,

$$\begin{aligned}\mu_{i,j:m:n+1}^{(R_1,\ldots,R_j+1,\ldots,R_m)} &= \frac{2}{(R_j+2)}\frac{A_{n+1;\tilde{R}_{j-1}}}{A_{n;\tilde{R}_{j-1}}}\Big[(R_j+1)\mu_{i,j:m:n}^{(R_1,\ldots,R_m)} - \mu_{i:m:n}^{(R_1,\ldots,R_m)}\\ &- \; (n-R_1-\ldots-R_{j-1}-j+1)\mu_{i,j-1:m-1:n}^{(R_1,\ldots,R_{j-1}+R_j+1,\ldots,R_m)}\\ &+ \; (n-R_1-\ldots-R_j-j)\mu_{i,j:m-1:n}^{(R_1,\ldots,R_j+R_{j+1}+1,\ldots,R_m)}\\ &- \; \frac{A_{n;\tilde{R}_j}}{2A_{n+1;\tilde{R}_{j-1}}}\mu_{i,j:m-1:n+1}^{(R_1,\ldots,R_{j-1},R_j+R_{j+1}+2,\ldots,R_m)}\\ &+ \; \frac{A_{n;\tilde{R}_{j-1}}}{2A_{n+1;\tilde{R}_{j-2}}}\mu_{i,j-1:m-1:n+1}^{(R_1,\ldots,R_{j-2},R_{j-1}+R_j+2,\ldots,R_m)}\Big]. \end{aligned}\tag{2.11}$$

**Proof**

From (2.10), we write

$$\mu_{i:m:n}^{(R_1,\ldots,R_m)} = E[X_{i:m:n}^{(R_1,\ldots,R_m)}\{X_{j:m:n}^{(R_1,\ldots,R_m)}\}^0]$$

$$= A_{n;\tilde{R}_{m-1}} \int_{0<x_1<...<x_{j-1}<x_{j+1}<...<x_m<\infty} \cdots \int \cdots \int x_i$$
$$\times \left\{ \int_{x_{j-1}}^{x_{j+1}} x_j^0 f(x_j)[1-F(x_j)]^{R_j} dx_j \right\} f(x_1)[1-F(x_1)]^{R_1}$$
$$\times \ldots \times f(x_{j-1})[1-F(x_{j-1})]^{R_{j-1}} f(x_{j+1})[1-F(x_{j+1})]^{R_{j+1}}$$
$$\times \ldots \times f(x_m)[1-F(x_m)]^{R_m} dx_1 ... dx_{i-1} dx_{i+1} ... dx_m, \tag{2.12}$$

which upon using (1.5) and integrating the innermost integral by parts and simplify, we get (2.11).

**Relation 2.5**

For $1 \leq i \leq m-1$ and $m \leq n$,

$$\mu_{i,m:m:n+1}^{(R_1,...,R_m+1)} = \frac{2}{(R_m+2)} \frac{A_{n+1;\tilde{R}_{m-1}}}{A_{n;\tilde{R}_{m-1}}} \Bigg[ -\mu_{i:m:n}^{(R_1,...,R_m)+R_m+2)} + (R_m+1)\mu_{i,m:m:n}^{(R_1,...,R_m)}$$
$$- (n-R_1-...-R_{m-1}-m+1)\mu_{i,m-1:m-1:n}^{(R_1,...,R_{m-1}+R_m+1)}$$
$$+ \frac{A_{n;\tilde{R}_{m-1}}}{2A_{n+1;\tilde{R}_{m-2}}} \mu_{i,m-1:m-1:n+1}^{(R_1,...,R_{m-1}} \Bigg]. \tag{2.13}$$

**Proof**

The proof can be done easily by following the same manner as in Relation 2.4.

## 2.3 Deductions

For some different censoring schemes, we deduce some special cases from the recurrence relations for the single and product moments established by Relations 2.1-2.5 as follows:

1. When $R_1 = ... = R_m = 0$, so that $m = n$, in which case the progressively censored order statistics reduce to the usual order statistics $X_{1:n}, X_{2:n}, ..., X_{n:n}$, then

   (a) Relation 2.1 reduces to

$$\mu_{1:n}^{(k)} = \frac{1}{(k+1)}\mu_{1:n}^{(k+1)} + \frac{(n-1)}{(k+1)}\mu_{1:n-1:n}^{(1,...,0)^{(k+1)}}$$
$$- \frac{n(n-1)}{2(k+1)(n+1)}\mu_{1:n-1:n+1}^{(2,...,0)^{(k+1)}} - \frac{1}{(k+1)}\frac{n}{n+1}\mu_{1:n:n+1}^{(1,0,...,0)^{(k+1)}} \tag{2,14}$$

and

$$\mu_{1:1:n+1}^{(1)^{(k+1)}} = 2(\mu_{1:1:n}^{(k+1)} - \frac{(k+1)}{n}\mu_{1:1:n}^{(k)}). \tag{2.15}$$

(b) Relation 2.2 reduces to

$$\begin{aligned}\mu_{i:n:n+1}^{(0,\ldots,1,\ldots,0)(k+1)} &= \frac{(n+1)}{(n-i+1)}\left\{\mu_{i:n:n}^{(k+1)} - (k+1)\mu_{i:n:n}^{(k)} + (n-i)\mu_{i:n-1:n}^{(0,\ldots,1,\ldots,0)(k+1)}\right\} \\ &- \frac{(n-i)}{2}\mu_{i:n-1:n+1}^{(0,\ldots,2,\ldots,0)(k+1)} - (n+1)\mu_{i-1:n-1:n}^{(0,\ldots,1,\ldots,0)(k+1)} \\ &+ \frac{1}{2}(n-i+2)\mu_{i-1:n-1:n+1}^{(0,\ldots,2,\ldots,0)(k+1)}. \end{aligned} \tag{2.16}$$

(c) Relation 2.3 reduces to

$$\begin{aligned}\mu_{n:n}^{(k+1)} &= [(k+1)\mu_{n:n}^{(k)} + \mu_{n-1:n-1:n}^{(0,\ldots,1)(k+1)} - \frac{2}{(n+1)}\mu_{n-1:n-1:n+1}^{(0,\ldots,2)(k+1)} \\ &+ \frac{1}{(n+1)}\mu_{n:n:n+1}^{(0,\ldots,1)(k+1)}]. \end{aligned} \tag{2.17}$$

(d) Relation 2.4 reduces to

$$\begin{aligned}\mu_{i,j:n:n+1}^{(0,\ldots,1,\ldots,0)} &= \frac{(n+1)}{(n-j+1)}[\mu_{i,j:n} - \mu_{i:n}] - (n+1)\mu_{i,j-1:n-1:n}^{(0,\ldots,1,\ldots,0)} \\ &+ \frac{(n+1)}{(n-j+1)}(n-j)\mu_{i,j:n-1:n}^{(0,\ldots,1,\ldots,0)} - \frac{(n-j)}{2}\mu_{i,j:n-1:n+1}^{(0,\ldots,2,\ldots,0)} \\ &+ \frac{(n-j+2)}{2}\mu_{i,j-1:n-1:n+1}^{(0,\ldots,2,\ldots,0)}]. \end{aligned} \tag{2.18}$$

(e) Relation 2.5 reduces to

$$\mu_{i,n:n:n+1}^{(0,\ldots,1)} = (n+1)[-\mu_{i:n} - \mu_{i,n-1:n-1:n}^{(0,\ldots,1)} + \mu_{i,n:n}] + \mu_{i,n-1:n-1:n+1}^{(0,\ldots,2)}. \tag{2.19}$$

2. When $R_1 = R_2 \ldots = R_{j-1} = 0$, so that there is no censoring before the time of the $j$-th failure, then the first $j$ progressively Type-II right censored order statistics are simply the first $j$ usual order statistics. Based on such censoring scheme, we deduce the following:

(a) Relation 2.1 reduces to

$$\mu_{1:n+1}^{(k+1)} = 2(\mu_{1:n}^{(k+1)} - \frac{(k+1)}{n}\mu_{1:n}^{(k)}), \tag{2.20}$$

which is the relation established by Balakrishnan (1985).

(b) Relation 2.2 reduces to

$$\begin{aligned}(k+1)\mu_{i:n}^{(k)} &= \frac{(n-i+1)}{2(n+1)}[2(n+1)\mu_{i:n}^{(k+1)} \\ &+ (n-i+2)\mu_{i-1:n+1}^{(k+1)}] - (n-i+1)\mu_{i-1:n}^{(k+1)}) \\ &- \frac{(n-i+1)(n-i+2)}{2(n+1)}\mu_{i:n+1}^{(k+1)}. \end{aligned} \tag{2.21}$$

hence, we have

$$(n+1)\mu_{i:n}^{(k+1)} = i\mu_{i+1:n+1}^{(k+1)} + (n-i+1)\mu_{i:n+1}^{(k+1)}, \tag{2.22}$$

[see Balakrishnan and Cohen (1991, p.24)], also from (2.22), we have

$$(n+1)\mu_{i-1:n}^{(k+1)} = (i-1)\mu_{i:n+1}^{(k+1)} + (n-i+2)\mu_{i-1:n+1}^{(k+1)}. \tag{2.23}$$

Thus,

$$\begin{aligned} 2(n+1)\mu_{i:n}^{(k+1)} + (n-i+2)\mu_{i-1:n+1}^{(k+1)} &= 2i\mu_{i+1:n+1}^{(k+1)} + 2(n-i+1)\mu_{i:n+1}^{(k+1)} \\ &+ (n+1)\mu_{i-1:n}^{(k+1)} - (i-1)\mu_{i:n+1}^{(k+1)}. \end{aligned} \tag{2.24}$$

Substituting into(2.16), we get

$$\mu_{i+1:n+1}^{(k+1)} = \frac{1}{i}\left[\frac{(n+1)(k+1)}{n-i+1}\mu_{i:n}^{(k)} + \frac{n+1}{2}\mu_{i-1:n}^{(k+1)} - \frac{n-2i+1}{2}\mu_{i:n+1}^{(k+1)}\right] \tag{2.25}$$

which is the same relation established by Balakrishnan (1985).

(c) Relation 2.3 reduces to

$$\mu_{n+1:n+1}^{(k+1)} = \frac{1}{n}\left[(n+1)(k+1)\mu_{n:n}^{(k)} + \frac{n+1}{2}\mu_{n-1:n}^{(k+1)} + \frac{n-1}{2}\mu_{n:n+1}^{(k+1)}\right] \tag{2.26}$$

which is the same relation established by Balakrishnan (1985).

(d) Relation 2.4 reduces to

$$\mu_{i,j:n+1} = \mu_{i,j-1:n+1} - \frac{2(n+1)}{n-j+2}\{\mu_{i,j:n} - \mu_{i,j-1:n} - \frac{1}{n-j+1}\mu_{i:n}\} \tag{2.27}$$

which is the same relation established by Balakrishnan (1985).

## 3. ESTIMATION

In this section, we apply two different methods to estimate the location and scale parameters of the half logistic distribution.

### 3.1 Maximum likelihood estimates

In this part, we shall derive the maximum likelihood estimates of the location and scale parameters of the half logistic distribution. Let $X_{1:m:n}, X_{2:m:n}, \ldots, X_{m:m:n}$

denote a progressively Type-II right censored sample from the location-scale half logistic distribution with density function

$$f(x;\theta,\sigma) = \frac{2}{\sigma}\frac{e^{(x-\theta)/\sigma}}{(e^{(x-\theta)/\sigma}+1)^2}, \qquad \theta \le x < \infty, \quad \sigma > 0. \tag{3.1}$$

The likelihood function to be maximized for estimates of $\theta$ and $\sigma$(which will be denoted by $\hat{\theta}$ and $\hat{\sigma}$) is

$$L(\theta,\sigma) = (Constant)(\frac{1}{\sigma})^m \prod_{i=1}^{m} \frac{e^{(x_{i:m:n}-\theta)/\sigma}}{(e^{(x_{i:m:n}-\theta)/\sigma}+1)^{R_i+2}}, \quad x_{1:m:n} \ge \theta. \tag{3.2}$$

Upon inspection of this likelihood, it is evident that it is an increasing function of $\theta$. Therefore, the maximum must lie on the line

$$\hat{\theta} = x_{1:m:n}. \tag{3.3}$$

Differentiating likelihood function given in (3.2) with respect to $\sigma$, the resulting equation to be solved for the maximum likelihood estimate of $\sigma$ is given by,

$$m + \sum_{i=1}^{m} \frac{(x_{i:m:n}-x_{1:m:n})}{\sigma}\{1 - \frac{(R_i+2)e^{(x_{i:m:n}-x_{1:m:n})/\sigma}}{(e^{(x_{i:m:n}-x_{1:m:n})/\sigma}+1)}\} = 0. \tag{3.4}$$

Since (3.4) cannot be solved analytically for $\sigma$, some numerical methods such as bisection method must be employed.

### 3.2 Approximate best linear unbiased estimate

Following the technique proposed by Balakrishnan and Aggarwala(2000), we calculate the approximate best linear unbiased estimation (ABLUEs) for the location and scale parameter $\theta^*$ and $\sigma^*$ as follows

$$\theta^* = \sum_{i=1}^{n} A_i Y_{i:m:n}^{(R_1,\ldots R_m)}, \tag{3.5}$$

$$\sigma^* = \sum_{i=1}^{n} B_i Y_{i:m:n}^{(R_1,\ldots R_m)}, \tag{3.6}$$

where $A_i$ and $B_i$ are the coefficient of the ABLUEs.

## 4. NUMERICAL ILLUSTRATIONS

A progressively Type-II right censored samples from half logistic distribution with location parameter $\theta = 0.0$, and scale parameter $\sigma = 1.0$ was generated based on 10,000 Monte Carlo runs, we calculate the coefficients of the ABLUEs of the location and scale parameters $\theta$ and $\sigma$ in Table 4.2. Also we calculate the MSEs and the Bias of the location and scale parameters $\theta$ and $\sigma$ based on the ABLUEs in Table 4.3.

**Table 4.1.** Sample sizes and censoring schemes from half logistic distribution

| n | m | censoring scheme |
|---|---|---|
| 8 | 3 | **Sch1** = $(1, 0, 4)$ |
| 10 | 4 | **Sch2** = $(1, 0, 1, 4)$ |
| 15 | 5 | **Sch3** = $(4, 0, 0, 0, 6)$ |
| 20 | 6 | **Sch4** = $(7, 0, 1, 0, 0, 6)$ |
| 30 | 7 | **Sch5** = $(10, 1, 0, 2, 0, 0, 10)$ |
| 35 | 8 | **Sch6** = $(10, 3, 1, 1, 0, 2, 0, 10)$ |

**Table 4.2.** Coefficients of the ABLUEs of $\theta$ and $\sigma$ from half logistic distribution.

| $n$ | $m$ | $A_i$ | $B_i$ |
|---|---|---|---|
| 8 | 3 | 1.42016 | -1.91947 |
| | | -.01685 | 14636 |
| | | -.40330 | 1.77311 |
| 10 | 4 | 1.28339 | -1.61811 |
| | | -.00167 | .07404 |
| | | -.04684 | .30371 |
| | | -.23488 | 1.24036 |
| 15 | 5 | 1.17579 | -1.47210 |
| | | .00072 | 04373 |
| | | -.00452 | 07687 |
| | | -.00969 | 10888 |
| | | -.16232 | 1.24262 |
| 20 | 6 | 1.12306 | -1.36474 |
| | | .00109 | 02849 |
| | | -.01040 | 14534 |
| | | -.00449 | 07528 |
| | | -.00735 | 09831 |
| | | -.10192 | 1.01733 |
| 30 | 7 | 1.10576 | -1.70793 |
| | | -.00357 | 08679 |
| | | -.00024 | 02974 |
| | | -.01192 | 20843 |
| | | -.00247 | 05837 |
| | | -.00364 | 07298 |
| | | -.08391 | 1.25161 |
| 35 | 8 | 1.07967 | -1.80177 |
| | | .19969 | -.34153 |
| | | .08948 | -.00691 |
| | | .04059 | 07030 |
| | | .03717 | -.01023 |
| | | -.02854 | 24769 |
| | | .00244 | 07243 |
| | | -.42051 | 1.77002 |

As a check of the entries of Table 4.2, for the coefficients of the ABLUEs, we see that $\sum_{i=1}^{n} A_i \simeq 1$ and $\sum_{i=1}^{n} B_i \simeq 0$.

**Table 4.3.** Bias and MSEs for the ABLUEs of $\theta$ and $\sigma$ from half logistic distribution

| n | m | scheme | BIAS($\theta^*$) | BIAS($\sigma^*$) | MSE($\theta^*$) | MSE($\sigma^*$) |
|---|---|---|---|---|---|---|
| 8 | 3 | **Sch1** | -0.00502 | 0.03910 | 0.06535 | 0.37507 |
| 10 | 4 | **Sch2** | -0.00730 | 0.03866 | 0.03961 | 0.24651 |
| 15 | 5 | **Sch3** | -0.00225 | 0.02608 | 0.01793 | 0.18331 |
| 20 | 6 | **Sch4** | -0.00162 | 0.02351 | 0.01011 | 0.14412 |
| 30 | 7 | **Sch5** | -0.00094 | 0.01223 | 0.00462 | 0.12587 |
| 35 | 8 | **Sch6** | -0.00126 | 0.01273 | 0.00337 | 0.10787 |

From Table 4.3, we see that as n increases, the mean square errors $MSE(\theta^*)$ and $MSE(\sigma^*)$ decrease for all censoring schemes.

**Example:**
A progressively Type-II right censored sample of size $m = 4$ from a sample of size $n = 10$ from half logistic distribution with $\theta = 0.0$, $\sigma = 1.0$, and censoring scheme $Sch = (1, 0, 1, 4)$, was simulated using SUBROUTINES RNUN and BLINF in IMSL, based on the algorithm given in Balakrishnan and Aggarwalla (2000) page 34. The simulated progressively Type-II right censored sample is

$$0.1097, 0.1718, 0.5012, 0.5122.$$

By making use of equations (3.5) and (3.6), then using the coefficients $A_i$ and $B_i$ for $n = 10$, $m = 4$, in Table 4.2 we determine the ABLUEs of $\theta$ and $\sigma$ as follows:

$$\begin{aligned}\theta^* &= (1.28339 \times 0.1097) + (-.00167 \times 0.1718) + (-.04684 \times 0.5012) \\ &\quad +(-.23488 \times 0.5122) = -0.0033\end{aligned}$$

and

$$\begin{aligned}\sigma^* &= (-1.61811 \times 0.1097) + (.07404 \times 0.1718) \\ &\quad +(.30371 \times 0.5012) + (1.24036 \times 0.5122) = 0.6227,\end{aligned}$$

so that we obtain the standard errors of the estimates $\theta^*$ and $\sigma^*$ to be

$$\begin{aligned}SE(\theta^*) &= \sigma^*(Var(\theta^*))^{1/2} = 0.6227 \times (0.0396)^{1/2} = 0.1239, \\ SE(\sigma^*) &= \sigma^*(Var(\sigma^*))^{1/2} = 0.6227 \times (0.2449)^{1/2} = 0.3082.\end{aligned}$$

Starting from Table 4.1, the MSEs and the Bias of the location and scale parameters $\theta$ and $\sigma$ based on the MLEs are given in Table 4.4.

**Table 4.4.** Bias and MSEs of the MLEs of $\theta$ and $\sigma$ from half logistic distribution

| n | m | scheme | BIAS($\hat{\theta}$) | BIAS($\hat{\sigma}$) | MSE($\hat{\theta}$) | MSE($\hat{\sigma}$) |
|---|---|---|---|---|---|---|
| 8 | 3 | **Sch1** | 0.22689 | -0.32862 | 0.09499 | 0.19655 |
| 10 | 4 | **Sch2** | 0.18218 | -0.26453 | 0.06233 | 0.14035 |
| 15 | 5 | **Sch3** | 0.12622 | -0.22104 | 0.03036 | 0.10692 |
| 20 | 6 | **Sch4** | 0.09595 | -0.19169 | 0.01758 | 0.08662 |
| 30 | 7 | **Sch5** | 0.06439 | -0.17404 | 0.00802 | 0.07579 |
| 35 | 8 | **Sch6** | 0.05503 | -0.15308 | 0.00596 | 0.06512 |

From Table 4.4, we see that as n increases, the mean square errors $MSE(\hat{\theta})$ and $MSE(\hat{\sigma})$ decrease for all censoring schemes.
In order to compare the performance of the ABLUES and MLEs, we use the definition of the relative efficiency proposed by Balakrishnan and Lee (1998) as follows

$$\text{Eff}(\theta) = \frac{MSE(\hat{\theta})}{Var(\theta^*)} \times 100, \tag{4.1}$$

and

$$\text{Eff}(\sigma) = \frac{MSE(\hat{\sigma})}{Var(\sigma^*)} \times 100. \tag{4.2}$$

The values of the relative efficiency in (4.1)and (4.2) can be interpreted as follows:

- if $\text{Eff}(\theta) > 100$, we conclude that the estimation of $\theta$ based on the ABLUEs is more efficient than that based on the MLEs.
- if $\text{Eff}(\theta) < 100$, we conclude that the estimation of $\theta$ based on the MLEs is more efficient than that based on the ABLUES.

The relative efficiency between the above methods of estimation for the half logistic distribution are computed in Table 4.5. By using (4.1)and (4.2) in Tables 4.3 and 4.4, we have the relative efficiency for the ABLUEs and MLEs as given in Table 4.5

**Table 4.5.** Relative efficiency between the ABLUEs and the MLEs for half logistic distribution

| n | Eff($\theta$) | Eff($\sigma$) |
|---|---|---|
| 8 | 145.42 | 51.98 |
| 10 | 157.55 | 57.55 |
| 15 | 169.38 | 59.01 |
| 20 | 173.93 | 59.96 |
| 30 | 173.90 | 59.13 |
| 35 | 176.77 | 59.57 |

From Table 4.5 we see that the estimation of $\theta$ based on the ABLUEs is more efficient than that based on the MLEs and the estimation of $\sigma$ based on the MLEs is more efficient than that based on the ABLUEs.

## 5. CONCLUSION

Recurrence relations for the single and product moments of progressively Type-II right censored order statistics from half logistic distribution are derived. In addition, some well known results are deduced as special cases. Next, the maximum likelihood estimators (MLEs) of the location and scale parameters of the half logistic distribution are calculated. In addition, the setup proposed by Balakrishnan and Aggarwala (2000) was used to compute the approximate best linear unbiased estimates (ABLUEs) of the location and scale parameters. Next, a simulation study to compare between the efficiency of the techniques considered for the estimation was carried out.

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