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More on Quick Analysis of Unreplicated Factorial Designs Avoiding Shrinkage and Inflation Deficiencies

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Abstract: Effective and quick methods that are easy to carry out even by hand, or easy to be programmed by hand-held calculators are needed for assessing the sizes of contrasts of unreplicated 2^p factorial designs. Moreover, they have the advantage to use the original numerical measurements which makes the analysis easier to explain. Basically, Lenth (1989) is one of the most familiar of such quick and powerful methods. Later on, Aboukalam (2001) proposes under constant effects an alternative sophisticated method to Lenth's method. The proposed method is the supreme from two considerable powers. The first utmost indicates less inflation deficiency while the other utmost indicates less shrinkage deficiency. Also under constant effects, Al-Shiha (2006) introduces an alternative quick method which is less shrinkage deficiency while the inflation deficiency is the same. If effects are random, Aboukalam (2005) introduces an alternative quick method in which the first power is favored as long as the second power is within a small margin.

In the spirit of quickness and fixed effects, this article adds another method which is supreme from the two considerable powers. The method is based on a one step of the scale-part of a suggested M-estimate for location. Explicitly, we suggest adapting the skipped median (ASKM) estimate. Critical values of ASKM–method, for several sample sizes often used, are empirically computed.

Keywords: *M-estimators for scale; powers; fractional factorials.*

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1. INTRODUCTION

The response in many practical experiments may be affected by a large number of factors, but the most factors or factor interactions are not active. Taking into the concern the experimentation cost, time, effort, and/or limitation of data resources, researchers usually apply for estimating and testing main and interaction effect of each factor at two levels and do not replicate experiments. Therefore, researchers employ unreplicated 2^p factorial experiments and use only one observation per treatment combination. There are many methods in the literature for analyzing unreplicated 2^p factorial experiments. For example, some of these methods are discussed in Lenth (1989), Hamada and Balakrishnan (1995), Loughin and Noble (1997) and Haaland and O'Connell (1995). It was shown that Lenth's method is the most popular technique. In addition to its computations being quick and simple, it is either quite comparable or it provides the best performance among several robust methods. However given constant effects, Aboukalam (2001) proposes a sophisticated and slow technique that dominates Lenth technique by two kinds of proposed powers. Al-Shiha (2006) competes with a first quick alternative that dominates lenthe's technique by the second power only, whereas the first power is almost the same. In the second and for given random effects, Aboukalam (2005) considers the first power as essential and suggests a quick technique that dominates Lenth technique, whereas the second power is neglected as long as it is within a small margin.

In this paper, we consider again constant effects and the two kinds of powers and propose a competing quick technique to Lenth technique. Extensive numerical simulations were conducted under a given level of significance and two selected sample sizes often used in the field, to compare the powers of the new method with Lenth's method. The powers of the suggested method were the supreme. As a result, the Lenth's method is more likely than the proposed method to be exposed to the deficiencies of the shrinkage or the inflation. The shrinkage (inflation) is a deficiency that causes some inactive (active) effects to come out as actives (inactive) effects.

At last, several critical points for different levels of significance and sample sizes were empirically simulated in order to be in the experimenter hand.

2. THEORETICAL OBJECTIVES

Suppose that $\hat{\beta}_1$, $\hat{\beta}_2$,....., $\hat{\beta}_n$ are the best linear unbiased estimators (BLUE) of the main and interaction effects obtained from an unreplicated 2^p factorial experiment in the standard order, where the effects are denoted by $\beta_1, \beta_2, ..., \beta_n$. Under the usual assumptions about the experimental errors: normality, independence, and a common unknown variance σ^2, the estimates $\hat{\beta}_1$, $\hat{\beta}_2$,....., $\hat{\beta}_n$ are independently normally distributed random variables with a common unknown variance $\tau^2=\sigma^2/2^p$ and means $\beta_1, \beta_2, ..., \beta_n$, respectively.

The statistical inference problem may be stated formally as follows. There are n independent normal distributions with unknown means $\beta_1, \beta_2, ..., \beta_n$ and a common unknown variance τ^2. From each distribution, a single observation $\hat{\beta}_j$ (j=1, 2, ..., n) is obtained. The objective is to infer, if any of the β_j significantly differs from zero. In other words, the objective is to determine whether the hypothesis H_o: β_j =0 can be significantly rejected under a level of significance α. The test is performed on the basis of the observation $\hat{\beta}_j$ and a scale estimate S for τ. The hypothesis H_o is rejected if the j-th standardized absolute estimate $SAE_j = |\hat{\beta}_j| / S$ is significantly large, say larger than the critical point, $cr(\alpha, n)$, of the statistic SAE_j . As a result, once a good scale estimate has been chosen, those active effects corresponding to significantly large values of SAE_j could be identified.

3. ADAPTED SKIPPED MEDIAN SCALE (S_{ASKM})

Let $\hat{\beta}_1$, $\hat{\beta}_2, ...,\hat{\beta}_n$ be n estimated effects from the scale model $F(\hat{\beta}/\tau)$, S_0 be an initial robust scale estimate, like $S_0 = 1.4826 \times med|\hat{\beta}_j|$. The one step scale part of M-estimates (see Huber (1981), p.147) is given by the equation:

$$S^2 = S_0^2 \sum_{j=1}^{n} \Psi^2(\hat{\beta}_j/S_0)/[(n-1)\times B] \qquad (1)$$

where $\Psi(.)$ is a skew symmetric function used in the location step and the constant B is chosen to be equal to E(Ψ^2) for consistency at the normality. The squared $\Psi_{SKM}^2(x)$ of the so-called 2.5-skipped median function is defined by : $\Psi_{SKM}^2(x) = 1 \;\; as\, |x| \in (0 , 2.5)$. We propose to adapt this function as follow: $\Psi_{ASKM}^2(x) = 1$, 0.25 as $|x| \in (0 , 1)$ and , $|x| \in (1 , 2.5)$ respectively. For convenience, the divisors B and (n-1) in Expression (1) are corrected before computing the scale S_{ASKM}. As the observations bellow 2.5 are only considered by Ψ_{ASKM}^2, so the divisor (n-1) will be replaced by the number (n_0) of the observations $\hat{\beta}_j$ under $2.5 S_0$. In other words, $n_0 = \# \hat{\beta}_j$; $|\hat{\beta}_j| \le 2.5 S_0$. Moreover, if the divisor B is removed, then nothing is changed in testing the hypothesis H_o because the critical point $cr(\alpha, n)$ is adapted to this removal. Under these considerations, S_{ASKM} is simply equal to $0.5\, S_0 \sqrt{1 + 3n_1 / n_0}$, where $n_1 = \# \hat{\beta}_j$ such that $|\hat{\beta}_j| \le S_0$. Finally, the skipping at the point 2.5 is only to facilitate the comparison with

Lenth-scale, S_L, which takes the easy expression $S_L = 1.5 \times med\{|\hat{\beta}_j| : |\hat{\beta}_j| \le 2.5 S_l\}$, where $S_l = 1.5 \times med|\hat{\beta}_j|$.

4. SIMULATION STUDIES AND RESULTS

4.1 Critical points of the *ASKM* test statistic

Scales are usually compared under similar probabilities (=α) of rejecting the null hypothesis H_o: $\beta_j = 0$ given that all the n affects β_j are not active; i.e. H_o is true. To fulfill this requirement, we should compute the empirical critical point, $cr(\alpha, n)$, of the test statistics SAE_j under H_o. The value of $cr(\alpha, n)$ is the 100(1−α)% quantile point of the distribution of the test statistic. The distribution function of this statistic using S_{ASM} is built empirically on the basis of 10000 samples of size n from the standard normal. Hence, the empirical distribution includes 10000×n values for SAE_j. For some values of α and some selected sample sizes n, the empirical critical points of the test statistics SAE_j based on S_{ASKM} are given in Table 4.1 below.

Table 4.1 Critical points of *ASKM* test statistics

n	$\alpha = 0.20$	$\alpha = 0.15$	$\alpha = 0.10$	$\alpha = 0.05$
15	1.417	1.625	1.913	2.408
31	1.445	1.639	1.901	2.327
63	1.452	1.639	1.885	2.277
127	1.458	1.643	1.884	2.263
255	1.460	1.641	1.878	2.246

4.2 Power comparisons

Both quick techniques of *ASKM* and Lenth will be assessed here on the basis of powers of types I and II (pow_I and pow_{II}) of Aboukalam and Al-Shiha (2001). Pow_I (Pow_{II}) is the probability of declaring an effect as active (inactive) given that it is active (inactive) effect. Hence, under H_o it is fulfilled that $\alpha = 1 - pow_{II}$. In the comparison between any two techniques both powers should be the supreme. Moreover, if pow_I (pow_{II}) declines then this indicates that the related values of the scale have inflated (shrunk) many times. So, the supreme powers scale will save the analysis more from the deficiencies of inflation and shrinkage. Indeed, the shrinkage causes many false active effects, and consequently the further investigations on the data will cost more. On the other hand, the inflation may drop some effects that are actives.

Here, pow_I and pow_{II}, multiplied by 100, of both techniques will be computed and plotted in Figures 4.1 and 4.2. The relevant powers were empirically assessed on the basis of 10000 runs (nr =10000) of standard normal random samples of sizes n = 15 and 31. Let na be the number of active effects introduced in the sample. The first *na* effects were made active effects. An effect β is made active if a possible shift Δ (= 2, 3, 4, 5 and 6) is added to the observation $\hat{\beta}$ (i.e. $\hat{\beta} + \Delta$). The selected number *na* was given the values 1, 2, 4, 6 and 7 as n =15; the values 3, 6, 9, 12 and 15 as n =31. The significant level α is given the relatively high value 0.2 in order to offer the desired active effects more chance to appear in further investigations. Eventually, the powers were calculated as follows:

$$Pow_I(Pow_{II}) = \frac{Num\ of\ declared\ actives(inactives) \mid they\ are\ actives(inactives)}{na \times nr\,\{(n-na) \times nr\}}$$

It is clear in Figures 4.1 and 4.2 that the powers of the *ASKM* - technique (solid) are comparable or greater than those of the Lenth's technique (dashed).

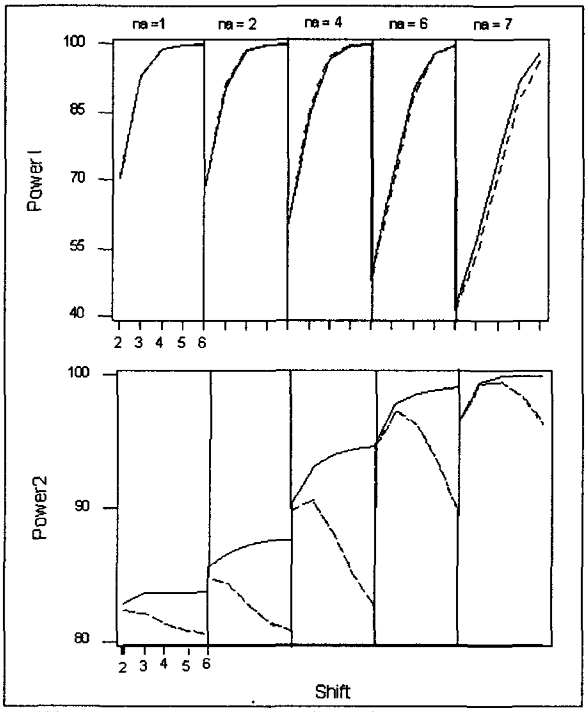


Figure 4.1 Powers I and II of sample size n = 15

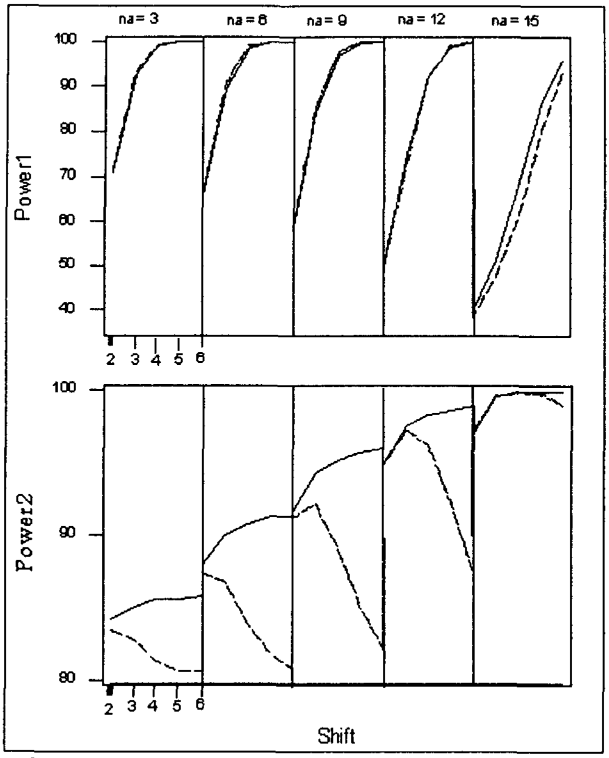
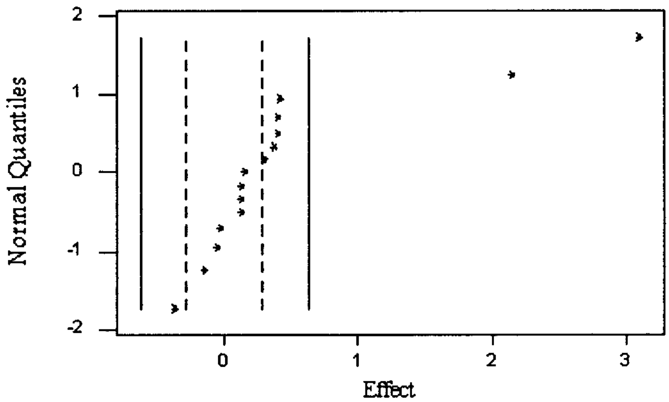


Figure 4.2 Powers I and II of sample size n=31

5. EXAMPLES AND CONCLUDING REMARKS

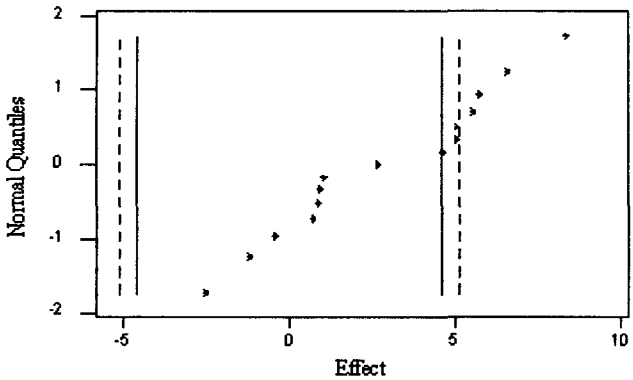
Example 1: Figure 5.1 represents the normal probability plot of Example II of Box and Meyer (1986). The ordered absolute effects of this example are 0.03, 0.05, 0.13, 0.13, 0.13, 0.15, 0.15, 0.30, 0.37, 0.37, 0.40, 0.40, 0.42, 2.15, and 3.10. There are 15 effects, where the last two of them are likely to be real active effect. Lenth's scale starts from $S_l = 1.5 \times med|\hat{\beta}_j| = 1.5 \times 0.3 = 0.45$, and then $S_L = 1.5 \times med\{|\hat{\beta}_j| : |\hat{\beta}_j| \le 2.5 S_l\} = 0.225$. So Lenth bands = $\pm S_l \times$ critical point $= \pm 0.225 \times 1.26 = \pm\ 0.284$. Whereas S_{ASKM} initiates from $S_0 = 1.4826 \times med|\hat{\beta}_j| = 0.44478$, so $n_0 = 13 = \#\hat{\beta}_j ; \{|\hat{\beta}_j| \le 1.11195\}$ and $n_1 = 13 = \#\hat{\beta}_j ; \{|\hat{\beta}_j| \le 0.44478\}$ and hence $S_{ASKM} = 0.5\ S_0\sqrt{1 + 3n_1 / n_0} = 0.44478$. As a result, *ASKM*-bands $= \pm S_{ASKM} \times$ critical point $= \pm 0.44478 \times 1.42 = \pm\ 0.63$. Figure 5.1 shows also the bands of Lenth and *ASKM*- analysis. This example explains how the Lenth's scale may be shrunk which, as a result, caused 6 false active alarms, whereas the *ASKM* –scale was saved from the shrinkage deficiency, and so it detected the two active effects properly.



Effects: **-0.37, -0.15, -0.05, 0.03, 0.13, 0.13, 0.13, 0.15, 0.30, 0.37, 0.40, 0.40, 0.42, 2.15, and 3.10.**
Bands of Lenth (dashed line) = **(-0.283, 0.283)** and Band of *ASKM* (solid line) = **(-0.63, 0.63).**
Figure 5.1: Shrinkage of Lenth's technique

Example 2: Figure 5.2 represents the normal probability plot of the ordered absolute effects: 0.42, 0.73, 0.94, 0.98, 1.11, 1.19, 2.50, 2.73, 4.70, 5.10, 5.11, 5.58, 5.80, 6.65, and 8.42. The data includes 15 effects and 7 of them are actives. Lenth scale S_L= 4.1, and Lenth's bands $= \pm S_l \times$ critical point $= \pm 4.1 \times 1.26 = \pm 5.16$. The *ASKM*- scale has initiated S_0= 4.05. The central interval [0, S_0] = [0, 4.05] contains n_1=8 absolute effects, and the part $[0, 2.5 \times S_0]$=[0, 10.12] contains n_0= 15 effects. As a result, the scale and the bands of the *ASKM*-technique are: $S_{ASKM} = 0.5 \times 4.05\sqrt{1 + 3 \times 8/15}$ = 3.27 and the bands= ±4.62. The active effects has inflated the Lenth's scale and, as a result, let Lenth's scale to drop 3 active effects, whereas *ASKM*- scale has detected all the active effects properly.

To summarize, *ASKM*-technique is quick and is more save from the deficiencies of the shrinkage or the inflation that may happen as Lenth' technique is applied.



Effects: **–2.50, –1.19, –0.42, 0.73, 0.94, 0.98, 1.11, 2.73, 4.70, 5.10, 5.11, 5.58, 5.80, 6.65, and 8.42.**

Bands of Lenth (dashed line)= **(–5.16, 5.16)** and Band of *ASKM* (solid line)= **(–4.62, 4.62).**

Figure 5.2 Inflation of Lenth's technique

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