On asymptotic Stability in nonlinear differential system†

안 정 향*
(Jeong Hyang An)

Abstract We investigate various $\Phi(t)$-stability of comparison differential equations and we abtain necessary and/or sufficient conditions for the uniform asymptotic and exponential asymptotic stability of the nonlinear differential equation $x' = f(t, x)$.

Key Words : local lipschitz conditions, quasimonotone, $\Phi(t)$-stability, $\Phi(t)$-boundedness, $\Phi(t)$-exponential asymptotic stability

1. Preliminaries and Definitions

Lyapunov second methods are now well established subjects as the most powerful techniques of analysis for the stability and qualitative properties of nonlinear differential equations $x' = f(t,x)$, $x(t_0) = x_0 \in R^N$.

One of the original Lyapunov theorems is as follows:

Lyapunov Theorems. For $x' = f(t,x)$, assume that there exists a function $V: R_+ \times S_\rho \to R_+$ such that

(i) V is C^1-function and positive definite,

(ii) V is decresent,

(iii) $\frac{d}{dt}V(t,x) = V_t(t,x) + V_x \cdot f(t,x) \leq -a(\|x\|)$

for $t \geq 0$, $x \in S_\rho$, where $S_\rho = \{x \in R^N \| x \| < \rho\}$ for $\rho > 0$, $a(r)$ is strictly increasing function with $a(0) = 0$.

Then the trivial solution $x(t) \equiv 0$ is uniformly asymptotically stable.

The advantage of the method is that is that it does not require the knowledge of solutions to analyse the stability of the equations. However in practical sense, how to find suitable Laypunov functions V for given equations are the most difficult questions. Hence weakening the conditions (i), (ii), and (iii), and enlarging the class of Lyapunov functions are basic trends in Lyapunov stability theory [2, 3, 4, 5, 6, 11].

In the unified comparison frameworks, Ladde [7] analysed the stability of comparison differential equations by using vector Lyapunov function methods.

Lakeshmikantham and Leela [9] initiated the cone valued Lyapunov function methods to avoid the *quasimonotonicity assumption of comparison* equations. They obtained various useful differential inequalities with cone-valued Lyapunov functions, Akpan and Akinyele [1] extended and generalized the results of [7, 8] to the ϕ_0-stability of the comparison differential equations by using the cone-valued Lyapunov functions.

Here we generalize, in some sense, the results

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* 대구한의대학교 정보보호학과 부교수

of [1] to the $\phi(t)$-stabilities of comparison equations below.

Let R^n denote the n-dimensional Euclidean space with any equivalent norm $\| \cdot \|$, and scalar product $(\ ,\ )$.

$R_+ = [0,\infty)$. $C[R_+ \times R^n, R^n]$ denotes the space of continuous functions from $R_+ \times R^n$ into R^n.

Definition 1.1 ([11]). A proper subset K of R^n is called a cone if (i) $\lambda K \subset K$, $\lambda \geq 0$; (ii) $K+K \subset K$; (iii) $K = \overline{K}$; (iv) $K^\circ \neq \emptyset$; (v) $K \cap (K) = \{0\}$, where $\overline{K}$ and K° denote the closure and interior of K, respectively and ∂K denotes the boundary of K. The order relation on R^n induced by the cone K is defined as follow:

For $x,y \in R^n$, $x \leq_k y$ iff $x-y \in K$, and $x \leq_{k^\cdot} y$ iff $y-x \in K^\circ$.

Definition 1.2 ([11]). The set $K^* = \{\phi \in R^n : (\phi,x) \geq 0)$, for all $x \in K\}$ is called the *adjoint cone* of K if $K*$ itself satisfies Definition 1.1.

Note that $x \in \partial K$ if and only if $(\phi,x) = 0$ for some $\phi \in K^*$, where $K_0 = K - \{0\}$.

Consider the differential equation

$$x' = f(t,x),\ x(t_0) = x_0,\ t_0 \geq 0 \qquad (1)$$

where $f \in C[R_+ \times R^N, R^N]$ and $f(t,0) = 0$ for all $t \geq 0$. Let $S_\rho = \{x \in R^N : \| x \| < \rho\}$, $\rho > 0$. Let $K \subset R^n$ be a cone in R^n, $n \leq N$. For $V \in C[R_+ \times S_\rho, K]$ at $(t,x) \in R_+ \times S_\rho$, let $D^+ V(t,x) = \lim_{h \to 0^+} \frac{1}{h}[V(t+h, x+hf(t,x)) - V(t,x)]$ be a Dini derivative of V along the solution curves of the equations (1).

Consider a comparison differential equation

$$u' = g(t,u),\ u(t_0) = u_0,\ t_0 \geq 0 \qquad (2)$$

where $g \in C[R_+ \times K, R^n]$, $g(t,0) = 0$ for all $t \geq 0$ and K is a cone in R^n.

Let $S(\rho) = \{u \in K : \| u \| < \rho\}$, $\rho > 0$. for $v \in C[R_+ \times S(p), K]$, at $(t,u) \in R_+ \times S(p)$, let $D^+ v(t,u) = \lim_{h \to 0^+} \frac{1}{h}[v(t+h, u+hg(t,u)) - v(t,u)]$ be a Dini derivative of v along solution curves of the equation (2).

Definition 1.3 ([11]). A function $g: D \to R^n$, $D \subset R^n$ is said to be *quasimonotone* nondecreasing relative to the cone K when it satisfies that if $x,y \in D$ with $x \leq_K y$ and $(\phi_0, y-x) = 0$ for some $\phi_0 \in K_{0^*}$, then $(\phi, g(y) - g(x)) \geq 0$.

Definition 1.4 ([8,10]). The trivial solution $x = 0$ of (1) is (S_1) *equistable* if for each $\varepsilon > 0$, $t_0 \in R_+$, there exists a positive function $\delta = \delta(t_0, \varepsilon)$ such that the inequality $\| x_0 \| < \delta$ implies $\| x(t,t_0,x_0) \| < \varepsilon$, for all $t \geq t_0$.

Other stability notions ($S_2 \sim S_8$) can be similarly defined [8,10].

Now we give cone-valued $\phi(t)$-stability definitions of the trival solution of (2).

Let $\phi : [0,\infty] \to K^*$ be a cone-valued function.

Definition 1.5 ([12]). The trivial solution $u = 0$ of (2) is

(S_1^*) $\phi(t)$-equistable if for each $\varepsilon > 0$, $t_0 \in R_+$, there exists a positive function $\delta = \delta(t_0, \varepsilon)$ such that the inequality $(\phi(t_0), u_0) < \delta$ implies $(\phi(t), r(t)) < \varepsilon$, for all $t \geq t_0$ where $r(t)$ is a maximal solution of (2);

(S_2^*) *uniformly* $\phi(t)$-stable if the δ in (S_1^*) is independent of t_0;

Other $\phi(t)$-stability notions ($S_3^* \sim S_8^*$) can be similarly defined [12].

Definition 1.6 The trivial solution $u = 0$ of (2) is

(B_1^*) $\phi(t)$-equibounded if for each $\alpha \geq 0$, $t_0 \in R_+$ there exist $\beta = \beta(t_0, \alpha)$ such that the inequality $(\phi(t_0), u_0) \leq \alpha$ implies $(\phi(t), r(t)) < \beta$ for all $t \geq t_0$, where $r(t)$

is maximal solution of (2);

(B_2^*) $\phi(t)$-uniform bounded if the β in (B_1^*) is independent of to ;

The definition (B_3^*~B_8^*) may be formulated similarly.

Lemma 1.7 ([1]). Let $\|\cdot\|_P: K \to K$ be a generalized norm and let $g \in C[R_+ \times K, R^n]$, $\| g(t,u_1) - g(t,u_2) \|_P$, $L \in C[R_+, R_+]$ for $(t,u_1), (t,u_2) \in R_+ \times K$.

If u, v are two solutions of (2) through (t_0,u_0) and (t_0,v_0), respectively, then for $t \geq t_0$ we have

$$\| u_0 - v_0 \|_P \exp\left[-\int_{t_0}^{t} L(s)ds \right]$$
$$\leq_K \| u - v \|_P$$
$$\leq_K \| u_0 - v_0 \|_P \exp\int_{t_0}^{t} L(s)ds$$

2. Stability Theorems

Theorem 2.1 ([12]). Assume that

(i) $V \in C[R_+ \times S_\rho, K]$, $V(t,x)$ is locally Lipschitzian in x relative to K and for $(t,x) \in R_+ \times S_\rho$, $D^+ V(t,x) \leq_K g(t, V(t,x))$,

(ii) $g \in C[R_+ \times K, R^n]$ and $g(t,u)$ is quasimonotone in u relative to K for each $t \in R_+$,

(iii) there exist $a,b \in K$ such that for some $\phi(t) \in K_{0*}$, for each $x \in S_P$, $b(\| x \|) \leq (\phi(t), V(t,x)) \leq a(\| x \|)$, $t \geq t_0 \geq 0$

Then the trivial solution $x=0$ of (1) has the corresponding one of the stability (S_1~S_8) properties if the trivial solution $u=0$ of (2) has each one of the $\phi(t)$-stability (S_1^*~S_8^*) properties in Definition 1.5.

Theorem 2.2 Let conditions (i) and (ii) of Theorem 2.1 hold. Assume further that for $c>0$, $d>0$, $(\phi(t),u(t)) \leq \| x_0 \|^d$ and $c\| x \|^d \leq (\phi(t), V(t,x))$. If the trivial solution $u=0$ of (2) is exponentially asymptotically $\phi(t)$-stable, then the trivial solution $x=0$ of (1) is exponentially asymptotically stable.

Proof) Let $x(t,t_0,x_0)$ be any solution of (1) such that $V(t_0,x_0) \leq_K y_0$. Then by Theorem 3.1 in [9], we have $V(t,x) \leq_K r(t)$. Since the trivial solution $u=0$ of (2) s exponentially asymptotically $\phi(t)$-stable, then there exist $\sigma > 0$, $\alpha > 0$ both real number such that $(\phi(t), r(t)) \leq \sigma(\phi(t), u(t)) \exp[-\alpha(t-t_0)]$, $t \geq t_0$ and $C\| x \|^d \leq \sigma(\phi(t), u(t)) \exp[-\alpha(t-t_0)]$

This implies that

$$\| x \| \leq M \| x_0 \| \exp[-\beta(t-t_0)], \quad t \geq t_0, \quad \frac{\sigma}{c} = M, \quad \frac{\alpha}{d} = \beta.$$

Theorem 2.3 Assume that

(i) $g \in C[R_+ \times K, R^n]$, $g(t,0)=0$ and $g(t,u)$ is quasimonotone in u relative to K for each $t \in R_+$ and for $(t,u),(t,v) \in R_+ \times K$ and

$L \in C[R_+, R_+]$,
$\| g(t,u) - g(t,v) \|_P \leq_K L(t) \| u - v \|_P$.

If the trivial solution $u=0$ of (2) is generalized exponentially asymptotically $\phi(t)$-stable, then there exists a cone-valued Lyapunov function v with the following properties :

(i) $v \in C[R_+ \times S^*(\rho), K]$, $v(t,0)=0$, and $v(t,u)$ is locally Lipschitzian in u relative to K for each $t \in R_+$ and for a continuous function $\beta(t) \geq 0$.

(ii) $(\phi(t), \| r(t) \|_P) \leq (\phi(t), v(t,u))$
$\leq \sigma(t,t_0)(\phi(t), \| r(t) \|_P)$
for some $\phi(t) \in K_0^*$, $\sigma \in C[R_+ \times R_+, R_+]$, $(t,u) \in R_+ \times S^*(\rho)$.

(iii) $D^+ v(\phi(t), v(t,u)) \leq_K -p'(t)(\phi(t), v(t,u))$, for $(t,u) \in R_+ \times S^*(\rho)$, $p'(t)$ exists, $p(t)$ is bounded and increasing.

Proof) Define a cone-valued function as

$$v(t,u) = sup_{\delta \geq 0}\{ \| u(t+\delta, t, u) \|_P$$

$$\times \exp(-p(t+\delta)+p(t))\},$$

where $u(t,t_0,u_0)$ are solutions of (2) passing through (t_0,u_0) and by (i) are continuous. Obviously $v(t,0)=0$.

Now for $(t,u_1),(t,u_2)\in R_+\times S^*(\rho)$, we have by Lemma 1.7 that

$$\begin{aligned}
&\| v(t,u_1)-v(t,u_2)\|_P \\
&= \| sup_{\delta\ge 0}\{\| u_1(t+\delta,t,u_1)\|_P \\
&\qquad \times \exp(-p(t+\delta)+p(t))\} \\
&\quad - \| sup_{\delta\ge 0}\{\| u_2(t+\delta,t,u_2)\|_P \\
&\qquad \times \exp(-p(t+\delta)+p(t))\}\|_P \\
&\le_K \| sup_{\delta\ge 0}\{\| u_1(t+\delta,t,u_1)-u_2(t+\delta,t,u_2)\|_P \\
&\qquad \times \exp(-p(t+\delta)+p(t))\}\|_P \\
&\le_K \| sup_{\delta\ge 0}\{\exp(-p(t+\delta)+p(t))\}\| u_1-u_2\|_P \\
&\qquad \times \exp\int_{t_0}^{t} L(s)ds \\
&= \beta(t)\| u_1-u_2\|_P ,
\end{aligned}$$

where

$$\beta(t)=sup_{\delta\ge 0}\{\exp(-p(t+\delta)+p(t))\} \times \exp\int_{t_0}^{t} L(s)ds \ge 0 .$$

For $\delta=0$, and by the uniqueness of solution of (2) we have $\| r(t)\|_P \le v(t,u)$ so that $(\phi(t),\| r(t)\|_P \le (\phi(t),v(t,u))$.

Since $u=0$ of (2) is generalized exponentially asymptotically $\phi(t)$-stable, we have, using Lemma 1.7 that

$$\begin{aligned}
&(\phi(t),v(t,u)= \\
&\le sup_{\delta\ge 0}\{\exp(-p(t+\delta)+p(t))(\phi(t),\| u(t+\delta,t,u\|_P)\} \\
&\le sup_{\delta\ge 0}\{\exp(-p(t+\delta)+p(t))(\phi(t),\| u(t+\delta,t,u\|_P)\} \\
&\le sup_{\delta\ge 0}\{\exp(-p(t+\delta)+p(t))M(t+\delta)(\phi(t),\| u_0\|_P) \\
&\qquad \times \exp(p(t_0)-p(t+\delta))\} \\
&= \beta(t,t_0)(\phi(t),\| u_0\|_P) \\
&\le \beta(t,t_0)(\phi(t),\| r(t)\|_P)\exp\int_{t_0}^{t} L(s)ds \\
&= \sigma(t,t_0)(\phi(t),\| r(t)\|_P),
\end{aligned}$$

where

$$\sigma(t,t_0)= sup_{\delta\ge 0}\{M(t+\delta)\exp(p(t_0)-p(t))\} \times \exp\int_{t_0}^{t} L(s)ds.$$

The proof of (c) is similar to that of (c) in Theorem 2.7 ([13]). This completes the proof of Theorem 2.3.

Let $H=\{a\in C[R_+,R_+] \mid a(t)$ is strictly increasing in t and $a(0)=0\}$.

Theorem 2.4 Assume that there exist function $V(t,x)$ and $g(t,u)$ with the following properties:

(i) $g\in C[R_+\times K,R^n]$, $g(t,0)=0$, $g(t,u)$ is quasimonotone in u relative to K

(ii) $V\in C[R_+\times S_\rho,K]$, $K\subset R^n$, $V(t,0)=0$, $V(t,x)\in R_+\times S_\rho$, $b(\| x\|)\le(\phi(t),V(t,x))$, $t\ge t_0\ge 0$ where $b\in H$ on the interval $0\le u\le \propto$ and $b(u)\to\infty$ as $u\to\infty$.

(iii) $D^+V(t,x)\le_k g(t,V(t,x))$, $(t,x)\in R_+\times S_\rho$.

Then, the $\phi(t)$-equiboundedness of the system (2) implies the equiboundedness of the system (1)

Proof) Let $\alpha\le 0$ and $t_0\in R_+$ be given, and let $\| x_0\|\le\alpha$.

Since the equation (2) is $\phi(t)$-equibounded, given $\alpha_1\ge 0$ and $t_0\in R_+$, there exists a $\beta_1=\beta_1(t_0,\alpha)$ that is continuous in t_0 for each α such that $(\phi(t),r(t,t_0,u_0))\le\beta_1$, $t\ge t_0$ provide $(\phi(t),u_0)\le\alpha_1$.

Moreover, as $b(u)\to\infty$ as $u\to\infty$, we can choose a $L=L(t_0,\alpha)$ verifying the relation $\beta_1(t_0,\alpha)\le b(L)$

Now let $u_0=V(t_0,x_0)$ in K.

Then, assumption (iii) and Lemma 1.4 show that

$$V(t,x(t,t_0,x_0))\le_k r(t,t_0,u_0),\ t\ge t_0$$

where $r(t,t_0,u_0)$ is the maximal solution of the system (2).

Suppose, if possible, that there is a solution $x(t,t_0,x_0)$ with $\| x_0 \| \le \alpha$ having that property that, for some $t_1 > t_2$, $\| x(t_1,t_0,x_0) \| = L$.

Then,

$b(L) = b(\| x \|) \le (\phi(t), V(t,x)) \le$

$(\phi(t), r(t,t_0,u_0)) < \beta_1(t_0,\alpha) \le b(L)$.

The proof is complete, since this contradiction implies that (B_1) holds.

Theorem 2.5 Let the conditions of Theorem 2.4 hold with $b(\| x \|) \le (\phi(t), V(t,x))$ being replaced by

$b(\| x \|) \le (\phi(t), V(t,x)) \le a(\| x \|)$

where $a \in H$.

Then, if the system (2) is $\phi(t)$-uniform bounded, the system (1) is likewise uniform bounded.

Proof) We choose $\alpha_1 = a(\alpha)$, which is independent of t_0. Since $\beta_1 = \beta_1(\alpha)$ in this case, it is easy to see from the choice of L that it is also independent of t_0.

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안 정 향 (Jeong Hyang An)

• 대구대학교 대학원 수학과(이학석사)
• 대구대학교 대학원 수학과(이학박사)
• 대구한의대학교 정보학부 부교수
• 관심분야 : 해석학(미분시스템의 안정성 이론) 수치해석