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A Note on Interval Approximation of a Fuzzy Number

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Abstract

Chanas(2001) introduced the notion of interval approximation of a fuzzy number with the condition that the width of this interval is equal to the width of the expected interval. In this note, this condition is relaxed and the resulting formulae are derived for determining the approximation interval. This interval is compared with the expected interval and approximation interval of a fuzzy number as introduced by Chanas.

Keywords : Fuzzy sets, Interval approximation

1. Introduction

The notion of an approximation interval of a fuzzy number was introduced by Chanas(2001). It is the interval which fulfills two conditions. In the first, its width is equal to the width of a fuzzy number being approximated. In the second, the Hamming distance between this interval and the approximated number is minimal. But it is more natural to relax the condition that its width is equal to the width of the expected interval. In this note, this condition is relaxed and the formulae derived for determining the approximation interval for a fuzzy number. The introduced notion is compared with the notion of expected interval of a fuzzy number known in literature

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2. The expected interval of a fuzzy number

A fuzzy number A is a fuzzy set of the real line R characterized by means of a membership function $\mu_A(x)$, $\mu: R \to [0,1]$:

$$\mu_A(x) = \begin{cases} 0 & \text{for } x \le a, \\ f_A(x) & \text{for } a \le x \le b, \\ 1 & \text{for } c \le x \le d, \\ g_A(x) & \text{for } d \le x \le b, \\ 0 & \text{for } x \ge b, \end{cases}$$

where f_A and g_A are continuous functions, f_A is increasing, and g_A is decreasing.

We assume that

$$\int_{-\infty}^{+\infty} \mu_A(x)\,dx = w_A < \infty$$

and call the value w_A the width of the fuzzy number A.

The expected interval $EI(A)$ of a fuzzy number A is defined [1, 2, 3] as

$$EI(A) = [E_1^A, E_2^A] = [\int_0^1 f_A^{-1}(t)\,dt, \int_0^1 g_A^{-1}(t)\,dt],$$

where f_A^{-1} and g_A^{-1} are the inverse functions of the function f_A and g_A.

By Theorem 1 (Chanas(2001)), we have that

$$width(EI(A)) = E_2^A - E_1^A = width(A) = w_A.$$

A fuzzy number A is a $L-R$ type if its membership function μ_A from the real numbers mapped into the interval [0,1] has the following form:

$$\mu_A(t) = \begin{cases} R\left(\frac{t-a^+}{\beta}\right) & \text{for } a^+ \le t, \\ L\left(\frac{a_- - t}{\alpha}\right) & \text{for } t \le a_-, \\ 1 & \text{for } t \in [a_-, a^+], \end{cases}$$

where L and R are strictly decreasing and continuous function from $[0,\infty)$ to $[0,1]$ satisfying $L(0) = R(0) = 1$. The parameter α and β are non-negative real numbers. If A is a fuzzy number of the $L-R$ type then as in

Dubois(1987) :

$$EI(A) = [\, a_{-} - \alpha\lambda,\ a^{+} + \beta\rho\,], \tag{1}$$

where

$$\lambda = \int_0^{\infty} L(t)\,dt, \quad \rho = \int_0^{\infty} R(t)\,dt.$$

3. The interval approximation introduced by Chanas

Given a fuzzy number A, then among intervals $I_z = [z, z + w_A]$, $z \in R$, indicate the interval $I_{z_0} = [z_0, z_0 + w_A]$, located the nearest to the fuzzy number A with respect to the Hamming distance, i.e,

$$H(I_{z_0}, A) = \min_z H(I_z, A),$$

$H(I_z, A)$ is the Hamming distance I_z and A:

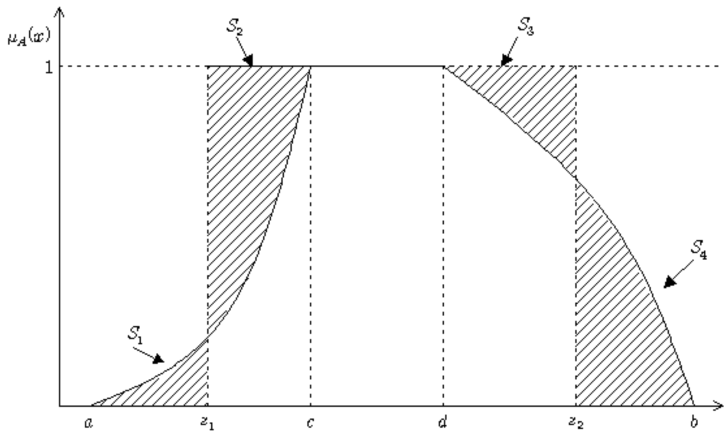
$$H(I_z, A) = \int_{-\infty}^{+\infty} |\chi_{I_z}(x) - \mu_A(x)|\, dx,$$

where χ_{I_z} is the characteristic function of interval I_z. We call the interval $I_{z_0} = [z_0, z_0 + w_A]$ an approximation interval of a fuzzy number A. The width of the interval is equal to the width of the expected interval and it is the nearest to the fuzzy number A, with respect to the Hamming distance, among all the intervals of the same width.

Using the result of Chanas (2001), the function $H(I_z, A)$ takes on its minimal value at the point z_0 fulfilling the following condition:

$$f_A(z_0) = g_A(z_0 + w_A), \tag{2}$$

from which the interval I_{z_0} is the r_0-cut A^{r_0} of the fuzzy number A, where $r_0 = f_A(z_0)$. If A is a fuzzy number of $L-L$ type, the approximation interval of a fuzzy number is equal to expected interval.



<Figure 1> The Hamming distance between I_z and fuzzy number A

4. The interval approximation under the relaxed condition

We now relax the condition that the width of the interval is equal to the width of the expected interval in Section 3.

A fuzzy number A is given. Among intervals $I_z = [z_1, z_2]$, $z_1 \le z_2$, $z_1, z_2 \in R$, indicate the interval $I_{z^*} = [z_1^*, z_2^*]$, located nearest to the fuzzy number A with respect to the Hamming distance, i.e,

$$H(I_{z^*}, A) = \min_z H(I_z, A).$$

The value of $H(I_z, A)$ is equal to the sum of the areas marked in Fig. 1:

$$\begin{aligned} H(I_z, A) &= S_1 + S_2 + S_3 + S_4 \\ &= \int_0^{z_1} f_A(x)\,dx + (c - z_1) - \int_{z_1}^{c} f_A(x)\,dx \\ &\quad + (z_2 - d) - \int_d^{z_2} g_A(x)\,dx + \int_{z_2}^{b} g_A(x)\,dx. \end{aligned}$$

Differentiating $H(I_z, A)$ with respect to z_1, z_2 with $z_1 < z_2$ we obtain

$$\frac{dH(I_z, A)}{dz_1} = 2 f_A(z) - 1$$

$$\frac{dH(I_z, A)}{dz_2} = 1 - 2g_A(z).$$

Then, for

$$f_A^{-1}(1/2) = z_1^*, \quad g_A^{-1}(1/2) = z_1^*, \tag{3}$$

(z_1^*, z_1^*) is the critical point of the two valued function $H(I_z, A)$. Now, the discriminant D

$$\begin{aligned} D &= \frac{d^2H(I_{z^*}, A)}{dz_1^2} \frac{d^2H(I_{z^*}, A)}{dz_2^2} - \left[\frac{d^2H(I_{z^*}, A)}{dz_1 dz_2} \right]^2 \\ &= -4f'_A(z_1^*)g'_A(z_2^*) > 0, \end{aligned}$$

since f_A is increasing and g_A is decreasing. Then by second derivative test, the function $H(I_z, A)$ takes on its minimal value at this point and the interval $I_{z^*} = [z_1^*, z_2^*]$ is the best interval approximation of A with respect to the Hamming distance. It results from condition (3) that the interval I_{z^*} is the 1/2-cut $A^{1/2}$ of the fuzzy number A. Let $w_A^* = z_2^* - z_1^*$. Comparing condition (2) with condition (3), we have that $I_{z^*} = I_{z_0}$ and $w_A = w_A^*$ if $f_A(z_0) = 1/2$, $I_{z^*} \subset I_{z_0}$ and $w_A \geq w_A^*$ if $f_A(z_0) \leq 1/2$, and $I_{z^*} \supset I_{z_0}$ and $w_A \leq w_A^*$ if $f_A(z_0) \geq 1/2$.

If A is a fuzzy number of $L-R$ type, then condition (3) takes the form

$$z_1^* = a_- - \alpha L^{-1}(1/2), \quad z_2^* = a^+ + \beta R^{-1}(1/2). \tag{4}$$

If L and R are linear functions, then $\lambda = L^{-1}(1/2)$ and $\rho = R^{-1}(1/2)$, and hence comparing (1) with (4), we have $EI(A) = I_{z_0} = I_{z^*}$ and $w_A = w_A^*$. If L and R are concave(convex) function, then $\lambda \geq (\leq) L^{-1}(1/2)$ and $\rho \geq (\leq) R^{-1}(1/2)$, and hence we have $EI(A) = I_{z_0} \supset (\subset) I_{z^*}$ and $w_A \geq (\leq) w_A^*$.

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