종방향 진동해석에 비구조적 유한요소 적용

Application of the Unstructured Finite Element to Longitudinal Vibration Analysis

김 치 경†

Kim, Chi-Kyung

(논문접수일 : 2005년 2월 10일 ; 심사종료일 : 2005년 12월 8일)

요 지

본 연구는 파 해석에 있어서 공간-시간 분할 개념을 도입하여 겔러킨 방법으로 해석하였다. 공간-시간 유한요소법은 오직 공간에 대해서만 분할하는 일반적인 유한요소법보다 간편하다. 비교적 큰 시간간격에 대해서 공간과 시간을 동시에 분할하는 방법을 제시하며 가중잔차법이 공간-시간 영역에서 유한요소 정식화에 이용되었다. 큰 시간 간격으로 인하여 문제의 해가 발산하는 경우가 동적인 문제에서 흔히 발생한다. 이러한 결점을 보완한 사각형 공간-시간 요소를 취하여 문제를 해석하고 해의 안정에 대해 기술하였다. 다수의 수치해석을 통하여 이 방법이 효과적 임을 알 수 있었다.

핵심용어 : 유한 요소법, 겔러킨 방법, 가중잔여, 공간-시간, 안정, 파 방정식, 종방향 진동

Abstract

This paper analyzes the continuous Galerkin method for the space-time discretization of wave equation. The method of space-time finite elements enables the simple solution than the usual finite element analysis with discretization in space only. We present a discretization technique in which finite element approximations are used in time and space simultaneously for a relatively large time period called a time slab. The weighted residual process is used to formulate a finite element method for a space-time domain. Instability is caused by a too large time step in successive time steps. A stability problem is described and some investigations for chosen types of rectangular space-time finite elements are carried out. Some numerical examples prove the efficiency of the described method under determined limitations.

keywords : *finite element method, Galerkin method, weighted residual, space-time, stability, wave equation, longitudinal vibration*

1. Introduction

Following the more and more wide-spread idea that finite element methods can be successfully applied to a great variety of problems, we try to use space-time finite elements to solve a problem in one-dimensional wave propagation. Most commonly used numerical methods for wave equations are based upon a space discretization which is independent of time, where we discretize in space using the finite element method and then uses some time-stepping method (Jang, *et al.*, 1998) for the resulting system of ordinary differential equations in time to obtain algebraic equations, which are then solved for the nodal values. This will be a stiff system which will pose extra requirements on the stability of the methods to be used for the time discetization. In order to solve such problems, numerical methods which treat both the space and time uniformly were proposed and numerically tested by many authors(Argyris *et al.*, 1989 ; Bajer, 1986 ; Bruch, *et al.*, 1974 ; Kaczkowski, 1975 ; Kim,

† 책임저자, 정회원 • 인천대학교 안전공학과 교수
전화: 032-770-8294 ; Fax: 032-770-4399
E-mail: kimchi@incheon.ac.kr

• 이 논문에 대한 토론을 2006년 6월 30일까지 본 학회에 보내주시면 2006년 9월호에 그 결과를 게재하겠습니다.

2001 ; Saitoh, *et al.*, 1994 ; Shakib *et al.*, 1991 ; Yu, *et al.*, 1985). They have been considered wide spectrum of problems, including heat conduction, elastodynamics, and advective-diffusive systems associated with fluid dynamics using space-time finite element methods. In this paper we consider simultaneous discretization of space and time variables to be uniform by formulating space-time finite element methods in wave equation. A space-time finite element is a finite element in which an additional time dimension is considered. A one-dimensional beam element has a two- dimensional shape in time and space(*et al.*, 1974). The idea is to use the method of weighted residuals to treat space and time in a uniform manner and thus integrate both the spatial and temporal variations of the unknown quantities simultaneously. A complete space-time finite element discretization eliminates the need for an additional ordinary differential equation solver to discretize time. The approximations are continuous with respect to the space variables for each fixed time at each time step. The discretization is repeated or adjusted for subsequent time slabs using continuous finite element approximations. In problems that involve two or more space dimensions, this allows the use of small elements in regions where the gradients of the solution are large and large elements where the solution has small gradients(Axelsson *et al.*, 1989). The rectangular division of the space-time is assumed and the method will be a conditionally stable one with regard to the time step. In this way we solve all the difficulties with the classical approach: stability, discretization error estimates and global error control are automatically satisfied (Hughes *et al.*, 1987).

2. Time-dependent Wave Propagation Problem

The partial differential equation for wave propagation can be written as

$$\rho u_{tt}(x,t) - \nabla \cdot (E \nabla (x,t)) = q(x,t) \text{ in } \Omega \text{ x} I \quad (1)$$

with boundary and initial conditions as follows:

$$u(0,t) = f(t) \text{ on } \Gamma_1 \text{ x} I$$
$$u_x(L,t) = g(t) \text{ on } \Gamma_3 \text{ x} I$$
$$u(x,0) = u_o(x) \text{ in } \Omega(0),\ x \subseteq \Omega$$
$$u_t(x,0) = v_o(x) \text{ in } \Omega(0),\ x \subseteq \Omega$$

where u(cm) is the displacement of any cross-section for $x \subseteq \Omega$ at time $x \subseteq I = [\,0\,, T\,]$ and where $\boldsymbol{T}$ is a given time, subscript variables indicate partial differentiation with respect to time and space, $\boldsymbol{E}$ is the elastic modulus (kgf/cm^2), $\rho = \rho(x) > 0$ is the density of material (kgf/sec^2/cm^4), q is the external force(kgf) and the boundary and the initial conditions $f(t)$ and $g(t)$ are the prescribed displacements respectively, $u_o(x)$ is the initial displacement, $v_o(x)$ is the initial velocity. The quantity c_x (cm/sec)= $\sqrt{E/\rho}$ is called the velocity of elastic wave propagation. Γ_1 and Γ_3 denote a nonoverlapping subdivision of the boundary Γ of Ω.

3. Discretization in Space and Time

We shall now consider a domain for space and time. Let us consider a time interval $I = \{t : 0 < t < CT\}$ and be partitioned into N pieces by the partition $0 = t_o < t_1 < t_2 < \dots < t_N = T$ and let open spatial interval $\Omega(t)$ be described by $\Omega = \{ x : 0 < x < L \}$ and be partitioned into M pieces by the partition $0 = x_o^n \langle x_1^n \langle x_2^n \langle \cdots \langle x_i^n \langle x_{i+1}^n \langle x_M^n = L$ for $0 \leq i \leq M-1$ and $n \geq 0$. Then the space-time domain is the product space $\Omega \times I$. For the n-th space-time domain, let the spatial domain be subdivided into n_{el} elements, Ω_n^i, $i = 1, 2, 3, \cdots, n_{el}$. Then, for the n-th space-time element the domain is

$$G_n^i = \Omega_n^i \times I_n \quad i = 1,\ 2,\ 3,\ \cdots, n_{el} \qquad (2)$$

with boundary$\Gamma(t)$. Within each space-time element the trial solution and weighting functions are approximated by r-th order interpolation polynomials. These functions are assumed to be continuous within each space-time domain and across the interfaces of the space-time domains, namely at times $t_1, t_2, \cdot\cdot\cdot, t_{N-1}$. We denote by G_n^i an arbitrary bilinear element with vertices $u_{i,n}, u_{i+1,n}, u_{i+1,n+1}$ and $u_{i,n+1}$, where i is a space index and n a time index. For G_n^i we take the displacement field $u^e(x,t) \subseteq G_n^i$ to be approximated by

$$u^e(x,t) = \sum_{k=1}^{m} N_k(G_n^i)u_k \qquad (3)$$

where m is the number of node for each element, $N_k(G_n^i)$ are interpolating polynomials of degree r defined over the region G_n^i and u_k are the nodal values of the field $u(x,t)$. The weighting functions $\psi_k(G_n^i)$ is also defined over the space G_n^i.

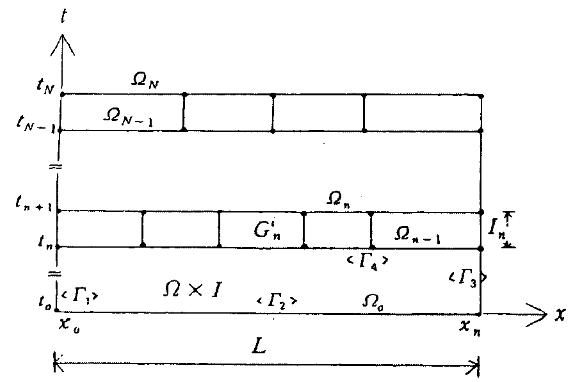

Fig. 1 The discretization of the domain in space and time

4. Weighted Residual Process

On an element basis, an algebraic relation among the u_k can be obtained by the method of weighted residuals. The Galerkin method for Eq.(1) can now be placed in a weighted residual formulation for seeking a function u^e:

$$\int_{G^e} \left(\frac{\partial}{\partial x}\left(E\frac{\partial u^e}{\partial x}\right) - \frac{\partial}{\partial t}\left(\rho\frac{\partial u^e}{\partial t}\right) + q\right)\psi_i^e\,(x,t)\,dG^e = 0$$
$$i = 1,2,3,\ldots,m \qquad (4)$$

where G^e is the domain for element e. ψ_i^e are the weight functions and m is the number of nodes in element G^e. If the function u^e were exact solution, then the term in the parenthesis would be identically zero for all choices of the weight function $\psi_i^e\,(x,t)$. This integral condition requires the error in satisfying the governing differential equation to be orthogonal to the weight functions ψ_i^e. If the weight functions are complete so that there exists no admissible function that is orthorgonal to all ψ_i^e except the null function, then u^e would be the exact solution. The first and the second term on the left-hand side of Eq.(4) contains second derivatives of the dependent variables Integration by parts for the first term on the left-hand side gives

$$\int_{G^e} \left(\frac{\partial}{\partial x}\left(E\frac{\partial u^e}{\partial x}\right)\right) \Psi_i^e\,(x,t)\,dG^e$$
$$= -\int_{G^e} E\frac{\partial u^e}{\partial x}\frac{\partial \psi_i^e}{\partial x}$$
$$dG^e + \int_{\Gamma^e} n_x E\frac{\partial u^e}{\partial x}\psi_i^e\,d\Gamma^e \qquad (5)$$

and hence Eq.(4) becomes in the rearranged form

$$\int_{G^e} E\frac{\partial u^e}{\partial x}\frac{\partial \psi_i^e}{\partial x}dG^e - \int_{\Gamma^e} n_x E\frac{\partial u^e}{\partial x}\psi_i^e\,d\Gamma^e + \int_{G^e}\rho\frac{\partial u^e}{\partial t}$$
$$\frac{\partial \psi_i^e}{\partial t}dG^e - \int_{\Gamma^e} n_t\rho\frac{\partial u^e}{\partial t}\psi_i^e\,d\Gamma^e - \int_{G^e} q\psi_i^e\,dG^e = 0$$
$$i = 1,2,\ 3,\ldots,m \qquad (6)$$

where Γ^e is the element boundary of G^e in $x-t$ plane, n_x and n_t are the outward unit normal vectors of Γ^e. Note that the secondary integral in Eq.(6) carries information about kinematic boundary conditions. Evidently, this second integral does not contribute anything to the equation at the internal grid points. When a grid point lies on a boundary and if

$\partial u/\partial n$ is specified on that boundary, then the integral can be evaluated. The last integral carries information about the forcing function.

5. Finite Element Formulation

The system of the space-time elements consists of four trapezoidal elements. The solution domain is discretized using two-dimensional bilinear quadrilateral elements for a general region specified by a global coordinates system(x,t).

$$x=\sum_{i=1}^{4} N_i\, x_i \quad t=\sum_{i=1}^{4} N_i\, t_i \tag{7}$$

where x_i, t_i are nodal coordinates in the global coordinate system. This transformation from the $x-t$ region to the 2×2 square in natural coordinate system(ξ,τ) can be expressed in matrix form as follows:

$$\begin{Bmatrix} x \\ t \end{Bmatrix} = \begin{pmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{pmatrix} \begin{pmatrix} u_{i,n} \\ t_{i,n} \\ u_{i+1,n} \\ t_{i+1,n} \\ u_{i+1,n+1} \\ t_{i+1,n+1} \\ u_{i,n+1} \\ t_{i,n+1} \end{pmatrix} \tag{8}$$

where the interpolation function $N_i(\xi,\tau)$ for node i has following form

$$N_i(\xi,\tau)=\frac{1}{4}(1+\xi\xi_i)(1+\tau\tau_i\,) \quad i=1,\ 2,\ 3,\ 4 \tag{9}$$

The axial displacement is the only unknown displacement in the bar element. The unknown displacement u^e in any point of the element interpolating by the same functions making this an isoparametric formulation is expressed in terms of the nodal values u_i.

$$u^e=\sum_{i=1}^{4} N_i\, u_i$$

The unknown displacement interpolated by the same functions making this an isoparametric formulation can be expressed in matrix form as follows:

$$u^e(x,t)=(\ N_1\ \ N_2\ \ N_3\ \ N_4) \begin{pmatrix} u_{i,n} \\ u_{i+1,n} \\ u_{i+1,n+1} \\ u_{i,n+1} \end{pmatrix} \tag{10}$$

The strain and stress are expressed by the relations

$$\epsilon=\frac{\partial u^e}{\partial x}=\frac{\partial N}{\partial x}u_i \tag{11}$$

$$\sigma=AE\ \epsilon=AE\frac{\partial u^e}{\partial x} \tag{12}$$

where AE is the axial stiffness coefficient. Deformations of the element in time can be determined as a velocity

$$\epsilon_t=\frac{\partial u}{\partial t}=\frac{\partial N}{\partial t}u_i \tag{13}$$

If the weighting functions $\psi_i^e\ (x,t)$ are taken as the interpolation functions N_i, then Eq.(6) becomes

$$\sum_{j=1}^{4}\int_{G^e} EN_{j,x}\,N_{i,x}\,u_j^e\,dG^e+\sum_{j=1}^{4}\int_{G^e}\rho N_{j,t}\,N_{i,t}\,u_j^e\,dG^e \\ -\sum_{j=1}^{4}\int_{\Gamma^e} En_x\,N_{j,x}\,N_i\,u_j^e\,d\Gamma^e-\sum_{j=1}^{4}\int_{\Gamma^e}\rho n_t\,N_{j,t}\,N_i\,u_j^e\,d\Gamma^e \\ -\int_{G^e} qN_i\,dG^e=0 \quad i\,,j=1,2,3,4 \tag{14}$$

Equation(14) is rewritten in matrix form as

$$[K]\{u\}+[M]-[BX]\ \{u\}-[BT]\ \{u\}-\{Q\}=0 \tag{15}$$

or

$$[A]\ \{u\}=\{Q\} \tag{16}$$

where

$$[A]=[K]+[M]-[BX]-[BT]$$
$$\{u\}=\{u_{i,n}u_{i+1,n}u_{i+1,n+1}u_{i+1,n}\}^T$$

The individual matrices are given as

$$[K]=\int_{G^e} E\,N_{j,x}\,N_{i,x}\,dG^e$$

$$[M] = \int_{G^e} \rho N_{j,t} N_{i,t}\, dG^e$$
$$[BX] = \sum_{j=1}^{4} \int_{\Gamma^e} E n_x N_{j,x} N_i u_j^e\, d\Gamma^e$$
$$[BT] = \sum_{j=1}^{4} \int_{\Gamma^e} \rho n_t N_{j,t} N_i u_j^e\, d\Gamma^e$$
$$\{Q\} = \int_{G^e} q N_i\, dG^e$$

The integration will be integrated numerically using Gauss-Legendre technique of selecting integration points and weighting factors in the $\xi-\tau$ space. Using this approach, the matrix $[K]$ only is considered. The other matrices can be performed in a same manner.

$$[K] = \sum_{i=1}^{p}\sum_{j=1}^{p} E W_i W_j [N^t(\xi_i,\tau_j)]^T \{A_1\}^T \{A_1\} [N^t(\xi_i,\tau_j)] \quad | J(\xi_i,\tau_j) | \qquad (17)$$

where

$$[A_1] = [J]^{-1}$$
$$[J] = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial t}{\partial \xi} \\ \frac{\partial x}{\partial \tau} & \frac{\partial t}{\partial \tau} \end{pmatrix}$$

where W_i and W_j are Gauss weights, ξ_i and τ_j are the coordinates of the Gauss points, and p is the number of Gauss points in each integration direction.

6. Assembling the Finite Element Equations

For each element we have equations of the form of Eq.(16). At $t=0$, we know all values of u. We need to assemble these individual element equations so that the values of u can be calculated at the end of the first time step. This continues so that during the n-th time step we know the values of u at $u_{i,n}$ and want to calculate $u_{i,n+1}$. Define a global nodal displacement and a nodal force vector as

$$\{U\} = \{u_1 u_2 u_3 \ldots u_N\}^T \qquad (18)$$

$$\{Q\} = \{q_1 q_2 q_3 \ldots q_N\}^T \qquad (19)$$

The assembled equations at the n-th time level will have the matrix form as

$$[S_2]\{U\}^{n+1} = \{Q\}^{n+1} - [S_1]\{U\}^n \qquad (20)$$

The assembled matrices $[S_1]$ and $[S_2]$ are tridiagonal. The elements of $[S_1]$, $[S_2]$ and $\{Q\}$ can be expressed respectively as :

$$[S_1] = \begin{bmatrix} s_{41}^1 & s_{42}^1 & & & \\ s_{31}^1 & s_{32}^1 + s_{41}^2 & s_{42}^2 & & \\ & s_{31}^2 & s_{32}^2 + s_{41}^3 & & \\ & & \cdot \quad \cdot \quad \cdot & & \\ & & & s_{32}^{i-2} + s_{41}^{i-1} & s_{42}^{i-1} \\ & & & s_{31}^{i-1} & s_{32}^{i-1} \end{bmatrix},$$

$$[S_2] = \begin{bmatrix} s_{44}^1 & s_{43}^1 & & & \\ s_{34}^1 & s_{33}^1 + s_{44}^2 & s_{43}^2 & & \\ & s_{34}^2 & s_{33}^2 + s_{44}^3 & & \\ & & \cdot \quad \cdot \quad \cdot & & \\ & & & s_{33}^{i-2} + s_{44}^{i-1} & s_{43}^{i-1} \\ & & & s_{34}^{i-1} & s_{33}^{i-1} \end{bmatrix},$$

$$\{Q\} = \begin{pmatrix} q_4^1 \\ q_3^1 + q_4^2 \\ q_3^2 + q_4^3 \\ \cdot \\ \cdot \\ q_4^{n-1} + q_3^{n-2} \\ q_3^{n-1} \end{pmatrix}$$

Where subscript j, k in s_{jk} ($j, k = 1,2,3,4$) stands for the coefficients of element stiffness matrix $[A]$ in Eq.16. The assembled matrices $[S]$ is related to time level n and $n+1$ and $\{Q\}$ is assemblage vector of nodal force. These equations can be eventually simplified into a set of linear algebraic equations for the unknown displacements in the time slab. The equation set which is not yet modified to included boundary conditions is rewritten in the final form as

$$[S_2]\{U\}^{n+1}+[S_1]\{U\}^n=\{Q\}^{n+1} \tag{21}$$

Such a formulation enables step-by-step solution. Since values of displacement are known at $n=t_o$, values at $n+1=t_o+\Delta t$ can be obtained by summing around the nodes at this latter step and then solving the system of simultaneous linear algebraic equations that results. At each new time step, an identical procedure is used until a required time is reached.

7. Numerical Solutions

To illustrate the formulation of the method and its efficiency the one-dimensional propagation of elastic wave in a bar with uniform cross section and finite length is chosen. A uniform bar of length L is initially stretched by an axial force($f_o=1.0$kgf) applied at the free end. The axial force is suddenly removed at time $t=0$. Bilinear elements of a beam are used in the solution of the problem. The structure has no internal force and has been made for fifty elements which are uniformly discretized and for time, ranging from 0.002 to 0.01 second. The physical properties used for computation are as follows: density ($\rho=8.0$ kgf · ms^2/mm^4), modulus of elasticity($\boldsymbol{E}=20{,}000$kgf/mm^2), area of cross section($A=1.0$mm^2) and length of bar ($\boldsymbol{L}=50.0$cm). The one-dimensional wave propagation problem to be solved is one governed by Eq.(1) and subject to the following boundary and initial conditions

$$u(L,t)=0,\ \ AE\frac{\partial u}{\partial x}(0,t)=1.0\text{kgf}$$

and

$$u(x,0)=0,\ \ \frac{\partial u}{\partial t}(x,0)=0$$

The analytical solutions for this problem is

$$u(x,t)=\frac{f_o}{E}(x-L)+\frac{8f_oL}{E\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos(2n-1)\frac{\pi}{2L}x\cos(2n-1)\frac{\pi c}{2L}t \tag{22}$$

The plotted results in Fig.2, that is, after 31 time steps, show how the method compare the exact solution with relatively large time increment $\Delta t=0.004$ microsecond. This time step is taken for numerical stability considerations. These results show oscillations and inaccuracies, particularly in the vicinity of the wave front. Even with $\Delta t=0.004$ microsecond which is double of characteristic time step these results are still within 2.2% of the exact solution at $t=0.124$ microsecond. An approximate size of the time step for stability can be found as $\Delta t=l/c=10/5{,}000=0.002$ms, where l is the length of each element. This size is often called the characteristic time step. To study the convergence of the present method, the results

Fig. 2 Particle velocity for wave propagation in bar with $\Delta t=0.004$ms.

Fig. 3 Particle velocity for wave propagation in bar with $\Delta t=0.002$ms.

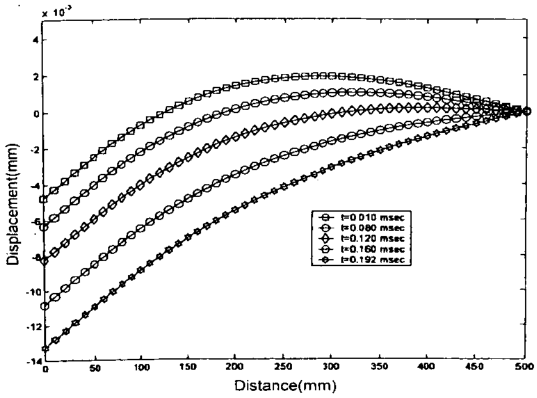

Fig. 4 Displacements for wave propagation in bar with $\Delta t = 0.002$ms.

in terms of particle velocity for time step of 0.002 ms at time t=0.1, 0.124, 0.204, 0.268ms were obtained. These results are plotted in Fig.3. There in no plottable difference between the analytical exact solution and the results of the present method. Fig.4 shows the displacement curves for various fixed values of the time t. The present method give close agreement to the analytical solution for a short time. It is noted that in all cases the solution is very accurately predicted, even with the large time step. However, for accurate prediction of the response, a finer finite element discretization and smaller space interval along with smaller space interval must be employed. No significant instability problems and much more rapid convergence to the analytical solution were experienced in this approach.

8. Conclusions

We have demonstrated the efficiency of using finite elements in both time and space where we the space-time domain as a whole in the generation of finite elements. The use of ordinary continuous finite element approximations enables us to use standard element packages for the space-time domain. In this method a complete space-time finite element discretization is generated which eliminates the need for any additional ordinary differential equation solver to resolve the temporal behavior of the problem. As we have seen, the numerical stability of time-stepping on the larger time steps is quite good. One disadvantage of any space-time finite element formulation is the additional dimension of time. Although more computer storage capacity was required to handle the present algorithm, this disadvantage was easily offset by the substantial savings in the CPU time due to rapid convergence of the solution. In this present approach, large time step increments were allowed for achieving stable solutions regardless of the maximum permissible time step in conventional numerical methods. Furthermore, numerical results have shown that the space-time Galerkin formulation is effective for localizing oscillations due to sharp gradients along with the fined mesh. The methods presented are not limited to one-dimensional problems. An obvious extension of the present method is the development of a similar scheme in multi-dimensional involving tetrahedrons and the combination of different types of elements in one single code.

Reference

Argyris, J.H., Scharpf, D.W.(1989) Finite element in time and space, *Nuclear Engineering and Design*, 10, pp.456~460.

Axelsson, O., Maubach, J.(1989) A time-space finite element discretization technique for the calculation of the electromagnetic field in ferromagnetic materials, *International Journal for Numerical Methods in Engineering*, 28, pp.2085~2111.

Bajer, C.I.(1986) Triangular and tetrahedral space-time finite elements in vibration analysis, *International Journal for Numerical Methods in Engineering*, 23, pp.2031~2048.

Bruch, J.C., Zyvoloski, G.(1974) A finite element weighted residual solution to one-dimensional field problems, *International Journal for Numerical Methods in Engineering*, 8, pp.481~494.

Hughes, T.J., Hulbert, G.W.(1987) Space-time finite element methods for elastodynamics formulations and error estimates, *Computer Methods in Applied Mechanics and Engineering*, 66, pp.339~363.

Jang, I.S. ***et al.***(1998) The Derivation of Error Estimates with Various Shape Functions for Time Integration Using Finite Element Approach, *Journal of the Computational Structural Engineering Institute of Korea*, 11, pp.187~196.

Kim, C.K.(2001) On the Numerical Computation for Solving the Two-Dimensional Parabolic Equations By Space and Time Finite Element Method, *JSME International Journal Series B*, 44(3), pp.434~438.

Kaczkowski, Z.(1975) The method of finitespace-time elements in dynamics of structures, *Journal of Technical Physics*, 16, pp.69~84.

Saitoh, T.S. ***et al.***(1994) Time-space method for multidimensional melting and freezing problems, *International Journal for Numerical Methods in Engineering*, 37, pp.793~1805.

Shakib, F., Hughes, T.J.(1991) A new finite element formulation for computational fluid dynamics: Ix. Fourier analysis of space-time Galerkin/Least-squares algorithms, *Computer Methods in Applied Mechanics and Engineering*, 66, pp.35~58.

Yu, L.R., Hsu, T.R.(1985) Analysis of heat conduction in solids by space-time finite element method, *International Journal for Numerical Methods in Engineering*, 21, pp.2001~2012.