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Some characterizations of interval-valued Choquet price functionals

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Abstract

In this paper, we define an interval-valued Choquet price functional which is a useful tool as the price of an insurance contract with ambiguity payoffs and investigate some characterizations of them. Moreover, we show that the insurance price with ambiguity payoffs has an interval-valued Choquet integral representation with respect to a capacity.

Key words : Choquet integrals, interval-valued functions, insurance price.

1. Introduction

Recently, in the papers([15,16,17,18]), they have been studied Choquet price in insurance as alternative as traditional pricing principles in insurance. We note that the Choquet integral is nonlinear generalization of the Lebesgue integral and has several properties that make it especially suitable for pricing insurance contracts.

We recall that the set-valued Choquet integral was first introduced by Jang, Kil, Kim and Kwon([5]) and restudied by Zhang, Guo and Lia([20]), and that theory about set-valued integrals has drawn much attention due to numerous applications in mathematics, economics, information theory, expected utility theory, and risk analysis. In many papers([5,6,7,8,9,10,11,19,20]), they have been studied some properties of set-valued Choquet integrals and interval-valued Choquet integrals.

In this paper, we will define interval-valued Choquet price functionals and investigate some characterizations of them. We also show that the insurance price with ambiguity payoffs has an interval-valued Choquet integral representation with respect to a capacity. This constraction is a useful tool to form Choquet price in insurance market with ambiguity payoffs.

2. Preliminaries and Definitions

In order to construct interval-valued Choquet price which describe the behavior of an insurance market with ambiguity payoffs, we list the following definitions and basic properties. We consider a two period market model where insurance contracts can be traded in the first period, and contracts ambiguously pay off in the second period. There are S possible states of the world at the second period, indexed by $s \in \Omega = \{1, 2, \cdots, S\}$. Then an ambiguity payoff can be represented by an interval-valued function $\bar{X} : \Omega \to I(R^+)\backslash\{\emptyset\}$, where $R^+ = [0, \infty)$ and

$$I(R^+) = \{\bar{a} = [a^-, a^+] \,|\, a^- \leq a^+, \;\; a^-, a^+ \in R^+\}.$$

We denote $\bar{\mathbf{B}}$ for the set of ambiguity payoffs which are represented by interval-valued functions, $\mathbf{B}$ for the set of payoffs which are represented by real-valued functions, 2^Ω for the power set of Ω.

Definition 2.1. (1) A set function μ is called a capacity if $\mu(\emptyset) = 0, \mu(\Omega) = 1$, and μ is satisfies monotonicity(with respect to set inclusion), that is,

$$A \subset B \Longrightarrow \mu(A) \leq \mu(B).$$

(2) A set function μ is said to be concave(convex) if for any $A, B \in 2^\Omega$,

$$\mu(A \cup B) + \mu(A \cap B) \leq (\geq)\mu(A) + \mu(B).$$

(3) A capacity μ is said to be lower semi-continuous if for any increasing sequence $\{A_n\} \subset 2^\Omega$, we have

$$\mu(\cup_{n=1}^\infty A_n) = \lim_{n\to\infty} \mu(A_n).$$

(4) A capacity μ is said to be upper semi-continuous if for any increasing sequence $\{A_n\} \subset 2^\Omega$ and $\mu(A_1) < \infty$, we have

$$\mu(\cap_{n=1}^\infty A_n) = \lim_{n\to\infty} \mu(A_n).$$

(5) A capacity μ is said to be continuous if it is both lower semi-continuous and upper semi-continuous.

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We note that a capacity μ is often referred to as a fuzzy measure with $\mu(\Omega) = 1$ and to as a distorted probability measure. It is well-known that if μ is a capacity, there exists a non-decreasing function $g : [0,1] \to [0,1]$ with $g(0) = 0$ and $g(1) = 1$, and a probability measure P such that $\mu = g \circ P$. Here, g is called a distortion function.

Let $C(R^+)$ be the class of closed subsets of R^+. We denote a real-valued function $X : \Omega \to R^+$, a closed set-valued function $\bar{X} : \Omega \to C(R^+) \setminus \{\emptyset\}$.

Definition 2.2. ([3,4,12,13,14]) Let X, Y be nonnegative measurable functions. We say that X and Y are comonotonic, in symbol $X \sim Y$ if

$$X(w) < X(w') \Rightarrow Y(w) \leq Y(w')$$

for all $w, w' \in \Omega$.

Theorem 2.3. ([3,4,12,13,14]) Let X, Y, Z be nonnegative measurable functions. Then we have
(1) $X \sim X$,
(2) $X \sim Y \Rightarrow Y \sim X$,
(3) $X \sim a$ for all $a \in R^+$,
(4) $X \sim Y$ and $Y \sim Z \Rightarrow X \sim (Y + Z)$.

Theorem 2.4. ([3,4,12,13,14]) Let X, Y be nonnegative measurable functions. Then we have the followings.
(1) If $X \leq Y$, then

$$(C)\int X d\mu \leq (C)\int Y d\mu.$$

(2) If $A \subset B$ and $A, B \in \mathbf{B}$, then

$$(C)\int_A X d\mu \leq (C)\int_B X d\mu.$$

(3) If $X \sim Y$ and $a, b \in R^+$; then

$$(C)\int (aX + bY) d\mu = a(C)\int X d\mu + b(C)\int Y d\mu.$$

(4) If $(X \vee Y)(w) = X(w) \vee Y(w)$ and $X \wedge Y)(w) = X(w) \wedge Y(w)$ for all $w \in \Omega$, then

$$(C)\int X \vee Y d\mu \geq (C)\int X d\mu \vee (C)\int Y d\mu$$

and

$$(C)\int X \wedge Y d\mu \leq (C)\int X d\mu \wedge (C)\int Y d\mu.$$

Definition 2.5. ([1,2]) A closed set-valued function $\bar{X}$ is said to be *measurable* if for each open set $O \subset R^+$,

$$\bar{X}^{-1}(O) = \{w \in \Omega | \bar{X}(w) \cap O \neq \emptyset\} \in \Omega.$$

Definition 2.6. The Choquet integral of $\bar{X}$ with respect to a capacity μ is equal to

$$(C)\int_A \bar{X} d\mu = \{(C)\int_A X d\mu | X \in S(\bar{X})\}$$

where $S(\bar{X})$ is the family of measurable selections of $\bar{X}$.

For completeness, we recall that the Choquet integral of $X \in \mathbf{B}$ with respect to μ (see[3,4,12,13,14]) is equal to

$$(C)\int X d\mu = \int_0^\infty \mu\{s \in \Omega | X(s) \geq t\} dt.$$

For any $a \in R^+$, we define $a = [a, a]$. Obviously, $a \in I(R^+)$.

Definition 2.7. If $\bar{a} = [a^-, a^+], \bar{b} = [b^-, b^+] \in I(R^+)$, then we define:
(1) $\bar{a} \wedge \bar{b} = [a^- \wedge b^-, a^+ \wedge b^+]$,
(2) $\bar{a} \vee \bar{b} = [a^- \vee b^-, a^+ \vee b^+]$,
(3) $\bar{a} \leq \bar{b}$ if and only if $a^- \leq b^-$ and $a^+ \leq b^+$,
(4) $\bar{a} < \bar{b}$ if and only if $\bar{a} \leq \bar{b}$ and $\bar{a} \neq \bar{b}$,
(5) $\bar{a} \subset \bar{b}$ if and only if $b^- \leq a^- \leq a^+ \leq b^+$.

Clearly, we have the following theorem for multiplication and Hausdorff metric on $I(R^+)$.

Theorem 2.8. (1)If we define $\bar{a} \cdot \bar{b} = \{x \cdot b | x \in \bar{a}, y \in \bar{b}\}$ for $\bar{a} \cdot \bar{b} \in I(R^+)$, then we have

$$\bar{a} \cdot \bar{b} = [a^- \cdot b^-, a^+ \cdot b^+].$$

(2) If $d_H : I(R^+) \times I(R^+) \to [0, \infty)$ is a Hausdorff metric, then we have

$$d_H(\bar{a}, \bar{b}) = \max\{|a^- - b^-|, |a^+ - b^+|\}.$$

Definition 2.9. Let $\bar{X}, \bar{Y}$ be ambiguity payoffs. We say that $\bar{X}$ and $\bar{Y}$ are comonotone, in symbol $\bar{X} \sim \bar{Y}$ if

$$\bar{X}(s) < \bar{X}(s') \Rightarrow \bar{Y}(s) \leq \bar{Y}(s') \text{ for all } s, s' \in \Omega.$$

We recall that $\bar{X} = [X^-, X^+] \sim \bar{Y} = [Y^-, Y^+]$ if and only if $X^- \sim Y^-$ and $X^+ \sim Y^+$ (see [11]).

3. Main results

In this section, we will define an interval-valued Choquet price functional which represents the price of an insurance contract with ambiguity payoffs and discuss some characterizations of them. We recall that for any capacity μ, there exists a non-decreasing function

$$g : [0,1] \to [0,1], \quad \text{with } g(0) = 0 \text{ and } g(1) = 1,$$

and a probability measure P such that $\mu = g \circ P$.

Since a capacity μ on a finite set Ω is continuous, by Theorem 3.16([20)],

$$(C)\int \bar{X} d\mu = [(C)\int X^- d\mu, (C)\int X^+ d\mu]. \quad (3.1)$$

In order to establishing interval-valued Choquet price functional, we need the following new definitions which are generalized concepts of Axioms 1-4 in [17].

Definition 3.1. (1) An interval-valued price functional

$$\bar{H} : \bar{\mathbf{B}} \to I(R^+) \setminus \{\emptyset\}$$

is said to be conditional state independent if it depends only on its distribution.

(2) $\bar{H}$ is said to be monotone if for any $\bar{X}, \bar{Y} \in \bar{\mathbf{B}}$,

$$\bar{X} \leq \bar{Y} \Rightarrow \bar{H}(\bar{X}) \leq \bar{H}(\bar{Y}).$$

(3) $\bar{H}$ is said to be comonotonic additive if for any $\bar{X}, \bar{Y} \in \bar{\mathbf{B}}$,

$$\bar{X} \sim \bar{Y} \Rightarrow \bar{H}(\bar{X} + \bar{Y}) = \bar{H}(\bar{X}) + \bar{H}(\bar{Y}).$$

(4) $\bar{H}$ is said to be continuous if for any $\bar{X}, \in \bar{\mathbf{B}}$,

$$\lim_{(d_1,d_2)\to(0^+,0^+)} d_H[\bar{H}(\bar{X} - [d_1,d_2])_+, \bar{H}(\bar{X})] = 0$$

and

$$\lim_{(d_1,d_2)\to(0,0)} d_H[\bar{H}(\bar{X} \wedge [d_1,d_2]), \bar{H}(\bar{X})] = 0,$$

where $(\bar{X} - [d_1,d_2])_+ = [\max(X^- - d_1, 0), \max(X^+ - d_2, 0)]$.

The decumulative distribution function(ddf) of a payoff X is denoted as

$$S_X(t) = \mu\{w | X(w) > t\}, \quad t \geq 0,$$

and ddf of an interval-valued payoff $\bar{X}$ is denoted as

$$S_{\bar{X}}(t) = [S_{X^-}(t), S_{X^+}(t)]. \quad (3.2)$$

We note that Definition 3.1(1) means the following property

$$\bar{H}(\bar{X}) = \bar{H}(\bar{X}') \text{ if } S_{\bar{X}} = S_{\bar{X}'}.$$

Clearly, we have the following basic characterizations of the interval-valued Choquet price functional.

Theorem 3.2. If we define the interval-valued Choquet price functional $\bar{H}_\mu : \bar{\mathbf{B}} \to I(R^+)\setminus\{\emptyset\}$ by

$$\bar{H}_\mu(\bar{X}) = (C)\int \bar{X} d\mu,$$

then it is conditional state independent, monotone, comonotonic additive.

Proof. Equation (3.1) implies that

$$\bar{H}_\mu(\bar{X}) = [(C)\int X^- d\mu, (C)\int X^+ d\mu].$$

By the definition of Choquet integrals, it is conditional state independent. If $S_{\bar{X}} = S_{\bar{X}'}$, by Equation (3.2),

$$S_{X^-} = S_{X'^-} \text{ and } S_{X^+} = S_{X'^+}.$$

By Axiom 1 in [17], we have

$$(C)\int X^- d\mu = (C)\int X'^- d\mu$$

and

$$(C)\int X^+ d\mu = (C)\int X'^+ d\mu.$$

Thus we have

$$(C)\int \bar{X} d\mu = (C)\int \bar{X}' d\mu.$$

By Equation (3.1), Theorem 3.5(2) ([10]) and Theorem 4.3([10]), we obtain that $\bar{H}$ is monotone and comonotonic additive.

Theorem 3.3. If an interval-valued price functional

$$\bar{H} = [H^-, H^+] : \bar{\mathbf{B}} \to I(R^+)\setminus\{\emptyset\}$$

satisfies Definition 3.1 (1),(2),(3),and (4), respectively, then H^- and H^+ are price functional from $\mathbf{B}$ to R^+ as satisfy Axiom 1,2,3,4 in [17], respectively.

Proof. Suppose that $\bar{H}$ satisfies Definition 3.1 (1). Let $X_1, X_2, Y_1, Y_2 \in \mathbf{B}$, $S_{X_1} = S_{Y_1}$, and $S_{X_2} = S_{Y_2}$. If we put

$$\bar{X} = [X_1, X_2] \text{ and } \bar{Y} = [Y_1, Y_2],$$

then $\bar{X}, \bar{Y} \in \bar{\mathbf{B}}$ and

$$S_{\bar{X}} = [S_{X_1}, S_{X_2}] = [S_{Y_1}, S_{Y_2}] = S_{\bar{Y}}.$$

Thus, by Definition 3.1 (1),

$$\bar{H}(\bar{X}) = \bar{H}(\bar{Y}).$$

This implies

$$H^-(X_1) = H^-(Y_1) \text{ and } H^+(X_2) = H^+(Y_2).$$

That is, H^- and H^+ satisfy conditional state independence which is Axiom 1 in [17].

Suppose that $\bar{H}$ satisfies Definition 3.1 (2). Let $X_1, X_2, Y_1, Y_2 \in \mathbf{B}$, $X_1 \le Y_1$ and $X_2 \le Y_2$. If we put

$$\bar{X} = [X_1, X_2] \text{ and } \bar{Y} = [Y_1, Y_2],$$

then $\bar{X}, \bar{Y} \in \bar{\mathbf{B}}$ and by Definition 3.1 (2),

$$\bar{X} \le \bar{Y}.$$

Thus, we have

$$H^-(X_1) \le H^-(Y_1) \text{ and } H^+(X_2) \le H^+(Y_2).$$

This implies H^- and H^+ satisfy monotonicity which is Axiom 2 in [17].

Suppose that $\bar{H}$ satisfies Definition 3.1 (3). Let $X_1, X_2, Y_1, Y_2 \in \mathbf{B}$, $X_1 \sim Y_1$ and $X_2 \sim Y_2$. If we put

$$\bar{X} = [X_1, X_2] \text{ and } \bar{Y} = [Y_1, Y_2],$$

then $\bar{X}, \bar{Y} \in \bar{\mathbf{B}}$ and $\bar{X} \sim \bar{Y}$ By Definition 3.1 (3),

$$\bar{H}(\bar{X} + \bar{Y}) = \bar{H}(\bar{X}) + \bar{H}(\bar{Y}).$$

Thus we have

$$H^-(X_1 + Y_1) = H^-(X_1) + H^-(Y_1)$$

and

$$H^-(X_2 + Y_2) = H^-(X_2) + H^-(Y_2).$$

This implies that H^- and H^- satisfy comonotonic additivity which is Axiom 3 in [17].

Suppose that $\bar{H}$ satisfies Definition 3.1 (4). Let $X_1, X_2 \in \mathbf{B}$. If we put $\bar{X} = [X_1, X_2]$ and $d_1, d_2 \ge 0$, then $\bar{X} \in \bar{\mathbf{B}}$ and

$$\lim_{(d_1,d_2)\to(0^+,0^+)} d_H[\bar{H}(\bar{X} - [d_1, d_2])_+, \bar{H}(\bar{X})] = 0$$

and

$$\lim_{(d_1,d_2)\to(0,0)} d_H[\bar{H}(\bar{X} \wedge [d_1, d_2]), \bar{H}(\bar{X})] = 0.$$

This implies that

$$\lim_{d_1\to 0^+} d_H[H^-(X_1 - d_1)_+, H^-(X_1)] = 0$$

and

$$\lim_{d_1\to 0^+} d_H[H^-(X_1 \wedge d_1)_+, H^-(X_1)] = 0,$$

$$\lim_{d_2\to 0^+} d_H[H^-(X_2 - d_2)_+, H^-(X_2)] = 0$$

and

$$\lim_{d_2\to 0^+} d_H[H^-(X_2 \wedge d_2)_+, H^-(X_2)] = 0.$$

That is, H^- and H^+ satisfy continuity which is Axiom 4 in [17].

Theorem 3.4. If an interval-valued price functional

$$\bar{H} : \bar{\mathbf{B}} \to I(R^+)\backslash\{\emptyset\}$$

satisfies Definition 3.1 (2)-(4), then there exists a unique monotone interval-valued set function $\bar{\gamma} = [\gamma_1, \gamma_2]$ on $\bar{\mathbf{B}}$ such that

$$\bar{H}(\bar{X}) = (C) \int \bar{X} d\bar{\gamma}$$

where γ_1 and γ_2 are monotone set functions on $\mathbf{B}$ and

$$(C) \int \bar{X} d\bar{\gamma} \equiv [(C) \int X^- d\gamma_1, (C) \int X^+ d\gamma_2].$$

Proof. Suppose that $\bar{H}$ satisfies Definition 3.1 (2)-(4). Then Theorem 3.3 implies that H^- and H^+ satisfy Axiom 2-4 in [17]. Thus, by Theorem 1([17]), there exist uniquely monotone set-functions γ_1 and γ_2 such that

$$H^-(X) = (C) \int X d\gamma_1, \quad \text{for all } X \in \mathbf{B}$$

and

$$H^+(X) = (C) \int X d\gamma_2, \quad \text{for all } X \in \mathbf{B}.$$

If we put $\bar{\gamma} = [\gamma_1, \gamma_2]$, then clearly, $\bar{\gamma}$ is monotone interval-valued set functions on $\bar{\mathbf{B}}$. For any $X = [X_1, X_2] \in \bar{\mathbf{B}}$,

$$\begin{aligned} \bar{H}(\bar{X}) &= [H^-(X^-), H^+(X^+)] \\ &= [(C) \textstyle\int X^- d\gamma_1, (C) \int X^+ d\gamma_2] \\ &= (C) \textstyle\int \bar{X} d\bar{\gamma}. \end{aligned}$$

By the uniqueness of γ_1 and γ_2, we obtain the uniqueness of $\bar{\gamma}$.

We remark that Theorem 3.4 means insurance price with ambiguity payoffs is represented by interval-valued Choquet price functional.

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