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CONVERGENCE OF GENERALIZED NETS AND FILTERS IN GRADED L-FUZZY TOPOLOGICAL SPACES

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ABSTRACT. In this paper, an idea of graded fuzzy net is introduced; its convergence is studied; Relations between g-net and g-filter are established in L-fuzzy setting.

0. INTRODUCTION

In 1979 a theory of convergence of fuzzy filters was developed by Lowen [9] for laminated spaces and afterwards it was extended to arbitrary fuzzy topological (Chang) spaces by Warren [17]. In 1995 W. Gahler [6, 7] introduced an idea of graded fuzzy filter in lattice valued setting (which he called L-fuzzy filter) and studied its convergence in Chang fuzzy topological spaces. Later on in the year of 1999 M. H. Burton, M. Muraleetharan and J. Gutieŕrez Garcia [1, 2] considered another type of graded fuzzy filter named as generalized filter (g-filter) by relaxing a condition imposed by W. Gahlar [6, 7]. In [12] we studied convergence of fuzzy filters and g-filters in graded L-fuzzy topological spaces.

On the other hand, Pu Pao Ming and Liu Ying Ming [14] introduced the concept of fuzzy net and studied its convergence in a Chang fuzzy topological space (CFTS). In [15] Ramadan, El Deeb & Abdel-Sattar studied the convergence of a fuzzy net in a smooth topological space using crisp points as well as fuzzy points.

In [11] we studied the graded convergence of a fuzzy net in a graded L-fts. In this paper, our intention is to introduce such a generalized version of fuzzy net (we shall call it a g-net) such that its associated fuzzy filter becomes a generalized fuzzy filter.

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In fact this paper is a continuation of our previous papers [11, 12] on fuzzy convergence.

The organization of the paper is as follows:

In section 2, we study the characteristic property of limit sets of g-filters, (the concept of which was introduced in [12]) in terms of q-neighborhoodness.

In section 3, we introduce a concept of generalized net (briefly called g-net) and study its generalized convergence in a graded L-fuzzy topological space. Relation of this g-net with corresponding g-filter has been established. Also characteristic property of a gp-map in terms of g-nets has been studied.

1. NOTATION AND PRELIMINARIES

In this paper X denotes a non-empty set; unless otherwise mentioned, L denotes a completely distributive order dense complete lattice with an order reversing involution $'$ whereas $L_0 = L - \{0\}$. Let 0 and 1 denote respectively the least and the greatest elements of L; L^X, the collection of all L-fuzzy subsets of X and $Pt(L^X)$, the set of all L-fuzzy pts of X. M(L) denotes the set of all molecules of L whereas $M(L^X)$ denotes the set of all molecule pts of L^X. By $\tilde{0}$ and $\tilde{1}$ we denote the constant L-fuzzy subsets of X taking values 0 and 1 respectively. For $p_x \in Pt(L^X)$ and $A, B \in L^X$ we say $p_x q A$ if $p_x \notin A^c$ and AqB if $A \not\subseteq B^c$. For other notations we follow [18].

Definition 1.1 ([16]). A function $\tau : L^X \to L$ is called an L-fuzzy topology on X if it satisfies the following conditions:

(O1) $\tau(\tilde{0}) = \tau(\tilde{1}) = 1$

(O2) $\tau(A_1 \wedge A_2) \geq \tau(A_1) \wedge \tau(A_2)$, for $A_1, A_2 \in L^X$

(O3) $\tau\Big(\vee_{i\in\Delta} A_i\Big) \geq \wedge_{i\in\Delta}\tau(A_i)$ for any $\{A_i\}_{i\in\Delta} \subset L^X$.

The pair (X, τ) is called an L-fuzzy topological space and τ is also called a gradation of openness (GO) on X.

Definition 1.2 ([16]). A function $\mathcal{F} : L^X \to L$ is called an L-fuzzy co-topology of X if it satisfies the following conditions :

(C1) : $\mathcal{F}(\tilde{0}) = \mathcal{F}(\tilde{1}) = 1$

(C2) : $\mathcal{F}(A_1 \vee A_2) \geq \mathcal{F}(A_1) \wedge \mathcal{F}(A_2)$, for $A_1, A_2 \in L^X$

(C3) : $\mathcal{F}\Big(\wedge_{i\in\Delta} A_i\Big) \geq \wedge_{i\in\Delta}\mathcal{F}(A_i)$ for any $\{A_i\}_{i\in\Delta} \subset L^X$.

The pair $(X, \mathcal{F})$ is called an L-fuzzy co-topological space and $\mathcal{F}$ is also called a gradation of closedness (GC) on X.

Definition 1.3 ([11]). Let (X, τ) be an L-fuzzy topological space and let $Q : Pt(L^X) \times L^X \to L$ be a mapping defined by $Q(p_x, A) = \vee\{\tau(U);\ p_x q U \subseteq A\}$. Then Q is said to be a gradation of q-neighbourhoodness in (X, τ).

Proposition 1.4 ([11]). *Let Q be a gradation of q-neighbourhoodness in an L-fuzzy topological space (X, τ). Then*

(QN1) : $Q(p_x, \tilde{1}) = 1, Q(p_x, \tilde{0}) = 0\ \forall\ p_x \in Pt(L^X)$,

(QN2) : $Q(p_x, A) \leq Q(p_x, B)$ *if* $A, B \in L^X,\ A \subseteq B$

(QN3) : $Q(p_x, A \wedge B) = Q(p_x, A) \wedge Q(p_x, B)\ \forall\ p_x \in M(L^X)\ \&\ A, B \in L^X$.

(QN4) : $Q(p_x, A) \not\leq k \Rightarrow \exists B_p \in L^X$ *s.t* $p_x q B_p \subseteq A\ \&\ \wedge\{Q(r_y, B_p);\ r_y \in Pt(L^X);\ r_y q B_p\} \not\leq k$.

Proposition 1.5 ([11]). *Let $Q : Pt(L^X) \times L^X \to L$ be a mapping satisfying (QN1) - (QN3) of Proposition 1.5. Let $\bar{\tau} : L^X \to L$ be defined by $\bar{\tau}(A) = \wedge\{Q(p_x, A);\ p_x \in M(L^X)\ \&\ p_x q A\}$. Then $(X, \bar{\tau})$ forms an L-fuzzy topological space. If further the condition (QN4) of Proposition 1.4 is satisfied by Q then the mapping $\bar{Q} : Pt(L^X) \times L^X \to L$ defined by $\bar{Q}(p_x, A) = \vee\{\bar{\tau}(U)\ ;\ p_x q U \subset A\}$ is identical with Q.*

Proposition 1.6 ([11]). *Let Q be a gradation of q-neighbourhoodness in an L-fuzzy topological space (X, τ) and $\bar{\tau} : L^X \to L$ be defined by $\bar{\tau}(A) = \wedge\{Q(p_x, A);\ p_x \in M(L^X)\ \&\ p_x q A\}$ then $\bar{\tau}$ is an L-fuzzy topology on X and $\bar{\tau} = \tau$.*

Definition 1.7 ([10]). Let (X, τ) be an L-fuzzy topological space and $e \in Pt(L^X)$. The q-neighbourhood system of the fuzzy pt e w.r.t the Chang fuzzy topology τ_r, denoted by $\tilde{Q}_r(e)$, is defined by $\tilde{Q}_r(e) = \{U \in L^X;\ \exists\ V \in \tau_r$ satisfying $eqV \subseteq U\}$.

Definition 1.8 ([11]). Let (X, τ) be an L-fuzzy topological space and $N : Pt(L^X) \times L^X \to L$ be a mapping defined by $N(p_x, A) = \vee\{\tau(U);\ p_x \in U \subseteq A\}$. Then N is said to be a gradation of neighbourhoodness in (X, τ).

Definition 1.9 ([11]). Let (X,τ) be an L-fuzzy topological space and $e \in Pt(L^X)$. The neighbourhood system of the fuzzy pt e w.r.t the Chang fuzzy topology τ_r, denoted by $\tilde{N}_r(e)$, is defined by $\tilde{N}_r(e) = \{U \in L^X;\ \exists\ V \in \tau_r \text{ satisfying } e \in V \subseteq U\}$.

Definition 1.10 ([18]). Let L be a complete lattice. Define a relation ' $<<$' in L as follows: $\forall\ a,b \in L, a << b$ if and only if $\forall\ S \subset L,\ \vee S \geq b \Rightarrow \exists\ s \in S$ such that $s \geq a$. $\forall a \in L$, denote $\beta(a) = \{b \in L; b << a\}$, $\beta^0(a) = M(\beta(a))$.

Definition 1.11 ([12]). Let X be a non-empty crisp set. A fuzzy filter on L^X is a non-empty family $\mathcal{G}$ of L-fuzzy subsets of X such that

(i) $\tilde{0} \notin \mathcal{G}$

(ii) $\mathcal{G}$ is closed under finite intersection.

(iii) if $B \in \mathcal{G}$ & $B \subset A$ then $A \in \mathcal{G}$, $\forall A, B \in L^X$.

Definition 1.12 ([12]). Let (X,τ) be an L-fuzzy topological space $\mathcal{G} \subset L^X$ be a fuzzy filter on $L^X, e \in Pt(L^X)$. Then e is called a cluster point of $\mathcal{G}$ of upper grade l and lower grade k , denoted by $\mathcal{G}\infty^l e$ and $\mathcal{G}\infty_k e$ respectively if $l' = \wedge\{r \in L_0 :\ U \cap A \neq \tilde{0},\ \forall\ U \in \tilde{Q}_r(e)\ \&\ A \in \mathcal{G}\ \}$ and $k' = \vee\{r \in L_0 :\ \exists\ U \in \tilde{Q}_r(e)$, and $\exists A \in \mathcal{G}$ such that $A \cap U = \tilde{0}\}$. e is called a limit point of $\mathcal{G}$ of upper grade l and lower grade k, denoted by $\mathcal{G} \rightarrow^l e$ and $\mathcal{G} \rightarrow_k e$ respectively, if $l' = \wedge\{r \in L_0 :\ \tilde{Q}_r(e) \subset \mathcal{G}\}$ and $k' = \vee\{r \in L_0 :\ \tilde{Q}_r(e) \not\subset \mathcal{G}\}$.

Definition 1.13 ([1]). Let $\mathcal{G} : L^X \rightarrow L$ be a mapping satisfying

(GF1) : $\mathcal{G}(\tilde{0}) = 0;\ \mathcal{G}(\tilde{1}) = 1$,

(GF2) : $\mathcal{G}(A_1 \wedge A_2) \geq \mathcal{G}(A_1) \wedge \mathcal{G}(A_2);\ \forall\ A_1, A_2 \in L^X$,

(GF3) : $\mathcal{G}(B) \geq \mathcal{G}(A)$ if $A \subset B;\ A, B \in L^X$,

then $\mathcal{G}$ is said to be a generalised filter (g-filter) on L^X.

Definition 1.14 ([12]). Let $\mathcal{G}$ be a g-filter in an L-fuzzy topological space (X,τ) and $e \in Pt(L^X)$. Call e a limit pt of $\mathcal{G}$, denoted by $\mathcal{G} \rightarrow e$ if $Q(e,U) \leq \mathcal{G}(U)\ \forall U \in L^X$, where Q is the gradation of q-neighbourhoodness in (X,τ). Denote the join of all limit points of $\mathcal{G}$ by $lim\mathcal{G}$. Call e a cluster point of $\mathcal{G}$, denoted by $\mathcal{G}\infty e$ if $\mathcal{G}(A) \not\leq Q'(e,U) \Rightarrow A \cap U \neq \tilde{0},\ \forall\ A, U \in L^X$, where Q is the gradation of q-neighbourhoodness on (X,τ). Denote the join of all cluster points of $\mathcal{G}$ by $clu\mathcal{G}$.

Definition 1.15 ([11]). Let $(X,\mathcal{F})$ be an L-fuzzy co-topological space with $\mathcal{F}$ as a GC on X. For each $r \in L_0$ and for each $A \in L^X$ we define $cl(A,r) = \wedge\{D \in$

L^X ; $A \subseteq D$; $D \in \mathcal{F}_r\}$ where $\mathcal{F}_r = \{C \in L^X$; $\mathcal{F}(C) \geq r\}$. cl is said to be L-fuzzy closure operator in $(X, \mathcal{F})$.

Proposition 1.16 ([11]). *Let $(X, \mathcal{F})$ be an L-fuzzy co-topological space with $\mathcal{F}$ as a GC on X and let $cl : L^X \times L_0 \to L^X$ be the L-fuzzy closure operator in $(X, \mathcal{F})$. Then*

(CO1) : $cl(\tilde{0}, r) = \tilde{0}$; $cl(\tilde{1}, r) = \tilde{1}\ \forall\ r \in L_0$.
(CO2) : $cl(A, r) \supseteq A$, $\forall\ A \in L^X\ \&\ \forall\ r \in L_0$.
(CO3) : $cl(A, r) \subseteq cl(A, s)$ *if* $r \leq s$.
(CO4) : $cl(A_1 \vee A_2, r) = cl(A_1, r) \vee cl(A_2, r)$, $\forall\ r \in L_0$
(CO5) : $cl\Big(cl(A, r), r\Big) = cl(A, r)$, $\forall\ r \in L_0$.
(CO6) : *If* $l = \vee\{r \in L_0$; $cl(A, r) = A\}$ *then* $cl(A, l) = A$.

Proposition 1.17 ([11]). *A fuzzy point $p_x \in cl(A, m) \iff \forall\ U \in L^X$ satisfying $p_x q U \not{q} A$, we have $\tau(U) \not\geq m$.*

Corollary 1.18 ([11]). $p_x \notin cl(A, m) \iff \exists$ *at least one* $U \in \tilde{Q}_m(p_x)$ *s.t* $U \not{q} A$.

Proposition 1.19 ([11]). *In an L-fuzzy topological space (X, τ), $p_x \in cl(A, m) \iff \forall\ U \in \tau_m,\ p_x q U\ \Rightarrow\ U q A$.*

Definition 1.20 ([11]). Let (X, τ) be an L-fuzzy topological space and $e \in Pt(L^X)$. Let D be any directed set and $S : D \to Pt(L^X)$ be any fuzzy net. For $U \in L^X$ if $\exists\ m \in D$ s.t $S(n) q U\ \forall n \geq m$ holds then we say that SqU eventually; if for every $m \in D\ \exists\ n \in D$ s.t $n \geq m$ and $S(n) q U$ then we say SqU frequently. Call 'e' a cluster point of upper grade l, denoted by $S \infty^l e$ and of lower grade k, denoted by $S \infty_k e$, of a fuzzy net $S : D \to Pt(L^X)$, if $l' = \wedge\{r \in L_0;\ \forall\ U \in \tilde{Q}_r(e),\ U q S$ frequently$\}$ and $k' = \vee\{r \in L_0$; $\exists\ V \in \tilde{Q}_r(e)$ s.t $V \not{q} S$ eventually $\}$ respectively. Call 'e' a limit point of upper grade l, denoted by $S \to^l e$ and lower grade k, denoted by $S \to_k e$, of S if $l' = \wedge\{r \in L_0$; $\forall\ U \in \tilde{Q}_r(e)$, $U q S$ eventually $\}$ and $k' = \vee\{r \in L_0$; $\exists\ V \in \tilde{Q}_r(e)$ s.t $V \not{q} S$ frequently $\}$.

Lemma 1.21 ([11]). *Let (X, τ) & (Y, δ) be any two L-fts and let $Q, \hat{Q}$ be the gradation of q-neighbourhoodness in (X, τ) and (Y, δ) respectively. A mapping $f : (X, \tau) \to (Y, \delta)$ is gp iff $Q(e, f^{-1}(V)) \geq \hat{Q}(f(e), V), \forall\ e \in M(L^X)$ and $\forall\ V \in L^Y$.*

2. Some Results on g-Filters and Fuzzy Filters

In the paper 'Fuzzy Convergence Theory -II' we have given a definition of fuzzy filters and g-filters. In the same paper we have studied some important results of these two types of filters. Here we are going to study some more results keeping the view on the results of g-net which wiii be discussed in the next section.

Proposition 2.1. *Let $\mathcal{G}$ be a g-filter in an L-fts (X,τ), $e \in Pt(L^X)$ and $lim(\mathcal{G}) = \vee\{p_x \in Pt(L^X);\ \mathcal{G} \to p_x\}$ then $e \in lim(\mathcal{G}) \iff \mathcal{G} \to e$.*

Proof. The implication $\mathcal{G} \to e \Rightarrow e \in lim(\mathcal{G})$ is obvious. So we are to prove the converse part only.
Suppose there exists a fuzzy point e s.t $e \in lim(\mathcal{G})$ but $\mathcal{G} \not\to e$.
Then $\exists\ U \in L^X$ s.t $\mathcal{G}(U) \not\geq Q(e,U)$
$\Rightarrow \exists\ V \in L^X$ s.t $eqV \subseteq U$ and $Q(f,V) \not\leq \mathcal{G}(U)$, $\forall\ fqV$, by (QN4)
$\Rightarrow Q(f,V) \not\leq \mathcal{G}(V)$, $\forall\ fqV$ [since $V \subseteq U \Rightarrow \mathcal{G}(V) \leq \mathcal{G}(U)$]
$\Rightarrow \mathcal{G} \not\to f\ \forall\ fqV$.
This means if $\mathcal{G} \to f$ for some $f \in Pt(L^X)$ then $f \not{q} V$ i.e $f \in V^c$
so $\vee\{f \in Pt(L^X);\ \mathcal{G} \to f\} \subseteq V^c$
i.e $lim(\mathcal{G}) \subseteq V^c$.
Hence $e \in lim(\mathcal{G}) \Rightarrow e \in V^c \Rightarrow e \not{q} V$, a contradiction.
This completes the proof. □

Proposition 2.2. *Let $\mathcal{G}$ be a g-filter in an L-fts (X,τ), $e \in Pt(L^X)$ and $clu(\mathcal{G}) = \vee\{p_x \in Pt(L^X);\ \mathcal{G}\infty p_x\}$ then $e \in clu(\mathcal{G}) \iff \mathcal{G}\infty e$.*

Proof. We shall show only that $e \in clu(\mathcal{G}) \Rightarrow \mathcal{G}\infty e$ because the reverse part is straightforward.
If possible let $e \in clu(\mathcal{G})$ but $\mathcal{G} \not\infty e$. Then $\exists\ A,U \in L^X$ s.t $\mathcal{G}(A) \not\leq Q'(e,U)$ but $A \cap U = \tilde{0}$.
Now $Q(e,U) \not\leq \mathcal{G}'(A) \Rightarrow \exists\ V \in L^X$ s.t $eqV \subseteq U$ and $Q(f,V) \not\leq \mathcal{G}'(A)$, $\forall\ fqV$, by (QN4)
$\Rightarrow \mathcal{G}(A) \not\leq Q'(f,V)\ \forall\ fqV$ but $A \cap V = \tilde{0}$
$\Rightarrow \mathcal{G} \not\infty f\ \forall f \in Pt(L^X)$ with fqV
So, if $\mathcal{G}\infty f$ for some $f \in Pt(L^X)$ then $f \not{q} V$ i.e $f \in V^c$.
Hence $\vee\{f \in Pt(L^X);\ \mathcal{G}\infty f\} \subseteq V^c \Rightarrow clu(\mathcal{G}) \subseteq V^c$.

So, $e \in clu(\mathcal{G}) \Rightarrow e \in V^c \Rightarrow e \not{q} V$, a contradiction.
This completes the proof. □

Proposition 2.3. *Let (X, τ) be an L-fts, 'cl' be the closure operator in (X, τ), $A \in L^X$ and $e \in M(L^X)$. If $\exists$ a fuzzy filter $\mathcal{G}$ not containing A^c s.t $\mathcal{G} \to_l e$ for some $l \not\leq k \in L_0$ then $e \in cl(A, k')$.*

Proof. Let the given condition be satisfied and if possible let $e \notin cl(A, k')$ then $\exists$ at least one $U \in \tilde{Q}_{k'}(e)$ s.t $U \not{q} A$ i.e $U \subseteq A^c \Rightarrow A^c \in \tilde{Q}_{k'}(e)$ ———(i).
Now $\mathcal{G} \to_l e \Rightarrow l' = \vee\{r \in L_0;\ \mathcal{G} \not\supseteq \tilde{Q}_r(e)\} \not\geq k'$
$\Rightarrow \tilde{Q}_{k'}(e) \subseteq \mathcal{G}$ ——— (ii).
From (i) and (ii) we can say $A^c \in \mathcal{G}$, a contradiction.
Hence $e \in cl(A, k')$. □

Proposition 2.4. *Let (X, τ) be an L-fts, 'cl' be the closure operator in (X, τ), $A \in L^X$ and $e \in Pt(L^X)$. If $\exists$ a g-filter $\mathcal{G}$ on L^X satisfying $\mathcal{G}(A^c) \not\geq k$ but $\mathcal{G} \to e$ then $e \in cl(A, k)$.*

Proof. $\mathcal{G} \to e \Rightarrow \mathcal{G}(U) \geq Q(e, U),\ \forall\ U \in L^X$.
So in particular we have $\mathcal{G}(A^c) \geq Q(e, A^c)$.
So $\mathcal{G}(A^c) \not\geq k \Rightarrow Q(e, A^c) \not\geq k$
$\Rightarrow \vee\{\tau(U);\ eqU \subseteq A^c\} \not\geq k$
$\Rightarrow \vee\{\tau(U);\ eqU \not{q} A\} \not\geq k$
$\Rightarrow \tau(U) \not\geq k\ \forall\ U \in L^X$ satisfying $eqU \not{q} A$
$\Rightarrow e \in cl(A, k)$ □

Corollary 2.5. *For any g-filter $\mathcal{G}$ in an L-fts (X, τ) : $\lim(\mathcal{G}) \subseteq cl(A. k)$, if $\mathcal{G}(A^c) \not\geq k$.*

Proposition 2.6. *Let 'cl' be the closure operator in an L-fts (X, τ), $e \in Pt(L^X)$ and $A \in L^X$. If $e \in cl(A, k) \backslash A$ then $\exists$ a g-filter $\mathcal{G}$ on L^X and $\exists\ f \in M(cl(A, k) \backslash A)$ s.t $\mathcal{G}(A^c) \not\geq k$ but $\mathcal{G} \to f$.*

Proof. $e \in cl(A, k)$
$\Rightarrow \forall\ U \in L^X$ satisfying $eqU \not{q} A$, $\tau(U) \not\geq k$, by Proposition 1.18.
$\Rightarrow \tau(A^c) \not\geq k$ [as $e \notin A \Rightarrow eqA^c \not{q} A$]
$\Rightarrow \wedge_{f \in M(L^X)}\{Q(f, A^c);\ fqA^c\} \not\geq k$, by Proposition 1.7.

$\Rightarrow \exists\ f \in M(L^X)$ s.t fqA^c and $Q(f, A^c) \ngeq k$
$\Rightarrow f \notin A$ and $Q_f(A^c) \ngeq k$.

If we take $\mathcal{G} = Q_f$ then $\mathcal{G}(A^c) \ngeq k$ and obviously $\mathcal{G} \to f$.
Again $Q(f, A^c) \ngeq k$
$\Rightarrow Q(f, U) \ngeq k$, $\forall\ U \in L^X$ with $fqU \not{q}A$ [as $U \not{q}A \Rightarrow U \subseteq A^c \Rightarrow Q(f, U) \leq Q(f, A^c)$] $\Rightarrow f \in cl(A, k)$.
So, $f \in M(cl(A, k)\backslash A)$. Hence the proof. □

Proposition 2.7. *Let 'cl' be the closure operator in an L-fts (X, τ) and $A \in L^X$. If $\exists$ a g-filter on L^X s.t $\mathcal{G}(A^c) \ngeq k$ and $\mathcal{G} \to f\ \forall\ f \in \beta^0(e)$, then $e \in cl(A, k)$.*

Proof. $\mathcal{G} \to f$, $\forall\ f \in \beta^0(e)$
$\Rightarrow \mathcal{G}(U) \geq Q(f, U)\ \forall\ f \in \beta^0(e)$ and $\forall\ U \in L^X$. ———— (i)
If possible let $e \notin cl(A, k)$ then $\exists$ at least one $V \in L^X$ s.t $eqV \not{q}A$ and $\tau(V) \geq k$.
Now $e = \vee\{f \in \beta^0(e)\}$ and $eqV \Rightarrow \exists\ f^0 \in \beta^0(e)$ s.t $f^0qV \not{q}A$.
So, $f^0qV \subseteq A^c$ and $\tau(V) \geq k \Rightarrow Q(f^0, A^c) \geq k$.
Therefore by (i), $\mathcal{G}(A^c) \geq Q(f^0, A^c) \geq k$, a contradiction.
Hence $e \in cl(A, k)$. □

Proposition 2.8. *Let Q be the gradation of q-neighbourhoodness in an L-fts (X, τ). Then for every $e \in M(L^X)$, the mapping $Q_e : L^X \to L$, defined by $Q_e(U) = Q(e, U)$, $\forall\ U \in L^X$, is a g-filter on L^X and $Q_e \to e$.*

The proof follows from the properties (QN1) - (QN3) of Q.

Proposition 2.9. *Let (X, τ) be an L-fts with Q as the gradation of q-neighbourhoodness, Λ be the collection of all g-filters on L^X and $lim(\mathcal{G}) = \vee\{p_x \in Pt(L^X); \mathcal{G} \to p_x\}$ where $\mathcal{G} \in \Lambda$. Then the following properties are satisfied by $lim(\mathcal{G})$:*

(i) *for every $e \in M(L^X)$, $\exists\ \mathcal{G} \in \Lambda$ s.t $e \in lim(\mathcal{G})$,*
(ii) *for every $e \in M(L^X)$ $\wedge\{\mathcal{G}(U);\ \mathcal{G} \in \Lambda;\ e \in lim(\mathcal{G})\} > 0 \Rightarrow eqU$, $\forall\ U \in L^X$,*
(iii) *if for some $(e, U) \in M(L^X) \times L^X$ with eqU and for some $\mathcal{H} \in \Lambda, \mathcal{H}(U) \ngeq \wedge\{\mathcal{G}(U);\ \mathcal{G} \in \Lambda;\ e \in lim(\mathcal{G})\}$ then $\exists\ V \in L^X$ s.t $eqV \subseteq U$ and $\forall\ f \in Pt(L^X)$ with fqV, $f \notin lim(\mathcal{H})$.*

Proof. (i) Since for every $e \in M(L^X)$, Q_e is a g-filter and $Q_e \to e, then$ so $e \in lim(Q_e)$.
(ii) Since $e \in lim(Q_e)$, then $\wedge\{\mathcal{G}(U);\ \mathcal{G} \in \Lambda;\ e \in lim(\mathcal{G})\} > 0 \Rightarrow Q_e(U) > 0 \Rightarrow Q(e,U) > 0 \Rightarrow eqU$.
(iii) $\mathcal{H}(U) \not\geq \wedge\{\mathcal{G}(U);\ \mathcal{G} \in \Lambda;\ e \in lim(\mathcal{G})\}$
$\Rightarrow \mathcal{H}(U) \not\geq Q_e(U)$ [since $e \in lim(Q_e)$]
$\Rightarrow \mathcal{H}(U) \not\geq Q(e,U)$.
Then, by (QN4), $\exists\ V \in L^X$ s.t $eqV \subseteq U$ and $\forall\ fqV,\ Q(f,V) \not\leq \mathcal{H}(U)$
$\Rightarrow \forall\ fqV,\ f \notin lim(\mathcal{H})$. □

Proposition 2.10. *Let Λ_0 be a collection of g-filters on L^X and for every $\mathcal{G} \in \Lambda_0$, $lim(\mathcal{G})$ is a fuzzy subset of X satisfying*

(i) *for every $e \in M(L^X)$, $\exists\ \mathcal{G} \in \Lambda_0$ s.t $e \in lim(\mathcal{G})$,*
(ii) $\wedge\{\mathcal{G}(U);\ \mathcal{G} \in \Lambda_0;\ e \in lim(\mathcal{G})\} > 0 \Rightarrow eqU$,
(iii) *if for some $(e,U) \in M(L^X) \times L^X$ with eqU and for some $\mathcal{H} \in \Lambda_0$, $\mathcal{H}(U) \not\geq \wedge\{\mathcal{G}(U);\ \mathcal{G} \in \Lambda_0;\ e \in \lim(\mathcal{G})\}$ then $\exists\ V \in L^X$ s.t $eqV \subseteq U$ and $\forall\ fqV,\ f \notin lim(\mathcal{H})$.*

Then the mapping $\bar{Q} : Pt(L^X) \times L^X \to L$, given by

$$\bar{Q}(e,U) = \wedge\{\mathcal{G}(U); \mathcal{G} \in \Lambda_0; e \in lim(\mathcal{G})\}$$

if $e \in M(L^X) = \vee\{\bar{Q}(f,U); f \in \beta^0(e)\}$ if $e \in Pt(L^X)\backslash M(L^X)$, is a gradation of q-neighbourhoodness on X.

Proof. Let $e \in M(L^X)$; then by (i) the collection $\{\mathcal{G}(U);\ \mathcal{G} \in \Lambda_0\ \&\ e \in lim(\mathcal{G})\}$ is non-empty, $\forall\ U \in L^X$. Also $\mathcal{G}(\tilde{1}) = 1\ \forall\ \mathcal{G} \in \Lambda_0$ so $\wedge\{\mathcal{G}(\tilde{1});\ \mathcal{G} \in \Lambda_0\ \&\ e \in lim(\mathcal{G})\} = 1$ i.e $\bar{Q}(e,\tilde{1}) = 1$. Similarly, $\mathcal{G}(\tilde{0}) = 0,\ \forall\ \mathcal{G} \in \Lambda_0$ so $\bar{Q}(e,\tilde{0}) = 0$.

Next if $e \in Pt(L^X)\backslash M(L^X)$. Then $\bar{Q}(e,\tilde{1}) = \vee\{\bar{Q}(f,\tilde{1});\ f \in \beta^0(e)\} = 1$ and $\bar{Q}(e,\tilde{0}) = \vee\{\bar{Q}(f,\tilde{0});\ f \in \beta^0(e)\} = 0$.
So, (QN1) is verified.

(QN2) follows from (GF3).
$\bar{Q}(e,U \cap V) = \wedge\{\mathcal{G}(U \cap V);\ \mathcal{G} \in \Lambda_0;\ e \in lim(\mathcal{G})\}$
$\geq \wedge\{\mathcal{G}(U) \wedge \mathcal{G}(V);\ \mathcal{G} \in \Lambda_0;\ e \in lim(\mathcal{G})\}$
$\geq [\wedge\{\mathcal{G}(U);\ \mathcal{G} \in \Lambda_0;\ e \in lim(\mathcal{G})\}] \wedge [\wedge\{\mathcal{G}(V);\ \mathcal{G} \in \Lambda_0;\ e \in lim(\mathcal{G})\}]$
$= \bar{Q}(e,U) \wedge \bar{Q}(e,V)$.
Also by (QN2), $\bar{Q}(e,U \cap V) \leq \bar{Q}(e,U), \bar{Q}(e,V)$

$\Rightarrow \ \bar{Q}(e, U \cap V) \leq \bar{Q}(e, U) \wedge \bar{Q}(e, V)$.
So, (QN3) is verified.

To verify (QN4) we shall prove first the following result:
$\forall\ (e, U) \in M(L^X) \times L^X$,
$\wedge\ \{\mathcal{G}(U);\ \mathcal{G} \in \Lambda_0;\ e \in lim(\mathcal{G})\}$
$= \vee\{\wedge\{\wedge\{\mathcal{H}(V);\ \mathcal{H} \in \Lambda_0;\ f \in lim(\mathcal{H})\};\ f \in Pt(L^X);\ fqV\};\ V \in L^X;\ eqV \subseteq U\}$
— (A).
<u>Case I:</u> If $e \not{q} U$ then by (ii) $\wedge\{\mathcal{G}(U);\ \mathcal{G} \in \Lambda_0;\ e \in lim(\mathcal{G})\} = 0$.
Also $e \not{q} U$ implies $\exists$ no $V \in L^X$ s.t $eqV \subseteq U$,
so R.H.S of (A) becomes supremum over an empty collection hence it is zero.
i.e L.H.S = R.H.S.
<u>Case II:</u> If eqU then for any $V \in L^X$ satisfying $eqV \subseteq U$ as we have $\mathcal{H}(V) \leq \mathcal{H}(U),\ \forall\ \mathcal{H} \in \Lambda_0$
so, $\wedge\{\mathcal{H}(V);\ \mathcal{H} \in \Lambda_0;\ f \in lim(\mathcal{H})\} \leq \wedge\{\mathcal{H}(U);\ \mathcal{H} \in \Lambda_0;\ f \in lim(\mathcal{H})\},\ \forall\ f \in Pt(L^X)$
$\Rightarrow \ \wedge\{\wedge\{\mathcal{H}(V);\ \mathcal{H} \in \Lambda_0;\ f \in lim(\mathcal{H})\};\ f \in Pt(L^X);\ fqV\} \leq \wedge\{\mathcal{G}(U);\ \mathcal{G} \in \Lambda_0;\ e \in lim(\mathcal{G})\}$ [since eqV].
As this is true for all $V \in L^X$ satisfying $eqV \subseteq U$, then
$\vee\{\wedge\{\wedge\{\mathcal{H}(V);\ \mathcal{H} \in \Lambda_0;\ f \in lim(\mathcal{H})\};\ f \in Pt(L^X);\ fqV\};\ V \in L^X;\ eqV \subseteq U\} \leq \wedge\{\mathcal{G}(U);\ \mathcal{G} \in \Lambda_0;\ e \in lim(\mathcal{G})\}$.
If possible let L.H.S < R.H.S then
$\wedge\{\wedge\{\mathcal{H}(V);\ \mathcal{H} \in \Lambda_0;\ f \in Pt(L^X);\ f \in lim(\mathcal{H})\};\ fqV\}$
$< \wedge\{\mathcal{G}(U);\ \mathcal{G} \in \Lambda_0;\ e \in lim(\mathcal{G})\};\ \forall\ V$ satisfying $eqV \subseteq U$
$\Rightarrow\ \exists\ f^0 \in Pt(L^X)$ s.t f^0qV and
$\wedge\{\mathcal{H}(V);\ \mathcal{H} \in \Lambda_0;\ f^0 \in lim(\mathcal{H})\} \not\geq \wedge\{\mathcal{G}(U);\ \mathcal{G} \in \Lambda_0;\ e \in lim(\mathcal{G})\}$
$\Rightarrow\ \exists\ \mathcal{H}_0 \in \Lambda_0$ s.t $f^0 \in lim(\mathcal{H}_0);\ f^0qV$ and $\mathcal{H}_0(V) \not\geq \wedge\{\mathcal{G}(U);\ e \in lim(\mathcal{G})\},\ \forall\ V \in L^X$ with $eqV \subseteq U$.
In particular we have $eqU \subseteq U$ so, $\mathcal{H}_0(U) \not\geq \wedge\{\mathcal{G}(U);\ \mathcal{G} \in \Lambda_0;\ e \in lim(\mathcal{G})\}$.
Then by condition (iii), $\exists\ W \in L^X$ s.t $eqW \subseteq U$ and $\forall\ f \in Pt(L^X)$ with $fqW,\ f \notin lim(\mathcal{H}_0)$, which contradicts the existence of f^0.
Hence $\wedge\{\mathcal{G}(U);\ \mathcal{G} \in \Lambda_0;\ e \in lim(\mathcal{G})\}$
$= \vee\{\wedge\{\wedge\{\mathcal{H}(V);\ \mathcal{H} \in \Lambda_0;\ f \in lim(\mathcal{H})\};\ f \in Pt(L^X);\ fqV\};\ V \in L^X;\ eqV \subseteq U\}$
i.e $\bar{Q}(e, U) = \vee\{\wedge\{\bar{Q}(f, V);\ f \in Pt(L^X);\ fqV\};\ V \in L^X;\ eqV \subseteq U\}$.
Now $\bar{Q}(e, U) \not\leq k$
$\Rightarrow\ \vee\{\wedge\{\bar{Q}(f, V);\ f \in Pt(L^X);\ fqV\};\ V \in L^X;\ eqV \subseteq U\} \not\leq k$

$\Rightarrow \exists\, V^0 \in L^X$ s.t $eqV^0 \subseteq U$ and $\wedge\{\bar{Q}(f, V^0);\ f \in Pt(L^X);\ fqV^0\} \not\leq k$.
Hence (QN4) is verified for all $e \in M(L^X)$.
If $e \in Pt(L^X)\backslash M(L^X)$ then $\bar{Q}(e,U) \not\leq k \Rightarrow \vee\{\bar{Q}(f,U);\ f \in \beta^0(e)\} \not\leq k$
$\Rightarrow \exists\, V^0 \in L^X$ s.t $f^0qV^0 \subseteq U$ and $\wedge\{\bar{Q}(g,V^0);\ g \in Pt(L^X)\ \&\ gqV^0\} \not\leq k$
$\Rightarrow \exists\, V^0 \in L^X$ s.t $eqV^0 \subseteq U$ and $\wedge\{\bar{Q}(g,V^0);\ g \in Pt(L^X)\ \&\ gqV^0\} \not\leq k$.
Hence (QN4) is verified for all $e \in Pt(L^X)$. □

Proposition 2.11. *Let Q be a gradation of q-neighborhoodness on an L-fts (X,τ) and Ω be the collection of all q-nbd filters. Let Λ_0 be a collection of g-filters on L^X containing Ω and satisfying the conditions (i)-(iii) of Proposition 2.10. Let $lim(\mathcal{G})$ be defined by $lim(\mathcal{G}) = \vee\{p_x \in Pt(L^X);\ \mathcal{G} \to p_x\}$ for every $\mathcal{G} \in \Lambda_0$. Then the mapping $\bar{Q} : Pt(L^X) \times L^X \to L$, defined by $\bar{Q}(e,U) = \wedge\{\mathcal{G}(U);\ \mathcal{G} \in \Lambda_0;\ e \in lim(\mathcal{G})\}$ if $e \in M(L^X) = \vee\{\bar{Q}(f,U);\ f \in \beta^0(e)\}$ if $e \in Pt(L^X)\backslash M(L^X)$, is a gradation of q-neighborhoodness on (X,τ) and $\bar{Q} = Q$.*

Proof. Since $lim(\mathcal{G})$ is the join of all limit points of $\mathcal{G}$, then it satisfies all the properties of Proposition 2.9. Hence by Proposition 2.10, $\bar{Q}$ is a gradation of q-neighborhoodness on (X,τ).
To show $\bar{Q} = Q$ we see that the neighborhood filter $Q_e \to e$, $\forall\, e \in M(L^X)$, so $e \in lim(Q_e)$
$\Rightarrow \wedge\{\mathcal{G}(U);\ \mathcal{G} \in \Lambda_0;\ e \in lim(\mathcal{G})\} \leq Q_e(U)$, $\forall\, e \in M(L^X)\ \&\ \forall\, U \in L^X$.
i.e $\bar{Q}(e,U) \leq Q(e,U)$, $\forall\, e \in M(L^X)$.
If possible let $Q(e^0,U^0) > \bar{Q}(e^0,U^0)$ for some $(e^0,U^0) \in M(L^X) \times L^X$.
Then $\wedge\{\mathcal{G}(U^0);\ \mathcal{G} \in \Lambda_0;\ e^0 \in lim(\mathcal{G})\} < Q(e^0,U^0)$
$\Rightarrow \exists\, \mathcal{G}_0 \in \Lambda_0$ s.t $e^0 \in lim(\mathcal{G}_0)$ but $\mathcal{G}_0(U^0) \not\geq Q(e^0,U^0)$
$\Rightarrow e^0 \in lim(\mathcal{G}_0)$ but $\mathcal{G}_0 \not\to e^0$, contradictory to the Proposition 2.1.
So, $\bar{Q}(e,U) = Q(e,U)$, $\forall\, e \in M(L^X)$ and $\forall\, U \in L^X$.
Next let $e \in Pt(L^X)\backslash M(L^X)$ and if possible let $\bar{Q}(e,U) \not\geq Q(e,U)$ for some $U \in L^X$.
i.e $\vee\{\bar{Q}(f,U);\ f \in \beta^0(e)\} \not\geq Q(e,U)$
$\Rightarrow \vee\{Q(f,U);\ f \in \beta^0(e)\} \not\geq Q(e,U)$.
Then by (QN4) $\exists\, V \in L^X$ s.t $eqV \subseteq U$ and $\forall\, g \in Pt(L^X)$ with gqV,
$Q(g,V) \not\leq \vee\{Q(f,U);\ f \in \beta^0(e)\}$
$\Rightarrow Q(g,V) \not\leq \vee\{Q(f,V);\ f \in \beta^0(e)\}\ \forall\, g \in Pt(L^X)$ with gqV ———— (i)
Again $eqV \subseteq U \Rightarrow \exists\, h^0 \in \beta^0(e)$ s.t $h^0qV \subseteq U$,

so by (i) $Q(h^0, V) \not\leq \vee\{Q(f, V);\ f \in \beta^0(e)\}$
$\Rightarrow\ Q(h^0, V) \not\leq Q(h^0, V)$, a contradiction. So, $\bar{Q}(e, U) \geq Q(e, U),\ \forall\ U \in L^X$.
Again $\bar{Q}(e, U) = \vee\{\bar{Q}(f, U);\ f \in \beta^0(e)\}$
$= \vee\{Q(f, U);\ f \in \beta^0(e)\} \leq Q(e, U)$. since $f \leq e\ \forall\ f \in \beta^0(e)$.
Then $\bar{Q}(e, U) = Q(e, U),\ \forall\ e \in Pt(L^X)\backslash M(L^X)$. □

3. Generalized Net (g-Net)

In this section we introduce a concept of generalized net in fuzzy setting. This is done by assigning a grade to each value of a fuzzy net. The definition is as follows:

Definition 3.1. Let D be any directed set. Any mapping $s : D \to Pt(L^X) \times L$ given by $s(n) = (e^n, r^n)\ \forall\ n \in D$ is said to be a generalized fuzzy net (or g-net) if $\forall\ m, l \in D,\ \exists\ k \in D$ s.t $k \geq m \vee l$ and $r^k \geq r^m \wedge r^l$.
In particular if we take $r^n = 1,\ \forall n \in D$ then the g-net reduces to a fuzzy net.

Definition 3.2. Let Q be the gradation of q-neighbourhoodness in an L-fts (X, τ). A g-net $s : D \to Pt(L^X) \times L$ given by $s(n) = (e^n, r^n)$ is said to converge to a fuzzy pt e, symbolically written as $s \to e$ if for every $U \in L^X$ with $Q(e, U) > 0,\ \exists\ m \in D$ s.t $r^m \geq Q(e, U)$ and $e^n qU\ \forall\ n \geq m$. Denote the join of all limit points of s by $lim(s)$ i.e $lim(s) = \vee\{p_x \in Pt(L^X);\ s \to p_x\}$

Definition 3.3. Let $\mathcal{G} : L^X \to Pr(L)$ be a prime valued g-filter and let $D = \{(e, U) \in M(L^X) \times L^X; eqU\ \&\ \mathcal{G}(U) > 0\}$. If we define a relation '$\geq$' on D by $(e, U) \geq (f, V) \iff U \subseteq V\ \forall\ (e, U), (f, V) \in D$, then '$\geq$' directs the set D. Now let $s(\mathcal{G}) : D \to M(L^X) \times L$ be given by $s(\mathcal{G})(e, U) = (e, \mathcal{G}(U))$. $s(\mathcal{G})$ is said to be the molecule g-net associated with $\mathcal{G}$.

Lemma 3.4. *Let X be a non-empty set and $p_x \in Pt(L^X), U \in L^X$ then $p_x qU \Rightarrow \exists k \in M(L)$ s.t $k \leq p$ but $k_x qU$.*

Proof. $p_x qU \Rightarrow p \not\leq U^c(x)$. As M(L) is a join generating subset of L so $\exists$ a subset S of M(L) s.t $\vee S = p \Rightarrow \exists k \in S$ s.t $k \leq p$ but $k \not\leq U^c(x)\ \Rightarrow\ k \leq p$ but $k_x qU$. □

Proposition 3.5. *Let $\mathcal{G} : L^X \to Pr(L)$ be a prime valued g-filter in an L-fts (X, τ). Then $\mathcal{G} \to e \iff s(\mathcal{G}) \to e$.*

Proof. Let $\mathcal{G} \to e$ then for any $U \in L^X$ with $Q(e,U) > 0$, by lemma 3.4, $\exists\, f \in M(L^X)$ s.t $fqU \Rightarrow (f,U) \in D$. For $m = (f,U)$, $r^m = \mathcal{G}(U)$ and $e^m = f$, where $s(\mathcal{G})(m) = (e^m, r^m)$. Since $\mathcal{G} \to e$ so $r^m = \mathcal{G}(U) \geq Q(e,U)$, where Q is the gradation of q-neighbourhoodness in (X,τ).
Again $\forall\, (g,V) \in D$ with $(g,V) \geq (f,U)$, $s(\mathcal{G})(g,V) = (g, \mathcal{G}(V))$ and gqV
$\Rightarrow gqU$ $\Big($since $V \subseteq U\Big)$.
So, $s(\mathcal{G}) \to e$.
Conversely, let $s(\mathcal{G}) \to e$ and let $U \in L^X$ be any fuzzy subset of X.
If $Q(e,U) = 0$ then $\mathcal{G}(U) \geq Q(e,U)$ holds trivially.
So, let $Q(e,U) > 0$. As $s(\mathcal{G}) \to e$ so $\exists\, (g, A_0) \in D$ s.t $s(\mathcal{G})(g,A_0) = (g, \mathcal{G}(A_0))$ with $\mathcal{G}(A_0) \geq Q(e,U)$. Then $\forall\, (h,V) \in D$ with $(h,V) \geq (g,A_0)$, $s(\mathcal{G})(h,V) = (h, \mathcal{G}(V))$ and hqU.
Therefore for all hqA_0 as $(h,A_0) \geq (g,A_0)$ so $hqU \Rightarrow A_0 \subseteq U$
$\Rightarrow \mathcal{G}(U) \geq \mathcal{G}(A_0) \geq Q(e,U)$.
As $U \in L^X$ is arbitrary so $\mathcal{G} \to e$. □

Definition 3.6. Let Q be the gradation of q-neighbourhoodness in an L-fts (X,τ). A g-net $s : D \to Pt(L^X) \times L$ given by $s(n) = (e^n, r^n)$ is said to have a cluster pt e, symbolically written as $s \infty e$, if for every $U \in L^X$ with $Q(e,U) > 0$ and $\forall\, m \in D$ with $r^m \not\leq Q'(e,U)$ $\exists\, n \in D$ s.t $n \geq m$ and $e^n qU$. Denote the join of all cluster points of s by $clu(s)$ i.e $clu(s) = \vee\{p_x \in Pt(L^X);\ s \infty p_x\}$.

Proposition 3.7. *Let $\mathcal{G} : L^X \to Pr(L)$ be a prime valued g-filter in an L-fts (X,τ). Then $\mathcal{G} \infty e \iff s(\mathcal{G}) \infty e$.*

Proof. Let $\mathcal{G} \infty e$. Let $U \in L^X$ with $Q(e,U) > 0$ and $(f,V) \in D$ with $s(\mathcal{G})(f,V) = (f, \mathcal{G}(V))$ and $\mathcal{G}(V) \not\leq Q'(e,U)$.
Since $\mathcal{G} \infty e$ so $U \cap V \neq \tilde{0} \Rightarrow \exists\, x \in X$ s.t $(U \cap V)(x) > 0 \Rightarrow (U \cap V)^c(x) < 1$
$\Rightarrow \exists\, p \in M(L)$ s.t $p \not\leq (U \cap V)^c(x)$ [since M(L) is a join generating subset of L]
$\Rightarrow p_x \notin (U \cap V)^c \Rightarrow p_x q (U \cap V)$.
Let $g = p_x$ then $g \in M(L^X)$ and $gq(U \cap V)$.
So, $(g,V) \in D$ and $(g,V) \geq (f,V)$ as well as gqU.
Hence $s(\mathcal{G}) \infty e$.
Conversely, let $s(\mathcal{G}) \infty e$ and let $\mathcal{G}(A) \not\leq Q'(e,U)$ for some $A, U \in L^X$.
Then $\mathcal{G}(A) \not\leq Q'(e,U) \Rightarrow \mathcal{G}(A) \neq 0 \Rightarrow A \neq \tilde{0} \Rightarrow \exists\, f \in M(L^X)$ s.t fqA
$\Rightarrow (f,A) \in D$ and $s(\mathcal{G})(f,A) = (f, \mathcal{G}(A))$.

Let $(f, A) = m$ then as $s(\mathcal{G}) \infty e$ and $r^m = \mathcal{G}(A) \not\leq Q'(e, U)$, so $\exists\, (g, V) \in D$ s.t $(g, V) \geq (f, A)$ and gqU.
Again $(g, V) \geq (f, A) \Rightarrow V \subseteq A$.
Further $(g, V) \in D \Rightarrow gqV \Rightarrow gqA$ [since $V \subseteq A$]
Thus $g \in M(L^X)$, gqA and gqU so $gq(A \cap U)$
$\Rightarrow A \cap U \neq \tilde{0}$. Hence $\mathcal{G} \infty e$. □

Proposition 3.8. *Let $s : D \to M(L^X) \times L$ be a molecule g-net given by $s(n) = (e^n, r^n)$, then the mapping $\mathcal{G}(s) : L^X \to L$ defined by $\mathcal{G}(s)(U) = \vee\{r^m;\ s(m) = (e^m, r^m)\ \&\ e^n qU\ \forall\, n \geq m\}$ is a g-filter on L^X. It is called the associated g-filter.*

Proof. (i) $\mathcal{G}(s)(\tilde{0})$ = sup of an empty subset of $L = 0$
(ii) Let $U, V \in L^X$. If $e^n \not{q} U$ or $e^n \not{q} V$ frequently then correspondingly $\mathcal{G}(s)(U)$ or $\mathcal{G}(s)(V) = 0$ so, $\mathcal{G}(s)(U \cap V) \geq \mathcal{G}(s)(U) \wedge \mathcal{G}(s)(V)$ holds trivially.
So, let $\exists\, m, l \in D$ s.t $e^n qU\ \forall n \geq m$ and $e^n qV\ \forall\, n \geq l$ then obviously $\mathcal{G}(s)(U) \geq r^m$ and $\mathcal{G}(s)(V) \geq r^l$. Now from the definition of g-net $\exists\, k \in D$ s.t $k \geq m \vee l$ and $r^k \geq r^m \wedge r^l$.
Again $\forall\, n \geq k$ as $e^n qU$, $e^n qV$ & $e^n \in M(L^X)$ so $e^n q(U \cap V)$
$\Rightarrow \mathcal{G}(s)(U \cap V) \geq r^k \geq r^m \wedge r^l$.
As L is completely distributive so $\mathcal{G}(s)(U \cap V) \geq \mathcal{G}(s)(U) \wedge \mathcal{G}(s)(V)$.
(iii) $U \supseteq V \Rightarrow \mathcal{G}(s)(U) \geq \mathcal{G}(s)(V)$ is obvious from the definition of $\mathcal{G}(s)$. □

Definition 3.9. Let $s : D \to M(L^X) \times L$ be a molecule g-net given by $s(n) = (e^n, r^n)$ then the mapping $\mathcal{G}(s) : L^X \to L$ defined by $\mathcal{G}(s)(U) = \vee\{r^m;\ s(m) = (e^m, r^m)\ \&\ e^n qU\ \forall\, n \geq m\}$ is called the g-filter associated with the g-net s on L^X.

Proposition 3.10. *Let s be a molecule g-net in an L-fts (X, τ) and $e \in Pt(L^X)$. Then $s \to e \Rightarrow \mathcal{G}(s) \to e$ where $\mathcal{G}(s)$ is the associated g-filter with the molecule g-net s.*

Proof. Let $s : D \to M(L^X) \times L$ be given by $s(n) = (e^n, r^n)$ and $s \to e$ and let $U \in L^X$. To show $\mathcal{G}(s)(U) \geq Q(e, U)$.
If $Q(e, U) = 0$ then obviously $\mathcal{G}(s)(U) \geq Q(e, U)$.
So, let $Q(e, U) > 0$ then as $s \to e$ so $\exists\, m \in D$ s.t $r^m \geq Q(e, U)$ and $e^n qU\ \forall\, n \geq m$
$\Rightarrow \mathcal{G}(s)(U) = \vee\{r^k;\ s(k) = (e^k, r^k)\ \&\ e^n qU\ \forall\, n \geq k\} \geq Q(e, U)$.
As $U \in L^X$ is arbitrary so $\mathcal{G}(s) \to e$. □

Now to show that the converse of the above proposition is not true always we consider the following example.

Example 3.11. Let X be a non-empty set and $\tau : I^X \to I$ be given by $\tau(\tilde{0}) = \tau(\tilde{1}) = 1$& $\tau(A) = 0$ for all other fuzzy subset A of X, where $I = [0, 1]$.
Then τ is a gradation of openness on X.
Let $s : N \to Pt(I^X) \times I$ be given by $s(n) = (1_x, 1 - \frac{1}{n})$ for some fixed $x \in X$, where N is the set of all natural numbers.
Then $Q(e, U) = 0,\ \forall\ U \in L^X$ if $U \neq \tilde{1}$
and $Q(e, \tilde{1}) = 1,\ \forall\ e \in Pt(I^X)$. Again if $U \in I^X$ be such that $U(x) > 0$ then $\mathcal{G}(s)(U) = \vee\{r^k;\ s(k) = (e^k, r^k)\ \&\ e^n qU\ \forall\ n \geq k\} = \vee_{n \in N}\{1 - \frac{1}{n}\} = 1$.
$\mathcal{G}(s)(U) = 0$ for all other $U \in I^X$.
So, $\mathcal{G}(s)(U) \geq Q(e, U)\ \forall\ U \in I^X$ and for every $e \in Pt(I^X)$.
So, $\mathcal{G}(s) \to e$ for every $e \in Pt(I^X)$.
On the contrary although $Q(e, \tilde{1}) = 1 > 0\ \forall\ e \in Pt(I^X)$ but $1 - \frac{1}{n} \ngeq Q(e, \tilde{1})\ \forall\ n \in N$. Hence $s \not\to e\ \forall\ e \in Pt(I^X)$.

Proposition 3.12 *For any molecule g-net s in an L-fts (X, τ) and $e \in Pt(L^X)$, $s \infty e \Rightarrow \mathcal{G}(s) \infty e$ where $\mathcal{G}(s)$ is the associated g-filter with the molecule g-net s.*

Proof. We have $\mathcal{G}(s)(A) = \vee\{r^m;\ e^n qA,\ \forall n \geq m\}$ for any fuzzy subset A.
Let $U, V \in L^X$, such that $\mathcal{G}(s)(U) \nleq Q'(e, V)$
then $\vee\{r^m;\ e^n qU,\ \forall n \geq m\} \nleq Q'(e, V)$
$\Rightarrow \exists m \in D$ s.t $r^m \nleq Q'(e, V)$ and $e^n qU,\ \forall n \geq m$.
Again as $s \infty e$ so $\exists\ n_0 \in D$ s.t $n_0 \geq m$ and $e^{n_0} qV$.
So $e^{n_0} qV$ and $e^{n_0} qU$ and $e^{n_0} \in M(L^X)$
$\Rightarrow e^{n_0} q(U \cap V)$
$\Rightarrow U \cap V \neq \tilde{0}$
$\Rightarrow \mathcal{G}(s) \infty e$. □

Proposition 3.13. *Let s be a g-net in an L-fts (X, τ) and $e,\ f \in Pt(L^X)$. Then*

(i) $s \to e \Rightarrow s \infty e$.
(ii) $s \to e \geq f \Rightarrow s \to f$.
(iii) $s \infty e \geq f \Rightarrow s \infty f$.

The proof is straightforward.

Definition 3.14. Let $s : D \to Pt(L^X) \times L$ be a g-net, given by $s(n) = (e^n, r^n)$ and let E be any directed set then a g-net $T : E \to Pt(L^X) \times L$ is said to be a g-subnet of s if $\exists$ a mapping $N : E \to D$ s.t $T(n) = s(N(n)) \; \forall \; n \in E$.

Proposition 3.15. *Let $s : D \to Pt(L^X) \times L$ given by $s(n) = (e^n, r^n)$ be a g-net on L^X. Then $D_t = \{n \in D; \; s(n) = (e^n, r^n) \; \& \; r^{n'} \geq t\}$ is a directed subset of D for all $t \in L$ if D_t is a non-empty subset of D.*

Proof. Let $m, n \in D_t$ then $r^m, r^n \geq t$ where $s(m) = (e^m, r^m)$ and $s(n) = (e^n, r^n)$. So, from the definition of g-net $\exists \; k \in D$ s.t $k \geq m \vee n \; \& \; r^k \geq r^m \wedge r^n \geq t$ where $s(k) = (e^k, r^k)$. So, $k \in D_t \; \& \; k \geq m \vee n$. Hence D_t is a directed subset of D. □

Definition 3.16. Let $s : D \to Pt(L^X) \times L$ be a g-net on L^X and let $D_t = \{n \in D; \; r^n \geq t$ where $s(n) = (e^n, r^n)\}$. Then D_t is a directed subset of D provided it is non-empty. We define a mapping $s_t : D_t \to Pt(L^X)$ given by $s_t(n) = e^n$ if $s(n) = (e^n, r^n) \; \forall \; n \in D_t$. Then s_t is a fuzzy net on L^X, s_t is said to be the t-level projection of s over $Pt(L^X)$.

Proposition 3.17. *Let $s : D \to Pt(L^X) \times L$ be a g-net in an L-fts (X, τ), s_t be the t-level projection of s over $Pt(L^X)$ and $e \in Pt(L^X)$. Then $s \to e \; \Rightarrow \; s_t \to^l e$ for some $l \geq t'$*

Proof. As $s \to e$ so $\exists \; k \in D$ s.t $r^k \geq Q(e, \tilde{1})$ i.e $r^k = 1$
$\Rightarrow \; D_t = \{n \in D; \; s(n) = (e^n, r^n) \; \& \; r^n \geq t\} \neq \phi \; \forall \; t \in L_0$.
Let $U \in \tilde{Q}_t(e)$ then $Q(e, U) > 0$.
As $s \to e$ so $\exists \; m \in D$ such that $r^m \geq Q(e, U) \geq t \; \& \; e^n q U \; \forall \; n \geq m$.
This means $m \in D_t$ and $s_t(n) q U \; \forall \; n \in D_t$ with $n \geq m$.
Therefore $s_t \to^l e$ for some $l \geq t'$. □

Proposition 3.18. *Let $s : D \to Pt(L^X) \times L$ be a g-net in an L-fts (X, τ), s_t be the t-level projection of s over $Pt(L^X)$ and $e \in Pt(L^X)$. If for every $t \in L_0$ $s_t \to_l e$ for some $l \not\leq t'$ then $s \infty e$.*

Proof. Given that for each $t \in L_0$, $s_t \to_l e$ for some $l \not\leq t'$
$\Rightarrow \; l' = \vee\{r \in L_0; \; \exists \; U \in \tilde{Q}_r(e); \; s_t \not{q} U$ frequently $\} \not\geq t$.
i.e for each $t \in L_0$;
$\vee \{r \in L_0; \; \exists \; U \in \tilde{Q}_r(e); \; s_t \not{q} U$ frequently $\} \not\geq t$.——— (i)

Now let $V \in L^X$ be any fuzzy subset of X with $Q(e,V) > 0$ and $t \in L_0$ be such that $t \leq Q(e,V)$. i.e $t \leq \vee\{\alpha \in L_0;\ V \in \tilde{Q}_\alpha(e)\}$———— (ii)
Then from (i) and (ii) we get
$\vee\{r \in L_0;\ \exists\ U \in \tilde{Q}_r(e);\ s_t\ \not{q}U$ frequently $\} \not\geq \vee\{\alpha \in L_0;\ V \in \tilde{Q}_\alpha(e)\}$ where $t \leq Q(e,V)$.
$\Rightarrow\ \exists\ \alpha_0 \in L_0$ s.t $V \in \tilde{Q}_{\alpha_0}(e)$ and $\vee\{r \in L_0;\ \exists\ U \in \tilde{Q}_r(e);\ s_t\ \not{q}U$ frequently $\} \not\geq \alpha_0$
$\Rightarrow\ \forall\ U \in \tilde{Q}_{\alpha_0}(e);\ s_t qU$ eventually.
$\Rightarrow\ s_t qV$ eventually. ———— (iii)
Now let $r^m \not\leq Q'(e,W)$ for some $m \in D$ and $W \in L^X$.
Then $Q(e,W) > 0$. As $s_t qW$ eventually $\forall\ t \leq Q(e,W)$
so, $\exists\ n \in D$ s.t $n \geq m$ and $e^n qW$, where $s(n) = (e^n, r^n)$.
$\Rightarrow\ s \infty e$. □

Definition 3.19. Let (X,τ) and (Y,δ) be any two L-fts, $f : X \to Y$ be any mapping and $s : D \to Pt(L^X) \times L$, given by $s(n) = (e^n, r^n)$, be a g-net on X. We define the image g-net $f \odot s : D \to Pt(L^Y) \times L$ by $f(s(n)) = (f(e^n), r^n)$.

Proposition 3.20. *Let (X,τ) and (Y,δ) be any two L-fts, $f : X \to Y$ be a gp-map and $s : D \to Pt(L^X) \times L$, given by $s(n) = (e^n, r^n)$, be a g-net on X. Then $s \to e\ \Rightarrow\ f \odot s \to f(e)$ where $e \in M(L^X)$.*

Proof. Let Q and $\hat{Q}$ be the gradation of q-neighbourhoodness in (X,τ) and (Y,δ) respectively. Let $V \in L^Y$ be such that $Q(f(e),V) > 0$. Then $f(e)qV$ and hence $eqf^{-1}(V)$. Again as f is gp and $e \in M(L^X)$ so by Lemma 1.22 $Q(e,f^{-1}(V)) \geq \hat{Q}(f(e),V)\ \Rightarrow\ Q(e,f^{-1}(V)) > 0$.
Now as $s \to e$ so $\exists\ m \in D$ s.t $r^m \geq Q(e,f^{-1}(V))$ and $e^n qf^{-1}(V),\ \forall\ n \geq m$
$\Rightarrow\ r^m \geq \hat{Q}(f(e),V)$ and $f(e^n)qV,\ \forall\ n \geq m$
$\Rightarrow\ f \odot s \to f(e)$. □

Proposition 3.21. *Let (X,τ) and (Y,δ) be any two L-fts and $f : X \to Y$ be any mapping. If for any g-net s on X, $s \to p_x \Rightarrow\ f \odot s \to f(p_x)\ \forall\ p_x \in M(L^X)$ then f is a gp-map.*

Proof. Let Q and $\hat{Q}$ be the gradation of q-neighbourhoodness in (X,τ) and (Y,δ) respectively. As $p_x \in M(L^X)$ so the mapping $Q_{p_x} : L^X \to L$ given by $Q_{p_x}(U) = Q(p_x,U)$ is a g-filter on L^X and $Q_{p_x} \to p_x$. So, by Proposition 3.5, the associated g-net $s[Q_{p_x}] \to p_x$.

Then according to the given condition $f \odot s[Q_{p_x}] \to f(p_x)$.

Now $s[Q_{p_x}] : M(L^X) \times L^X \to L^X \times L$ is given by $s[Q_{p_x}](r_y, U) = (r_y, Q(p_x, U))$ where $r_y qU$.

Therefore $f \odot s[Q_{p_x}](r_y, U) = (f(r_y), Q(p_x, U))$.

Again $f \odot s[Q_{p_x}] \to f(p_x)$ means $\forall\ V \in L^Y$ with $\hat{Q}(f(p_x), V) > 0$, $\exists\ (r_y^0, U^0) \in M(L^X) \times L^X$ s.t $Q(p_x, U^0) \geq \hat{Q}(f(p_x), V)$ and $f(t_z)qV\ \forall\ (t_z, W) \geq (r_y^0, U^0)$ — ——(i)

Now we shall prove that $U^0 \subseteq f^{-1}(V)$.

For, $\forall\ t_z \in Pt(L^X)$ with $t_z qU^0$ as $t \not\leq (U^0)^c(z)$ and $M(L)$ is a join generating subset of L so $\exists\ t^0 \in M(L)$ s.t $t \geq t^0 \not\leq (U^0)^c(z) \Rightarrow t_z \geq t_z^0 qU^0 \Rightarrow (t_z^0, U^0) \in Dom(s[Q_{p_x}])$.

Again $(t_z^0, U^0) \geq (r_y^0, U^0) \Rightarrow f(t_z^0)qV$, by (i)

$\Rightarrow t_z^0 qf^{-1}(V) \Rightarrow t_z qf^{-1}(V)$ and as this is true for all $t_z \in Pt(L^X)$ with $t_z qU^0$.

So, $U^0 \subseteq f^{-1}(V)$.

Hence $Q(p_x, f^{-1}(V)) \geq Q(p_x, U^0) \geq \hat{Q}(f(p_x), V)$.

So by Lemma 1.22 f is a gp-map. □

Proposition 3.22. *Let 'cl' be the closure operator in an L-fts* (X, τ), $e \in Pt(L^X)$ *and* $A \in L^X$. *If* $e \in M(cl(A, k))$ *for some* $k \in L_0$ *then* $\exists$ *a molecule g-net s in* A *s.t* $s_k \to^l e$ *for some* $l \geq k'$.

Proof. Let $e \in M(cl(A, k))$. Then by Proposition 1.20, for every $U \in \tilde{Q}_k(e)$, UqA i.e for every $U \in \tilde{Q}_k(e)$, $\exists\ x^u \in X$ s.t $U(x^u) \not\leq A^c(x^u) \Rightarrow A(x^u) \not\leq U^c(x^u)$.

As M(L) is a join generating subset of L so $\exists\ p^u \in M(L)$ s.t $A(x^u) \geq p^u \not\leq U^c(x^u)$ $\Rightarrow p^u_{x^u} \in M(L^X)$, $p^u_{x^u} qU$ and $p^u_{x^u} \in A$.

Since $e \in M(L^X)$, then $\tilde{Q}_k(e)$ is a directed set with respect to the relation '$\geq$' defined by $U \geq V \iff U \subseteq V$, $\forall U, V \in \tilde{Q}_k(e)$.

So we define a molecule g-net $s : \tilde{Q}_k(e) \to M(L^X) \times L$ by $s(U) = (p^u_{x^u}, Q(e, U))$.

Then s is a molecule g-net in A.

Obviously $D_k = Dom(s_k) = \{U \in L^X; s(U) = (p^u_{x^u}, Q(e, U))\ \&\ Q(e, U) \geq k\} = \tilde{Q}_k(e)$ and as $p^u_{x^u} qV\ \forall\ V \in \tilde{Q}_k(e)$, so $s_k \to^l e$ for some $l \geq k'$. Hence the proof. □

Proposition 3.23. *Let* $f \in M(L^X)$ *and* $s[Q_f]$ *be the g-net associated with the neighbourhood g-filter* Q_f *and* $lim(s[Q_f]) = \vee\{p_x \in Pt(L^X);\ s[Q_f] \to p_x\}$ *then* $e \in lim(s[Q_f]) \iff s[Q_f] \to e$.

Proof. $s[Q_f] \to e \Rightarrow e \in lim(s[Q_f])$ is obvious. So we are to prove the converse part only.

Let $e \in lim(s[Q_f])$ and if possible let $s[Q_f] \not\to e$ then $Q_f \not\to e$, by Proposition 3.5.

$\Rightarrow \exists\, U \in L^X$ s.t $Q_f(U) \not\geq Q(e,U)$.

Then by (QN4), $\exists\, V \in L^X$ s.t $eqV \subseteq U$ but $Q(g,V) \not\leq Q_f(U)\ \forall\, g \in Pt(L^X)$ with gqV

$\Rightarrow Q(g,U) \not\leq Q_f(U)\ \forall\, g \in Pt(L^X)$ with gqV

$\Rightarrow Q_f \not\to g\ \forall\, g \in Pt(L^X)$ with gqV

$\Rightarrow$ if $Q_f \to g$ then $g \not{q} V$ i.e $g \in V^c$

$\Rightarrow lim(Q_f) \subseteq V^c \Rightarrow lim(s[Q_f]) \subseteq V^c$

$\Rightarrow e \in V^c \Rightarrow e \not{q} V$, a contradiction.

Hence the proof. □

The above Proposition 3.23 is, however, not valid for arbitrary g-net S. This is shown in the following Example:

Example 3.24. Let X be a non-empty crisp set and I be the unit closed interval [0,1]. Let $U_n \in I^X$ be defined by $U_n(x) = 0.5 + \frac{1}{n+2},\ \forall x \in X$ and let $g^n = (0.5 - \frac{1}{n+3})_\xi \in Pt(I^X),\ \forall n \in N$ and for some fixed $\xi \in X$. Then obviously $g^k q U_l\ \forall l \leq k$, where $k = 1, 2, 3, ...$

Let $\tau : I^X \to I$ be a gradation of openness defined by

$\tau(U_n) = 0.5 - \frac{1}{n+2},\ \forall n \in N$ and $\tau(\tilde{0}) = \tau(\tilde{1}) = 1$.

Let $S : N \to Pt(I^X) \times I$ be defined by

$$S(n) = (e^n, r^n) = \begin{cases} (0.1_\xi, 1) & \text{if } n = 1 \\ (0.8_\xi, 0.5 - \frac{1}{n+2}) & \text{if } n = 2, 3, ... \end{cases}$$

For $k \in N$, consider the fuzzy point g^k. Then for any $U \in I^X$ with $Q(g^k, U) > 0$, $\exists l \in N$ s.t $r^l \geq Q(g^k, U)$ and $e^n q U,\ \forall n \geq l$. So, $S \to g^k\ \forall k \in N$ and hence $g^k \in lim(S),\ \forall k \in N$.

So, $\vee_{k \in N} g^k = .5_\xi \in lim(S)$.

But $S \not\to .5_x$ because $Q(.5_\xi, \tilde{.9}) = \vee_{n\in N}(.5 - \frac{1}{n+2}) = 0.5$ and $\forall m \in N$, $r^m = 0.5 - \frac{1}{m+2} < 0.5$.

Proposition 3.25. *Let (X,τ) be an L-fts with Q as the gradation of q-neighbourhoodness, for every $e \in M(L^X)$, let $Q_e : L^X \to L$ be given by $Q_e(U) = Q(e,U)$, $\forall\, U \in L^X$. Let $s[Q_e]$ be the molecule g-net on L^X associated with Q_e and let $\mathcal{G}[s[Q_e]]$ be the g-filter on L^X associated with $s[Q_e]$ then $\mathcal{G}[s[Q_e]] = Q_e \ \forall\, e \in M(L^X)$.*

Proof. As $s[Q_e] : M(L^X) \times L^X \to L$ is given by $s[Q_e](f,V) = (f, Q(e,V))$ so $\mathcal{G}[s[Q_e]](U) = \vee\{Q(e,V);\ s[Q_e](f,V) = (f,Q(e,V))\ \&\ gqU,\ \forall\,(g,W) \geq (f,V)\}$.
Now let $(f,V) \in D$ be such that $\forall\,(g,W) \in D,\ (g,W) \geq (f,V) \Rightarrow gqU$.
Then $\forall\, h \in Pt(L^X)$ with $hqV\ \exists\, h^0 \in \beta^0(h)$ s.t h^0qV so, $(h^0,V) \in D$ and $(h^0,V) \geq (f,V) \Rightarrow h^0qU \Rightarrow hqU$
$\Rightarrow V \subseteq U$.
Therefore $\mathcal{G}[s[Q_e]](U) \leq \vee\{Q(e,V);\ V \subseteq U\}$
$= \vee\{\vee\{\tau(W);\ eqW \subseteq V\};\ V \subseteq U\}$
$= \vee\{\tau(W);\ eqW \subseteq U\}$
$= Q(e,U)$.
$\mathcal{G}[s[Q_e]](U) \geq Q(e,U)$, $\forall\, U \in L^X$ is obvious. Hence $\mathcal{G}[s[Q_e]] = Q_e\ \forall\, e \in M(L^X)$ □

Proposition 3.26. *Let (X,τ) be an L-fts with Q as the gradation of q-neighbourhoodness, Γ be the collection of all molecule g-nets on L^X and Λ be the collection of all g-filters associated with the members of Γ. Let $lim(\mathcal{G}[s]) = \vee\{p_x \in Pt(L^X);\ \mathcal{G}[s] \to p_x\}$ where $s \in \Gamma$. Then the following properties are satisfied by $lim(\mathcal{G}[s])$:*

(i) *for $e \in M(L^X)$, $\exists\, s \in \Gamma$ s.t $e \in lim(\mathcal{G}[s])$,*
(ii) *for $e \in M(L^X)$, $U \in L^X$, $\wedge\left\{\mathcal{G}[s](U);\ \mathcal{G}[s] \in \Lambda;\ e \in lim(\mathcal{G}[s])\right\} > 0 \Rightarrow eqU$,*
(iii) *if for some $(e,U) \in M(L^X) \times L^X$ with eqU and for some $\mathcal{H}[s_0] \in \Lambda$ $\mathcal{H}[s_0](U) \not\geq \wedge\left\{\mathcal{G}[s](U);\ \mathcal{G}[s] \in \Lambda;\ e \in lim(\mathcal{G}[s])\right\}$ hold then $\exists\, V \in L^X$ s.t $eqV \subseteq U$ and $\forall\, f \in Pt(L^X)$ with fqV, $f \notin lim(\mathcal{H}[s_0])$.*

Proof. (i) As by Proposition 2.8, Q_e is a g-filter on L^X and $e \in lim(Q_e)$, $\forall\, e \in Pt(L^X)$ and by Proposition 3.25 $\mathcal{G}[s[Q_e]] = Q_e$, $\forall\, e \in M(L^X)$.
So, (i) follows.

(ii) $\wedge\Big\{\mathcal{G}[s](U);\ \mathcal{G}[s]\in\Lambda;\ e\in lim(\mathcal{G}[s])\Big\}>0$
$\Rightarrow\ \mathcal{G}[s[Q_e]](U)>0\ \Big[\text{as } e\in lim(\mathcal{G}[s[Q_e]])\Big]$
$\Rightarrow\ Q_e(U)>0$, by Proposition 3.25.
$\Rightarrow\ eqU$.

(iii) For every $e\in M(L^X)$,
$\mathcal{H}[s_0](U)\not\geq\wedge\Big\{\mathcal{G}[s](U);\ \mathcal{G}[s]\in\Lambda;\ e\in lim(\mathcal{G}[s])\Big\}$
$\Rightarrow\ \mathcal{H}[s_0](U)\not\geq\mathcal{G}[s[Q_e]](U)\ \Big[\text{ as } e\in lim(\mathcal{G}[s[Q_e]])\Big]$
$\Rightarrow\ \mathcal{H}[s_0](U)\not\geq Q_e(U)$
$\Rightarrow\ \mathcal{H}[s_0](U)\not\geq Q(e,U)$.
Then by (QN4), $\exists\ V\in L^X$ s.t $eqV\subseteq U$ and $\forall\ f\in Pt(L^X)$ with fqV, $Q(f,V)\not\leq\mathcal{H}[s_0](U)$.
$\Rightarrow\ \forall\ fqV,\ f\notin lim(\mathcal{H}[s_0])$ □

Proposition 3.27. *Let Γ be the collection of all molecule g-nets on L^X and Λ be the collection of all g-filters on L^X associated with the members of Γ. If for every $s\in\Gamma$, $lim(\mathcal{G}[s])$ is a fuzzy subset of X satisfying*

(i) *for every $e\in M(L^X)$, $\exists\ s\in\Gamma$ s.t $e\in lim(\mathcal{G}[s])$,*
(ii) $\wedge\Big\{\mathcal{G}[s](U);\ \mathcal{G}[s]\in\Lambda;\ e\in lim(\mathcal{G}[s])\Big\}>0\ \Rightarrow\ eqU,\ \forall\ U\in L^X$ *and* $\forall\ e\in M(L^X)$,
(iii) *if for some $(e,U)\in M(L^X)\times L^X$ with eqU and for some $\mathcal{H}[s_0]\in\Lambda$, $\mathcal{H}[s_0](U)\not\geq\wedge\Big\{\mathcal{G}[s](U);\ \mathcal{G}[s]\in\Lambda;\ e\in \lim(\mathcal{G}[s])\Big\}$ then $\exists\ V\in L^X$ s.t $eqV\subseteq U$ and $\forall\ fqV,\ f\notin lim(\mathcal{H}[s_0])$.*

Then the mapping $\bar{Q}:Pt(L^X)\times L^X\to L$, given by

$$\bar{Q}(e,U)=\wedge\Big\{\mathcal{G}[s](U);\ \mathcal{G}[s]\in\Lambda;\ e\in lim(\mathcal{G}[s])\Big\},$$

if $e\in M(L^X)=\vee\Big\{\bar{Q}(f,U);\ f\in\beta^0(e)\Big\}$, if $e\in Pt(L^X)\backslash M(L^X)$ is a gradation of q-neighbourhoodness on X.

The proof follows from Proposition 2.10.

Proposition 3.28. *Let Q be a gradation of q-neighbourhoodness on an L-fts (X,τ), Γ be the collection of all molecule g-nets on L^X and Λ be the collection of all g-filters on L^X associated with the members of Γ. Let $lim(\mathcal{G}[s])$ is defined by $lim(\mathcal{G}[s])=\vee\{p_x\in Pt(L^X);\ \mathcal{G}[s]\to p_x\}$ for every $s\in\Gamma$. Then the mapping $\bar{Q}:Pt(L^X)\times L^X\to$*

L, defined by

$$\bar{Q}(e,U) = \wedge\Big\{\mathcal{G}[s](U);\ \mathcal{G}[s] \in \Lambda;\ e \in lim(\mathcal{G}[s])\Big\},$$

if $e \in M(L^X) = \vee\Big\{\bar{Q}(f,U);\ f \in \beta^0(e)\Big\}$, *if* $e \in Pt(L^X)\backslash M(L^X)$ *is a gradation of q-neighbourhoodness on* (X,τ) *and* $\bar{Q} = Q$.

Proof. By Proposition 3.25, Λ contains the associated filters Q_e, $\forall\ e \in M(L^X)$, so the proof follows from Proposition 2.11. □

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