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TIME DISCRETIZATION WITH SPATIAL COLLOCATION METHOD FOR A PARABOLIC INTEGRO–DIFFERENTIAL EQUATION WITH A WEAKLY SINGULAR KERNEL

CHANG HO KIM

ABSTRACT. We analyze the spectral collocation approximation for a parabolic partial integrodifferential equations(PIDE) with a weakly singular kernel. The space discretization is based on the spectral collocation method and the time discretization is based on Crank–Nicolson scheme with a graded mesh. We obtain the stability and second order convergence result for fully discrete scheme.

1. INTRODUCTION

Let Ω be a rectangular domain in $\mathbb{R}^2$ with boundary $\partial\Omega$ (typically $\Omega \equiv (-1,1)^2$), and let $T \in \mathbb{R}$ satisfy $0 < T < \infty$. We shall consider the spectral collocation method for the following partial integrodifferential equation with a weakly singular kernel

$$
\begin{aligned}
u_t - \Delta u &= \int_0^t K(t-s)\mathcal{B}u(\mathbf{x},s)\,ds + f(\mathbf{x},t), && (\mathbf{x},t) \in \Omega \times (0,T], \\
u &= 0, && \text{on } \partial\Omega, \quad t > 0, \\
u(\mathbf{x},0) &= u_0(\mathbf{x}), && \text{in } \Omega,
\end{aligned} \tag{1}
$$

where $\mathcal{B}$ is a general partial differential operator of second order with smooth and time independent coefficients:

$$\mathcal{B} = -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_j}\Big(b_{ij}(\mathbf{x})\frac{\partial}{\partial x_i}\Big) + \sum_{j=1}^{2} b_j(\mathbf{x})\frac{\partial}{\partial x_j} + b_0(\mathbf{x})I$$

and K is a weakly singular kernel such that

$$|K^{(i)}(t)| \le C_K t^{-i-\gamma} \qquad \text{with} \quad 0 \le \gamma < 1, \qquad \text{for } t > 0, \quad i = 0,1.$$

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Furthermore, throughout this paper, we shall assume that f is sufficiently smooth. Problems of this nature arise in several areas, such as the theory of linear viscoelasticity and heat conduction in material with memory; see, for example, the monograph of Renardy *et al.*([19]).

Some authors proposed and analyzed for the numerical semi discretized (spatially discretized) methods of (1): (also chen*et al.*([8]), Pani *et al.*([15]) for other spatial discretization methods) The spatial spectral and pseudo–spectral methods and first order time discretization method have been proposed and analyzed in the work of Kim & Choi ([11]) .

The numerical method considered in this paper will be obtained by discretizing in space by a spectral collocation method with Jacobi weights, followed by a finite difference and quadrature scheme for the time stepping. The spectral collocation methods have evolved as valuable technique for the solution of a broad class of problems(see Canuto *et al.*([5])). The popularity of such methods is due in part to their conceptual simplicity, wide applicability and ease of implementation. The advantages of spectral collocation methods over the other methods are that the calculation of the coefficients in the equations determining the approximate solution is very efficient, since no integrals need to be evaluated and the use of the fast Fourier transform allows a less expensive computation time of the derivatives and the nonlinear terms in collocation methods by Chebyshev polynomials.

The time discretization is very interesting because of the nature of "memory effect". The time discretization methods are derived essentially by replacing the time derivative in (10) by difference quotient and using a quadrature rule for the integral term. The difficulties involved in such time discretization are that all approximate values of $u(\mathbf{x}, \cdot)$ in (1) have to be retained, causing great demands for data storage. To overcome this difficulties, higher order quadrature formulas or quadratures based on the use of sparser set of time levels were proposed in many literature such as Pani *et al.* ([16],[17]) and Sloan *et al.*([20]) for the partial integrodifferential equation with smooth kernel. In the case of a weakly singular kernel, the regularity of the solution with respect to time is limited, which makes higher order quadrature formulas useless, as well as quadratures based on the use of sparser set of time levels: Formally, the solution of (1) satisfies

$$u_{tt} = \Delta u_t + K(t)\mathcal{B}u(0) + \int_0^t K(t-s)\mathcal{B}u_s\,ds + f_t$$

or

$$|u_{tt}| \leq C_K t^{-\gamma} |\mathcal{B}u(0)| + \text{ more regular terms in time, } 0 \leq \gamma < 1$$

In advance, we shall assume that the solution of (1) satisfies

$$\mathcal{A}3 \ : \ |u_{tt}| \leq G(x,t)t^{-\gamma} \ \text{ for some } \ G(x,t) \in L^\infty([0,T]; H^2_{\omega_\alpha}(\Omega)).$$

The aim of this paper is to show the stability and to obtain the error estimates for the scheme (15).

2. Preliminaries

We now introduce some definitions and recall some basic results which will be used throughout this paper. We first introduce the weighted Sobolev spaces on the square associated with the Jacobi weighted measure. For any $\mathbf{x} = (x_1, x_2) \in \Omega$, we set $\omega_\alpha(\mathbf{x}) = (1 - x_1^2)^\alpha (1 - x_2^2)^\alpha$, where $-1 < \alpha < 1$. We define

$$L^2_{\omega_\alpha}(\Omega) = \left\{ v \ : \ \Omega \to \mathbb{R} \text{ measurable}; \ \int_\Omega v^2(\mathbf{x}) \omega_\alpha(\mathbf{x}) \ d\mathbf{x} < +\infty \right\},$$

which is a Hilbert space for the scalar product

$$(u,v)_{\omega_\alpha} = \int_\Omega u(\mathbf{x}) v(\mathbf{x}) \omega_\alpha(\mathbf{x}) \ d\mathbf{x}.$$

For any integer $m \geq 0$, the weighted Sobolev space is defined by

$$H^m_{\omega_\alpha}(\Omega) = \left\{ v \in L^2_{\omega_\alpha}(\Omega) \ ; \ \frac{\partial^{p+q} v}{\partial x_1^p \partial x_2^q} \in L^2_{\omega_\alpha}(\Omega), \forall (p,q) \in \mathbb{N}^2, p + q \leq m \right\},$$

which is equipped with the norm

$$\|v\|_{m,\omega_\alpha,\Omega} = \left(\int_\Omega \sum_{p+q \leq m} \left(\frac{\partial^{p+q} v}{\partial x_1^p \partial x_2^q} \right)^2 \omega_\alpha(\mathbf{x}) d\mathbf{x} \right)^{1/2}$$

For a real number $s \geq 0$ which is not an integer, the Hilbert space $H^s_{\omega_\alpha}(\Omega)$ is defined by interpolation between $H^{[s]}_{\omega_\alpha}(\Omega)$ and $H^{[s]+1}_{\omega_\alpha}(\Omega)$ where $[s]$ is the integer part of s, and its norm is denoted by $\| \cdot \|_{s,\omega_\alpha,\Omega}$. We denote by $H^s_{\omega_\alpha,0}(\Omega)$ the closure in $H^s_{\omega_\alpha}(\Omega)$ of the space $D(\Omega)$ of all functions of C^∞ having a compact support in Ω. Whenever there is no confusion, we drop the subscript Ω for $\| \cdot \|_{m,\omega_\alpha,\Omega}$ and $(\cdot,\cdot)_{m,\omega_\alpha,\Omega}$. Throughout this paper, we denote C by the generic positive constant depending on K,T and grading exponent q.

For an integer $N > 0$, we set $\mathbb{P}_N = \tilde{\mathbb{P}}_N \times \tilde{\mathbb{P}}_N$, where $\tilde{\mathbb{P}}_N$ is the space of the polynomials of degree $\leq N$ in single variable. Furthermore, we set $\mathbb{P}^0_N(\Omega) = \{p \in \mathbb{P}_N \mid p(\mathbf{x}) = 0 \text{ on } \partial\Omega\}$.

We set

$$A_{\omega_\alpha}(u,v) = \int_\Omega \nabla u \cdot \nabla(v\omega_\alpha) d\mathbf{x} \quad \text{for any } u, v \in H^1_{\omega_\alpha,0}(\Omega). \tag{1}$$

A complete study of the properties of the bilinear form $A_{\omega_\alpha}(\cdot\ ,\cdot)$ has been done in the paper of Bernardi *et al.*([2]) for one or two–dimensional problems. For our work, we define the Ritz projection operator $\Pi^\alpha_N : H^1_{\omega_\alpha,0} \longmapsto \mathbb{P}^0_N(\Omega)$ by

$$A_{\omega_\alpha}(v - \Pi^\alpha_N v, \phi) = 0, \quad \forall \phi \in \mathbb{P}^0_N(\Omega). \tag{2}$$

The error estimate of the Ritz projection can be found in the paper of Bressan *et al.*([1]) and Bernardi *et al.*([2]); for all $v \in H^\sigma_{\omega_\alpha}(\Omega) \cap H^1_{\omega_\alpha,0}(\Omega)$, with $0 \le \mu \le \sigma$, $\sigma \ge 1$

$$\|v - \Pi^\alpha_N v\|_{\mu,\omega_\alpha} \le CN^{e(\mu)-\sigma}\|v\|_{\sigma,\omega_\alpha}, \tag{3}$$

where $e(\mu) = \mu$ if $\mu \le 1$ and $e(\mu) = 2\mu - 1$ if $\mu > 1$.

For the spectral collocation methods with Jacobi weights for $\alpha > -1$, we denote by $\zeta^\alpha_0(= -1)$, $\zeta^\alpha_1, \cdots$ and $\zeta^\alpha_N(= 1)$ the nodes of Gauss-Lobatto integration formula of degree N with respect to the Jacobi weight $\tilde{w}(\xi) = (1-\xi^2)^\alpha$ and $\omega^\alpha_0, \omega^\alpha_1, \cdots$, and ω^α_N its corresponding weights, respectively (see Bernardi *et al.*([2]) and Canuto *et al.*([5])). Then, we see that

$$\int_{-1}^1 p(\xi)\tilde{w}(\xi)\, d\xi = \sum_{j=0}^N p(\xi^\alpha_j)\omega^\alpha_j \qquad \text{for any } p \in \tilde{\mathbb{P}}_{2N-1}. \tag{4}$$

Next we set for $0 \le j,\ k \le N$,

$$\mathbf{x}^\alpha_{jk} = (\zeta^\alpha_j, \zeta^\alpha_k) \quad \text{and} \quad \mathbf{w}^\alpha_{jk} = \omega^\alpha_j\omega^\alpha_k,$$

and introduce the "grid" $\Xi^\alpha_N = \{\mathbf{x}^\alpha_{jk}\ ;\ 0 \le j,k \le N\}$. We define the interpolation operator $\mathbf{I}^\alpha_N : C^0(\bar{\Omega}) \longmapsto \mathbb{P}_N(\Omega)$ by

$$\mathbf{I}^\alpha_N v(\mathbf{x}) = v(\mathbf{x}), \qquad \forall \mathbf{x} \in \Xi^\alpha_N.$$

For any real μ and σ such that $0 \le \mu \le \sigma$, $\sigma > 1$, the interpolation error is estimated as follows (see Canuto *et al.*(1982)):

$$\|v - \mathbf{I}^\alpha_N v\|_{\mu,\omega_\alpha} \le CN^{2\mu-\sigma}\|v\|_{\sigma,\omega_\alpha} \qquad \text{for all } v \in H^\sigma_{\omega_\alpha}(\Omega). \tag{5}$$

We now define a discrete inner product:

$$(\phi, \psi)_N = \sum_{i,j=0}^N \phi(\mathbf{x}^\alpha_{ij})\psi(\mathbf{x}^\alpha_{ij})\mathbf{w}^\alpha_{ij}, \qquad \forall \phi,\ \psi \in C^0(\bar{\Omega}), \tag{6}$$

then, it follows from (4) that

$$(\phi,\psi)_N = (\phi,\psi)_{\omega_\alpha}, \qquad \forall \phi,\psi : \phi\cdot\psi \in \mathbb{P}_{2N-1}(\Omega).$$

We denote the error between discrete inner product $(\cdot\,,\,\cdot)_N$ and continuous inner product $(\cdot\,,\cdot)_{\omega_\alpha}$ by $(E(\phi),\psi)$:

$$(E(\phi),\psi) = (\phi\,,\psi)_N - (\phi\,,\psi)_{\omega_\alpha}, \qquad \forall \phi,\psi \in C^0(\bar{\Omega}).$$

It can be shown that (see Bressan *et al.*([1]) and Canuto *et al.*([5])): for all $\phi \in H^{\sigma}_{\omega_\alpha,0}(\Omega)$,

$$|(E(\phi),\psi)| \le CN^{-\sigma}\|\phi\|_{\sigma,\omega_\alpha}\|\psi\|_{\omega_\alpha}, \qquad \forall \psi \in \mathbb{P}_N(\Omega). \tag{7}$$

The discrete norm induced by (6)

$$\|\phi\|_N = (\phi,\phi)_N^{1/2}, \qquad \forall \phi \in C^0(\bar{\Omega})$$

is equivalent to the $L^2_{\omega_\alpha}$–norm, namely;

$$\|\phi\|_{\omega_\alpha} \le \|\phi\|_N \le 2\|\phi\|_{\omega_\alpha}, \qquad \forall \phi \in \mathbb{P}_N(\Omega) \tag{8}$$

(see [5]). The following is an easy consequence of (4); for any $\psi \in \mathbb{P}^0_N(\Omega)$ and $\phi \in \mathbb{P}_N(\Omega)$,

$$\left(\frac{\partial\phi}{\partial x_i},\psi\right)_N = -\left(\phi,\frac{1}{\omega_\alpha}\frac{\partial}{\partial x_i}(\psi\omega_\alpha)\right)_N.$$

We now define

$$a_N(\phi,\psi) := \left(\nabla\phi,\frac{1}{\omega_\alpha}\nabla(\psi\omega_\alpha)\right)_N$$

and

$$\begin{aligned} B^\alpha_N(\phi,\psi) :=& \sum_{i,j=1}^{2}\left(b_{ij}\frac{\partial\phi}{\partial x_i},\frac{1}{\omega_\alpha}\frac{\partial(\psi\omega_\alpha)}{\partial x_j}\right)_N \\ &+\sum_{j=1}^{2}\left(b_j\frac{\partial\phi}{\partial x_j},\psi\right)_N + \left(b_0\phi,\psi\right)_N, \quad \forall\phi,\psi\in\mathbb{P}^0_N(\Omega), \end{aligned}$$

which are, respectively, discrete analogues of $A_{\omega_\alpha}(\phi,\psi)$ and

$$\begin{aligned} B_{\omega_\alpha}(\phi,\psi) =& \sum_{i,j=1}^{2}\left(b_{ij}\frac{\partial\phi}{\partial x_i},\frac{1}{\omega_\alpha}\frac{\partial(\psi\omega_\alpha)}{\partial x_j}\right)_{\omega_\alpha} \\ &+\sum_{j=1}^{2}\left(b_j\frac{\partial\phi}{\partial x_j},\psi\right)_{\omega_\alpha} + \left(b_0\phi,\psi\right)_{\omega_\alpha}, \quad \forall\phi,\psi\in H^0_{\omega_\alpha}(\Omega). \end{aligned}$$

From (7), it can be easily shown that the error estimates for $a_N(\cdot,\cdot)$ and $B_N^\alpha(\cdot,\cdot)$ with respect to $A_{\omega_\alpha}(\cdot,\cdot)$ and $B_{\omega_\alpha}(\cdot,\cdot)$ are respectively, for $\phi \in H^\sigma_{\omega_\alpha}(\Omega)$ and for $\psi \in \mathbb{P}_N^0(\Omega)$,

$$
\begin{aligned}
|a_N(\phi,\psi) - A_{\omega_\alpha}(\phi,\psi)| &\le CN^{1-\sigma}\|\phi\|_{\sigma,\omega_\alpha}\|\psi\|_{1,\omega_\alpha},\\
|B_N^\alpha(\phi,\psi) - B_{\omega_\alpha}(\phi,\psi)| &\le CN^{1-\sigma}\|\phi\|_{\sigma,\omega_\alpha}\|\psi\|_{1,\omega_\alpha}.
\end{aligned} \tag{9}
$$

It has been proved (see Canuto *et al.*([5])) that the bilinear form $a_N(\cdot,\cdot)$ is continuous and coercive over $\mathbb{P}_N^0(\Omega)$, *i.e.*, there exist positive constants c_1, c_2 such that

$$|a_N(\phi,\psi)| \le c_1\|\phi\|_{1,N}\|\psi\|_{1,N}, \qquad \forall\phi,\ \psi \in \mathbb{P}_N^0(\Omega)$$

and

$$a_N(\phi,\phi) \ge c_2\|\phi\|_{1,N}^2, \qquad \forall\phi \in \mathbb{P}_N^0(\Omega),$$

where $\|\cdot\|_{1,N}$ is the discrete analogue of Sobolev norm $\|\cdot\|_{1,\omega_\alpha}$, (*i.e.* $\|\phi\|_{1,N}^2 = \|\phi\|_N^2 + \|\nabla\phi\|_N^2$).

3. Full Discrete Scheme

In this paper, we shall assume that there is a unique generalized solution of (1) satisfying the following regularity conditions:

$$
\begin{aligned}
\mathcal{A}1 \ &: \ u \in C([0,T]; H^2_{\omega_\alpha} \cap H^1_{\omega_\alpha,0}),\\
&\quad\ u_t \in C([0,T]; L^2_{\omega_\alpha}) \cap L^1((0,T]; H^2_{\omega_\alpha} \cap H^1_{\omega_\alpha,0}),\\
&\quad\ u_{tt} \in L^1([0,T]; L^2_{\omega_\alpha}).\\
\mathcal{A}2 \ &: \ u \in \mathcal{A}1 \cap L^1([0,T]; H^\sigma_{\omega_\alpha} \cap H^1_{\omega_\alpha,0}),\\
&\quad\ u_t \in L^1([0,T]; H^\sigma_{\omega_\alpha}), \text{ for some } \sigma \ge 2,
\end{aligned}
$$

As a starting point for the time discretization of (1), we define the semi-discrete solution of (1) as the function $U : (0,T] \longrightarrow \mathbb{P}_N(\Omega)$ such that

$$
\begin{aligned}
U_t(\mathbf{x},t) - \Delta U(\mathbf{x},t) &= \int_0^t K(t-s)\mathcal{B}_N U(\mathbf{x},s)\,ds + f(\mathbf{x},t), \quad \mathbf{x} \in \Xi_N^\alpha \cap \Omega,\\
U(\mathbf{x},t) &= 0 \qquad \text{on} \quad \Xi_N^\alpha \cap \partial\Omega, \quad t > 0,\\
U(\mathbf{x},0) &= u_0(\mathbf{x}), \qquad \text{in} \quad \Xi_N^\alpha \cap \Omega,
\end{aligned} \tag{10}
$$

where

$$
\begin{aligned}
\mathcal{B}_N U = &- \sum_{i,j=1}^2 \frac{\partial}{\partial x_j}\left[\mathbf{I}_N^\alpha\left(b_{ij}(\mathbf{x})\frac{\partial U}{\partial x_i}\right)\right]\\
&+ \sum_{j=1}^2 \mathbf{I}_N^\alpha\left(b_j(\mathbf{x})\frac{\partial U}{\partial x_j}\right) + \mathbf{I}_N^\alpha\Big(b_0(\mathbf{x})U\Big) \quad \forall\mathbf{x} \in \Xi_N^\alpha.
\end{aligned}
$$

For the time discretization of (10), we shall consider the graded mesh:(See also [3, 12])
Given $M \in \mathbb{N}$, let $\Pi_M := \{t_0, \cdots, t_M\}$ $(0 = t_0 < t_1 < \cdots < t_M = T)$ denote a partition of the interval $[0,T]$. With a given partition Π_M of $[0,T]$ we associate the quantities

$$h := \max_{n \le M} h_n,$$

where $h_n := t_n - t_{n-1} (n = 1, \cdots, M)$. If the mesh points $\{t_n\}_{n=0}^{M}$ are given by

$$t_n := \left(\frac{n}{M}\right)^q T \qquad (n = 0, \cdots, M), \tag{11}$$

then Π_M is called a *graded mesh*; in the present context, the so-called *grading exponent* $q \in \mathbb{R}$ will always satisfy $q \ge 1$. Let $U^k \in \mathbb{P}_N(\Omega)$ be the approximation of the exact solution of (10) at time t_k. The time discretization considered here will be based on the backward difference quotient $\bar{\partial}_t U^k = (U^k - U^{k-1})/h_k$. The integral term then has to be evaluated by numerical quadrature from the values of U^k's, but since the integrand is singular, we shall use the product integration: Let the integral term be denoted by

$$J_k(\phi) = \int_0^{t_k} K(t_k - s)\phi(s)\, ds. \tag{12}$$

We approximate ϕ in $J_k(\phi)$ by piecewise functions

$$\hat{\phi}(s) = \begin{cases} \phi^0, & 0 \le s \le t_1 \\ \phi^{j+1/2}, & t_j < s \le t_{j+1},\ 1 \le j \le k-2, \\ \\ \frac{t_{k-1/2} - s}{h_{k-1/2}}\phi^{k-3/2} + \frac{s - t_{k-3/2}}{h_{k-1/2}}\phi^{k-1/2}, & t_{k-1} < s \le t_k, \end{cases} \tag{13}$$

where we denote that $t_{k-1/2} := (t_k + t_{k-1})/2$, $\phi^{k-1/2} := (\phi^k + \phi^{k-1})/2$ and $h_{k-1/2} := (h_k + h_{k-1})/2$. Thus we write the quadrature for $J_k(\phi)$ as

$$\begin{aligned} \sigma^k(\phi) &= J_k(\hat{\phi}) = \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} K(t_k - s)\hat{\phi}(s)\, ds \\ &= \kappa_{k0}\phi^0 + \sum_{j=1}^{k-2} \kappa_{kj}\phi^{j+1/2} + \kappa_{k\,k-1}\phi^{k-3/2} + \kappa_{k\,k}\phi^{k-1/2} \text{ for } k \ge 2, \end{aligned} \tag{14}$$

and

$$\sigma^1(\phi) = \kappa_{10}\phi^0 = \int_0^{t_1} K(t_1 - s)\, ds\phi^0,$$

where

$$\kappa_{kj} = \begin{cases} \int_{t_j}^{t_{j+1}} K(t_k - s)\, ds, & \text{if } j \le k-2, \\ \int_{t_{k-1}}^{t_k} K(t_k - s)\frac{t_{k-1/2}-s}{h_{k-1/2}}\, ds, & \text{if } j = k-1, \\ \int_{t_{k-1}}^{t_k} K(t_k - s)\frac{s-t_{k-3/2}}{h_{k-1/2}}\, ds, & \text{if } j = k. \end{cases}$$

In the following lemma, the error estimate for our quadrature scheme (14) is stated.

Lemma 1. *Suppose $f \in C^2(0,T]$, $f' \in C[0,T]$ and $|f''(t)| \le Ct^{-\gamma}$. If the grading exponent q satisfies $q \ge \frac{2}{2-\gamma}$, then there is a constant C_T depending on f and T such that*

$$\left|\sum_{k=1}^{n} h_k(J_k(f) - \sigma^k(f))\right| = \left|\sum_{k=1}^{n} h_k\left(\int_0^{t_k} K(t_k - s)f(s)\, ds - \sigma^k(f)\right)\right| \le C_T(f)h^2.$$

Proof. Define $E_{k,j}(f)$ for $j = 0, \cdots, k-1$ by

$$E_{k,j}(f) := \begin{cases} \int_0^{t_1} K(t_k - s)(f(s) - f^0)\, ds, & \text{for } j = 0, \\ \int_{t_j}^{t_{j+1}} K(t_k - s)(f(s) - f^{j+\frac{1}{2}})\, ds, & \text{for } j = 1, \cdots, k-2, \\ \int_{t_{k-1}}^{t_k} K(t_k - s)(f(s) - \hat{f}(s))\, ds, & \text{for } j = k-1, \end{cases}$$

where we denote

$$\hat{f}(s) = \frac{t_{k-1/2} - s}{h_{k-1/2}} f^{k-3/2} + \frac{s - t_{k-3/2}}{h_{k-1/2}} f^{k-1/2}.$$

Using the mean value theorem for the integrations, we have

$$\begin{aligned} |E_{k,0}(f)| &\le C(t_k - t_1)^{-\gamma}\left|\int_0^{t_1} (f(s) - f^0)\, ds\right| \\ &\le C\,(t_k - t_1)^{-\gamma} h_1^2 |f'(\eta_0)| \\ &\le C\,(h_1)^{2-\gamma}|f'(\eta_0)|, \qquad \text{for some } \eta_0 \in [0, t_1] \\ &\le C_T(f)h^2, \qquad \text{by the hypothesis } q(2-\gamma) \ge 2. \end{aligned}$$

Moreover, from the error estimates for the trapezoidal rule, we obtain

$$\begin{aligned} |E_{k,j}(f)| &= \left|K(t_k - \eta_j)\int_{t_j}^{t_{j+1}} (f(s) - f^{j+\frac{1}{2}})\, ds\right|, \qquad \text{for some } \eta_j \in [t_j, t_{j+1}] \\ &\le C\,(t_k - t_{j+1})^{-\gamma} \max_{s\in[t_j,t_{j+1}]} |f''(s)| h_{j+1}^3 \\ &\le C\,(k-j)^{-\gamma} h_{j+1}^{3-\gamma}, \qquad \text{for } j = 1, \cdots, k-2. \end{aligned}$$

An integral estimate for the summation gives the following inequality:

$$\sum_{k=2}^{n} h_k \sum_{j=1}^{k-2} |E_{k,j}| \leq C_f\, h^2 \sum_{k=3}^{n} h_k \sum_{j=1}^{k-2} (k-j)^{-\gamma} h_{j+1}^{1-\gamma} \leq C_T(f) h^2.$$

To estimate $E_{k,k-1}$, we define a linear Lagrange interpolation polynomial

$$L_f^k(s) := \frac{s-t_{k-1}}{h_k} f^k + \frac{t_k - s}{h_k} f^{k-1}.$$

From the triangle inequality and the Lagrange interpolation theorem, we obtain

$$\begin{aligned} |f(s) - \hat{f}(s)| &\leq |f(s) - L_f^k(s)| + |\hat{f}(s) - L_f^k(s)| \\ &\leq \frac{h_k^2}{2} \max_{t\in[t_{k-1},t_k]} |f''(t)| + |\hat{f}(s) - L_f^k(s)|. \end{aligned}$$

Now, to estimate $|\hat{f}(s) - L_f^k(s)|$, using the well-known Taylor's formula with remainder;

$$\begin{aligned} f^k &= f^{k-1} + h_k f'(t_{k-1}) + \int_{t_{k-1}}^{t_k} (t_k - s) f''(t)\, dt, \\ f^{k-2} &= f^{k-1} - h_{k-1} f'(t_{k-1}) + \int_{t_{k-2}}^{t_{k-1}} (t_{k-1} - s) f''(t)\, dt, \end{aligned}$$

we have

$$\begin{aligned} \max_{s\in[t_{k-1},t_k]} |\hat{f}(s) - L_f^k(s)| &= |\hat{f}(t_{k-1}) - f^{k-1}| \\ &= \frac{1}{h_{k-1/2}} \left| \frac{h_k}{2} f^{k-3/2} + \frac{h_{k-1}}{2} f^{k-1/2} - h_k f^{k-1} \right| \\ &\leq C h_k h_{k-1} |t_{k-1}^{-\gamma}|. \end{aligned}$$

Hence, we obtain the estimate for $E_{k,k-1}$:

$$\begin{aligned} |E_{k,k-1}(f)| &\leq \max_{t\in[t_{k-1},t_k]} |f(s) - \hat{f}(s)| \left| \int_{t_{k-1}}^{t_k} K(t_k - s)\, ds \right| \\ &\leq C\, h_k^2 \max_{t\in[t_{k-1},t_k]} |f''(t)| h_k^{1-\gamma} \\ &\leq C(f)\, h^2 t_k^{-\gamma} h_k^{1-\gamma}. \end{aligned}$$

Noting that $\sum_{k=2}^{n} t_k^{-\gamma} h_k^{2-\gamma} \leq h_n^{1-\gamma} \sum_{k=2}^{n} t_k^{-\gamma} h_k \leq C_T h_n^{1-\gamma}$, we conclude that

$$\sum_{k=1}^{M} h_k \sum_{j=0}^{k-1} |E_{k,j}(f)| \leq C_T(f)\, h^2.$$

This completes the proof. □

Our fully discretized scheme based on the Crank–Nicolson scheme is now defined by

$$(15)\qquad \begin{aligned} \bar{\partial}_t U^k(\mathbf{x}) - \Delta U^{k-1/2}(\mathbf{x}) &= \tilde{\sigma}^k(\mathcal{B}_N U(\mathbf{x})) + f^{k-1/2}, \qquad k \geq 2,\\ \bar{\partial}_t U^1(\mathbf{x}) - \Delta U^1(\mathbf{x}) &= \sigma^1((\mathcal{B}_N U(\mathbf{x})) + f^1,\\ U^0(\mathbf{x}) &= u_0(\mathbf{x}),\\ U^k(\mathbf{x}) &= 0 \qquad \mathbf{x} \in \Xi_N^\alpha \cap \partial\Omega, \end{aligned}$$

where

$$\tilde{\sigma}^k(\mathcal{B}_N U(\mathbf{x})) := \frac{1}{2}\left\{\sigma^k(\mathcal{B}_N U(\mathbf{x})) + \sigma^{k-1}(\mathcal{B}_N U(\mathbf{x}))\right\} \qquad \text{for } k \geq 2.$$

4. Stability and Convergence

In this section, we discuss the stability and the convergence results for the fully discretized parabolic PIDE. Using (6), we rewrite (15) equivalently as follows:

$$\begin{aligned} (\bar{\partial}_t U^k, \phi)_N - (\Delta \tilde{U}^k, \phi)_N &= \tilde{\sigma}^k((\mathcal{B}_N U, \phi)_N) + (\tilde{f}^k, \phi)_N, \quad \forall \phi \in \mathbb{P}_N^0(\Omega),\\ (U^0, \phi)_N &= (u_0, \phi)_N, \end{aligned}$$

or

$$(1)\qquad \begin{aligned} (\bar{\partial}_t U^k, \phi)_N + a_N(\tilde{U}^k, \phi) &= \tilde{\sigma}^k(B_N^\alpha(U, \phi)) + (\tilde{f}^k, \phi)_N, \ \forall \phi \in \mathbb{P}_N^0(\Omega), \ k \geq 1,\\ (U^0, \phi)_N &= (u_0, \phi)_N, \end{aligned}$$

where we set $\tilde{\sigma}^1 = \sigma^1$,

$$(2)\qquad \tilde{U}^k = \begin{cases} U^k, & k = 0, 1\\ U^{k-1/2}, & k \geq 2, \end{cases}$$

and define $\tilde{f}^k$ similarly.

We now show the stability and obtain the error estimates for the fully discretized scheme (15) or (1).

Lemma 2. *Let* $\tau_{kj} := \int_{t_j}^{t_{j+1}} |K(t_k - s)|\, ds$ *and* $\partial\kappa_{kj} = \kappa_{kj} - \kappa_{k-1,j-1}$ *for* $0 \leq j < k \leq M$. *If the grading exponent* q *satisfies* $\frac{2}{2-\gamma} \leq q$ *for* $0 \leq \gamma < 1$, *then there is a positive constant* C *depending on* K, T, q *and* γ *such that*

$$(i)\qquad \tau_{kj} \leq C(k-j)^{-\gamma} h_{j+1}^{1-\gamma}, \quad j < k,$$

$$(ii)\qquad \sum_{k=2}^{n}\sum_{j=1}^{k} |\partial\kappa_{kj}| \leq C, \qquad \textit{for } n \leq M.$$

Proof. Using the mean value theorem, we can easily verify that

$$q(k-j)h_j \le (t_k - t_j) \le q(k-j)h_k \le qkh_k \le Ct_k, \text{ for } j \le k, \tag{3}$$

$$(h_k - h_j) \le C(k-j)\max\{\frac{1}{j}h_j, \frac{1}{k}h_k\}, \text{ for } k \ge 2. \tag{4}$$

Applying the mean value theorem to τ_{kj} for $j \le k-2$, it follows from (3) that

$$\tau_{kj} \le C_K(t_k - t_{j+1})^{-\gamma}h_{j+1} \le C(k-j)^{-\gamma}h_{j+1}^{1-\gamma}, \qquad \text{for } j = 0, \cdots, k-2.$$

A direct integration yields that

$$\tau_{k,k-1} \le C_K \int_{t_{k-1}}^{t_k} (t_k - s)^{-\gamma}\, ds \le Ch_k^{1-\gamma},$$

which completes the proof of (i).

To estimate the inequality (ii), for $1 \le j \le k-2$, keeping the mean value theorem in mind, we have

$$\begin{aligned}
|\partial\kappa_{kj}| \le\ & \int_0^1 |K(t_k - t_j - h_{j+1}u)|(h_{j+1} - h_j)\, du \\
& + \int_0^1 |K(t_k - t_j - h_{j+1}u) - K(t_{k-1} - t_{j-1} - h_j u)| h_j\, du \\
\le\ & C\frac{1}{j}\tau_{kj} + C_K \int_0^1 \big(t_{k-1} - t_{j-1} - h_j u\big)^{-1-\gamma}(h_k - h_j)h_j\, du \\
\le\ & C\Big(\frac{1}{k}h_k^{1-\gamma}(k-j)^{-\gamma}\frac{k}{j}\Big(\frac{h_{j+1}}{h_k}\Big)^{1-\gamma} + (h_k - h_j)(t_{k-1} - t_j)^{-1-\gamma}h_j\Big).
\end{aligned}$$

Using (3) and (4) for the second term of the right hand side, we show that

$$(h_k - h_j)(t_{k-1} - t_j)^{-1-\gamma}h_j \le C(k-j)^{-\gamma}h_j^{-\gamma}\frac{1}{k}h_k \max\left\{\frac{k}{j}\frac{h_j}{h_k}, 1\right\}. \tag{5}$$

Then summing $|\partial\kappa_{kj}|$ for $1 \le j \le k-2$, we have

$$\begin{aligned}
\sum_{j=1}^{k-2} |\partial\kappa_{kj}| \le\ & Ck^{-\gamma}h_k^{1-\gamma}\int_0^1 (1-s)^{-\gamma}\big(s^{(q-1)(1-\gamma)-1} + s^{-(q-1)\gamma}\big)\, ds, \\
\le\ & Ck^{-\gamma}h_k^{1-\gamma},
\end{aligned}$$

since the hypothesis $(q-1)\gamma < 1$. We now turn to the estimates for $|\partial\kappa_{k,k-1}|$ and $|\partial\kappa_{k\,k}|$:

$$
\begin{aligned}
|\partial\kappa_{k,k-1}| \le & \int_0^1 |K(h_k(1-u))| \left| \frac{h_k^2(1/2-u)}{h_{k-1/2}} - \frac{h_{k-1}^2(1/2-u)}{h_{k-3/2}} \right| du \\
& + \int_0^1 |K(h_k(1-u)) - K(h_{k-1}(1-u))| \left| \frac{h_{k-1}^2(1/2-u)}{h_{k-3/2}} \right| du \\
\le & \, C \int_0^1 |K(h_k(1-u))| \frac{1}{k} h_k \, du \\
& + C \int_0^1 (1-u)^{-1-\gamma} h_{k-1}^{-1-\gamma} (h_k - h_{k-1})(1-u) \left| \frac{h_{k-1}^2}{h_{k-3/2}} \right| du \\
\le & \, C \frac{1}{k} h_k^{1-\gamma}.
\end{aligned}
$$

Similarly, we can obtain the estimate:

$$
|\partial\kappa_{k\,k}| = |\kappa_{k\,k} - \kappa_{k-1,k-1}| \le C \frac{1}{k} h_k^{1-\gamma}.
$$

Thus, we conclude that for $n \le M$,

$$
\sum_{k=2}^{n} \sum_{j=1}^{k} |\partial\kappa_{k\,j}| \le C \sum_{k=2}^{n} \left(\frac{1}{k} + k^{-\gamma}\right) h_k^{1-\gamma} \le C \int_0^T t^{-\gamma} \, dt \le C,
$$

which completes the proof of (ii). □

Lemma 3. *Let the grading exponent q satisfy*

$$
\frac{2}{2-\gamma} \le q < 1 + \gamma^{-1}.
$$

For each $\epsilon > 0$, there is a constant C_ϵ independent of M such that

$$
\left| \sum_{k=1}^{n} h_k \sum_{j=0}^{k-1} (k-j)^{-\gamma} h_{j+1}^{1-\gamma} f_j f_k \right| \le \epsilon \sum_{k=1}^{n} h_k f_k^2 + C_\epsilon \sum_{k=0}^{n-1} (n-k)^{-\gamma} h_{k+1}^{1-\gamma} \sum_{j=0}^{k} h_{j+1} f_j^2.
$$

Proof. Applying the inequality $ab \le \epsilon a^2 + 1/(4\epsilon) b^2$ to the left hand side, we obtain

$$
\left| \sum_{k=1}^{n} h_k \sum_{j=0}^{k-1} (k-j)^{-\gamma} h_{j+1}^{1-\gamma} f_j f_k \right| \le \epsilon \sum_{k=1}^{n} h_k f_k^2 + \frac{1}{4\epsilon} \sum_{k=1}^{n} h_k \left(\sum_{j=0}^{k-1} (k-j)^{-\gamma} h_{j+1}^{1-\gamma} f_j \right)^2.
$$

It follows form Cauchy–Schwarz inequality that

$$
\begin{aligned}
h_k \left(\sum_{j=0}^{k-1} (k-j)^{-\gamma} h_{j+1}^{1-\gamma} f_j \right)^2 & \le h_k \left(\sum_{j=0}^{k-1} h_{j+1}^{-\gamma} (k-j)^{-\gamma} \right) \sum_{j=0}^{k-1} (k-j)^{-\gamma} h_{j+1}^{2-\gamma} f_j^2 \\
& \le C \sum_{j=0}^{k-1} (k-j)^{-\gamma} h_{j+1}^{2-\gamma} f_j^2,
\end{aligned}
$$

where the last inequality can be obtained by

$$h_k \sum_{j=0}^{k-1} h_{j+1}^{-\gamma}(k-j)^{-\gamma} \le C\big(h_k k\big)^{1-\gamma} \int_0^1 s^{-(q-1)\gamma}(1-s)^{-\gamma}\, ds \le C.$$

Changing the indices and the order of summations, and again changing the indices, we obtain

$$\begin{aligned}\sum_{k=1}^{n}\sum_{j=0}^{k-1}(k-j)^{-\gamma}h_{j+1}^{2-\gamma}f_j^2 &= \sum_{k=1}^{n}\sum_{j=1}^{k} j^{-\gamma}h_{k-j+1}^{2-\gamma}f_{k-j}^2\\ &= \sum_{j=1}^{n} j^{-\gamma}\sum_{k=j}^{n} h_{k-j+1}^{2-\gamma}f_{k-j}^2\\ &= \sum_{j=1}^{n}(n-j+1)^{-\gamma}\sum_{k=0}^{j-1} h_{k+1}^{2-\gamma}f_k^2.\end{aligned}$$

This completes the proof. □

It is well known discrete Gronwall's lemma: Let $\{\alpha_i\}$, $\{\beta_i\}$, and $\{\omega_i\}$ be any nonnegative sequences satisfying

$$\omega_n \le \alpha_n + \sum_{j=0}^{n-1}\beta_j\omega_j, \qquad n \ge 0.$$

Then, there is a positive constant C satisfying

$$\omega_n \le C_\beta\big\{\alpha_n + \sum_{j=0}^{n-1}\beta_j\alpha_j\big\}, \qquad n \ge 0. \tag{6}$$

In this paper, the following version of Gronwall's lemma, which will be frequently used to establish our theory, is needed.

Lemma 4. *Let $\boldsymbol{w} = \{w_n\}$ and $\boldsymbol{\alpha} = \{\alpha_n\}$ be sequences of nonnegative real numbers satisfying*

$$w_n \le \alpha_n + \sum_{j=0}^{n-1}(n-j)^{-\gamma}h_{j+1}^{1-\gamma}w_j, \qquad n \ge 0. \tag{7}$$

Then there is a positive constant C_T such that

$$w_n \le \alpha_n + C_T\sum_{j=0}^{n-1}(n-j)^{-\gamma}h_{j+1}^{1-\gamma}\alpha_j, \qquad 0 \le n \le M.$$

Proof. For a positive sequence $\boldsymbol{\beta} = \{\beta_k\}$, we define an iterated sum $I_n^s(\boldsymbol{\beta})$:

$$\begin{aligned} I_n^1(\boldsymbol{\beta}) &= \sum_{k=0}^{n-1}(n-k)^{-\gamma}h_{k+1}^{1-\gamma}\beta_k, \\ I_n^s(\boldsymbol{\beta}) &= \sum_{k=s-1}^{n-1}(n-k)^{-\gamma}h_{k+1}^{1-\gamma}I_k^{s-1}(\boldsymbol{\beta}), \qquad \text{for } s \geq 2. \end{aligned}$$

By induction, we can show that

$$\begin{aligned} I_n^{s+1}(\boldsymbol{\beta}) &\leq C_s\sum_{k=1}^{n-1}(n-k)^{-\gamma}h_{k+1}^{1-\gamma}\sum_{j=0}^{k-1}((k-j)h_k)^{(s-1)(1-\gamma)-1}h_k^{\gamma}h_{j+1}^{1-\gamma}\beta_j \\ &\leq C_s\sum_{j=0}^{n-1}\{(n-j)h_n\}^{s(1-\gamma)-1}h_n^{\gamma}h_{j+1}^{1-\gamma}\beta_j, \qquad \text{for } s \geq 1, \end{aligned} \tag{8}$$

and,

$$\begin{aligned} I_n^{s+1}(\boldsymbol{\beta}) &\leq C_s\sum_{j=0}^{n-1}(n-j)^{-\gamma}\{(n-j)h_n\}^{(s-1)(1-\gamma)}h_{j+1}^{1-\gamma}\beta_j, \\ &\leq C_s\sum_{j=0}^{n-1}(n-j)^{-\gamma}h_{j+1}^{1-\gamma}\beta_j \qquad \text{for } s \geq 1. \end{aligned} \tag{9}$$

Then, it follows directly from (7) that

$$w_n \leq \alpha_n + \sum_{t=1}^{s}I_n^t(\boldsymbol{\alpha}) + I_n^{s+1}(\boldsymbol{w}), \tag{10}$$

or

$$I_n^{s+1}(\boldsymbol{w}) \leq \sum_{t=0}^{s}I_n^{s+t+1}(\boldsymbol{\alpha}) + I_n^{s+1}(I^{s+1}(\boldsymbol{w})).$$

For $s(1-\gamma)-1 > 0$, it follows from (8) and (3) that

$$\begin{aligned} I_n^{s+1}(\boldsymbol{w}) &\leq C_s\sum_{k=0}^{n-1}h_n^{\gamma}h_{k+1}^{1-\gamma}\alpha_k + C_s\sum_{k=0}^{n-1}h_n^{\gamma}h_{k+1}^{1-\gamma}I_k^{s+1}(\boldsymbol{w}) \\ &\leq C_s\sum_{k=0}^{n-1}h_n^{\gamma}h_{k+1}^{1-\gamma}\alpha_k + C\sum_{k=0}^{n-1}t_{k+1}^{-\gamma}h_{k+1}I_k^{s+1}(\boldsymbol{w}). \end{aligned}$$

Applying ordinary discrete Gronwall's lemma (6) and noting $\sum t_{k+1}^{-\gamma}h_{k+1} \leq C_T$, we obtain

$$I_n^{s+1}(\boldsymbol{w}) \leq C_T\sum_{k=0}^{n-1}h_n^{\gamma}h_{k+1}^{1-\gamma}\alpha_k \leq C_T\sum_{k=0}^{n-1}(n-k)^{-\gamma}h_{k+1}^{1-\gamma}\alpha_k, \tag{11}$$

Thus, it follows from (10), (9), and (11) that

$$w_n \le \alpha_n + C_T \sum_{k=0}^{n-1} (n-k)^{-\gamma} h_{k+1}^{1-\gamma} \alpha_k,$$

which completes the proofs. □

We now give the stability result for the full discrete scheme:

Theorem 1. *Suppose that $h(\equiv \max_{n\le M} h_n)$ is so small that*

$$\tau = \int_0^h |K(s)|\, ds \le \frac{c_2}{\|B_N^\alpha\|}.$$

Then the scheme (15) is stable, i.e., there is a positive constant $C := C(K, T, q, \gamma)$ such that

$$\|U^n\|_N \le C\Big(\|U^0\|_N + \sum_{k=1}^{n} h_k \|f^k\|_N \Big).$$

Proof. For convenience, with the notation $\tilde{U}^k$ and $\tilde{f}^k$ defined in (2), it follows from Lemma 2 that

$$\begin{aligned} |\sigma^1(B_N^\alpha(U,\phi))| &\le C h_1^{1-\gamma} \|\tilde{U}^0\|_{1,N} \|\phi\|_{1,N}, \\ |\sigma^k(B_N^\alpha(U,\phi))| &\le C\Big\{ \sum_{j=0}^{k-1} (k-j)^{-\gamma} h_{j+1}^{1-\gamma} \|\tilde{U}^j\|_{1,N} \|\phi\|_{1,N} \Big\} \\ &\quad + \tau_{nn} \|B_N^\alpha\| \|\tilde{U}^k\|_{1,N} \|\phi\|_{1,N}. \end{aligned}$$

Taking $\phi = \tilde{U}^k$ in (1) and noting $|\tau_{nn}| \le \tau(2 - \frac{h_{n-1}}{h_n + h_{n-1}}) \le \tau(2 - 2^{-q})$, we have

$$\begin{aligned} &\frac{1}{2}\big(\|U^k\|_N^2 - \|U^{k-1}\|_N^2\big) + \frac{c_2}{2^{q+1}} h_k \|\tilde{U}^k\|_{1,N}^2 \\ &\le C\, h_k \|\tilde{U}^k\|_{1,N} \Big\{ \sum_{j=0}^{k-1} (k-j)^{-\gamma} h_{j+1}^{1-\gamma} \|\tilde{U}^j\|_{1,N} \Big\} + h_k \|\tilde{f}^k\|_N \|\tilde{U}^k\|_N, \end{aligned}$$

or

$$\begin{aligned} &\big(\|U^k\|_N^2 - \|U^{k-1}\|_N^2\big) + h_k \|\tilde{U}^k\|_{1,N}^2 \\ &\le C\, h_k \|\tilde{U}^k\|_{1,N} \Big\{ \sum_{j=0}^{k-1} (k-j)^{-\gamma} h_{j+1}^{1-\gamma} \|\tilde{U}^j\|_{1,N} \Big\} + C h_k \|\tilde{f}^k\|_N \|\tilde{U}^k\|_N. \end{aligned}$$

Summing from $k = 1$ to n and applying Lemma 3 with suitable ϵ, we have

$$\|U^n\|_N^2 + \sum_{k=1}^{n} h_k \|\tilde{U}^k\|_{1.N}^2 \le \|U^0\|_N^2 + C\sum_{k=1}^{n} h_k \|\tilde{f}^k\|_N \|\tilde{U}^k\|_N$$
$$+C\sum_{k=0}^{n-1}(n-k)^{-\gamma} h_{k+1}^{1-\gamma} \sum_{j=0}^{k} h_{j+1}\|\tilde{U}^j\|_{1.N}^2.$$

It follows from Lemma 4 with $w_n = \|U^n\|_N^2 + \sum_{j=0}^{n} h_{j+1}\|\tilde{U}^j\|_{1.N}^2$ that

$$\|U^n\|_N^2 \le C\left\{\|U^0\|_N + \sum_{k=1}^{n} h_k\|\tilde{f}^k\|_N\right\} \max_{k\le n}\|U^k\|_N.$$

Hence, we have

$$\|U^n\|_N \le \max_{k\le n}\|U^k\|_N \le C\left(\|U^0\|_N + \sum_{k=1}^{n} h_k\|f^k\|_N\right) \qquad \text{for } n \le M,$$

this completes the proof. □

Finally we obtain the second order convergence results for the fully discretized scheme (15).

Theorem 2. *Let u and $\{U^n\}$ be the solution of (1) and (15) respectively. We assume that u satisfies the regularity assumptions $\mathcal{A}1$, $\mathcal{A}2$ and $\mathcal{A}3$. Further we assume that $u \in C^3((0,T]; L^2_{\omega_\alpha}(\Omega))$. If the grading exponent q satisfies $q = \frac{2+p}{2-\gamma} \le 2$ for some $p > 0$, and $h = \max_{n\le M} h_n$ is so small that*

$$\int_0^h |K(s)|\, ds \le \frac{c_2}{\|B_N^\alpha\|}, \qquad \text{for all } k \ge 1,$$

then there exists a constant C_T independent of N and h such that

$$\|u^n - U^n\|_{\omega_\alpha} \le C_T N^{1-\sigma}\left(\|u_0\|_{\sigma,\omega_\alpha} + \int_0^{t_n} \|u_t\|_{\sigma,\omega_\alpha}\, ds\right)$$
$$+C_T\sum_{k=1}^{n} h_k\|E(f^k)\| + C_T(u)h^2.$$

Proof. Let $e^k = U^k - u^k$ and comparing (1) with the variational form of (1) at time $t = t_k$ and $t = t_{k-1}$, we have

$$(12) \qquad (\bar{\partial}_t e^k, \phi)_N + a_N(\tilde{e}^k, \phi) = \tilde{\sigma}^k(B_N^\alpha(e^\cdot, \phi)) + (E(\tilde{f}^k), \phi) + R_1^k + R_2^k + R_3^k,$$

where we denote R_1^k, R_2^k and R_3^k as follows:

$$\begin{aligned}
R_1^k &= (\tilde{u}_t^k, \phi)_{\omega_\alpha} - (\bar{\partial}_t u^k, \phi)_N = (\tilde{u}_t^k - \bar{\partial}_t u^k, \phi)_{\omega_\alpha} - (E(\bar{\partial}_t u^k), \phi), \\
R_2^k &= A_{\omega_\alpha}(\tilde{u}^k, \phi) - a_N(\tilde{u}^k, \phi), \\
R_3^k &= \tilde{\sigma}^k(B_N^\alpha(u, \phi) - B_{\omega_\alpha}(u, \phi)) + \tilde{\sigma}^k((\mathcal{B}u, \phi)_{\omega_\alpha}) - \tilde{J}_k((\mathcal{B}u, \phi)_{\omega_\alpha}).
\end{aligned}$$

Here, we set $\tilde{J}_1 = J_1$,

$$\tilde{J}_k(\,\cdot\,) = \frac{1}{2}(J_k(\,\cdot\,) + J_{k-1}(\,\cdot\,)), \quad \text{for } k \geq 2,$$

and, $\tilde{u}^k$ and $\tilde{e}^k$'s are similarly defined as in (2).

With the same argument as Theorem 1, taking $\phi = \tilde{e}^k$ in (12), then we have

$$\begin{aligned}
&\frac{1}{2}(\|e^k\|_N^2 - \|e^{k-1}\|_N^2) + \frac{c_2 h_k}{2^{q+1}}\|\tilde{e}^k\|_{1,N}^2 \\
&\leq C h_k \sum_{j=0}^{k-1}(k-j)^{-\gamma} h_{j+1}^{1-\gamma}\|\tilde{e}^j\|_{1,N}\,\|\tilde{e}^k\|_{1,N} \\
&\quad + h_k\|E(\tilde{f}^k)\|\,\|\tilde{e}^k\|_N + h_k|R_1^k + R_2^k + R_3^k|. \qquad (13)
\end{aligned}$$

We now turn to the estimates for R_1^k, R_2^k and R_3^k:
Since $u \in C^2((0, t_1]; L^2_{\omega_\alpha}(\Omega))$, using the Taylor formula with the integral form of the remainder gives

$$h_1|(u_t^1 - \bar{\partial}_t u^1, \tilde{e}^1)| \leq h_1\|\tilde{e}^1\|_N \int_0^{t_1} \|u_{tt}\|_{\omega_\alpha}\, ds \leq C(u) h_1^{2-\gamma}\|\tilde{e}^1\|_N \leq C(u) h^2 \|\tilde{e}^1\|_N.$$

If $u \in C^3((0, T]; L^2_{\omega_\alpha}(\Omega))$, then still by the Taylor formula, we obtain

$$\begin{aligned}
h_k|(u_t^{k-1/2} - \bar{\partial}_t u^k, \tilde{e}^k)| &\leq h_k^2\,\|\tilde{e}^k\|_N \int_{t_{k-1}}^{t_k} \|u_{ttt}\|_{\omega_\alpha}\, ds, \\
&\leq C_T(u)\|\tilde{e}^k\|_N h_k^3 t_k^{-1-\gamma} \quad \text{for } k \geq 2.
\end{aligned}$$

Summing up to n, we have

$$\begin{aligned}
\sum_{k=2}^n h_k^3 t_k^{-1-\gamma} &\leq C\sum_{k=2}^n \left(\frac{1}{k}\right)^{1+\gamma} h_k^{2-\gamma} \leq C\sum_{k=2}^n k^{-1+p}\left(\frac{1}{M}\right)^{2+p} \\
&\leq C h^2 \sum_{k=2}^n \frac{1}{n}\left(\frac{k}{n}\right)^{-1+p}\left(\frac{n}{M}\right)^p \leq C_T(u) h^2.
\end{aligned}$$

For the second term of R_1^k, we obtain

$$(E(\bar{\partial}_t u_t^k), \tilde{e}^k) \leq C\frac{1}{h_k} N^{1-\sigma}\|\tilde{e}^k\|_N \int_{t_{k-1}}^{t_k} \|u_t\|_{\sigma,\omega_\alpha}\, ds.$$

We immediately have

$$\sum_{k=1}^{n} h_k|R_1^k| \le \max_{k\le n} \|e^k\|_N \left(C_T(u)h^2 + CN^{1-\sigma} \int_0^{t_n} \|u_t\|_{\sigma,\omega_\alpha}\, ds \right). \tag{14}$$

Using (9), Lemma 1 and the inequality $ab \le \frac{1}{4\epsilon}a^2 + \epsilon b^2$ to obtain the estimates for R_2 and R_3, we show that

$$\begin{aligned} |R_2^k| + |R_3^k| &\le CN^{1-\sigma}\left(\|\tilde{u}^k\|_{\sigma.\omega_\alpha} \|\tilde{e}^k\|_{1,N} + \sum_{j=0}^{k-1} (k-j)^{-\gamma} h_{j+1}^{1-\gamma} \|\tilde{u}^j\|_{\sigma,\omega_\alpha} \|\tilde{e}^k\|_{1,N} \right) \\ &\quad + C(u)h^2 \|e^k\|_N \\ &\le C_\epsilon N^{2(1-\sigma)} \max_{j\le k} \|\tilde{u}^j\|_{\sigma,\omega_\alpha}^2 + \epsilon \|\tilde{e}^k\|_{1,N}^2 + C(u)h^2\|e^k\|_N. \end{aligned}$$

Summing (13) from $k = 1$ to n, and applying Lemma 3 with suitable ϵ, we have

$$\begin{aligned} &\|e^n\|_N^2 + \sum_{k=1}^{n} h_k \|\tilde{e}^k\|_{1,N}^2 \\ &\le \|e^0\|_N^2 + C\sum_{k=0}^{n-1} (n-k)^{-\gamma} h_{k+1}^{1-\gamma} \sum_{j=0}^{k} h_{j+1} \|\tilde{e}^j\|_{1,N}^2 \\ &\quad + \sum_{k=1}^{n} h_k \|E(\tilde{f}^k)\| \|\tilde{e}^k\|_N + C_\epsilon N^{2(1-\sigma)} \max_{j\le k} \|u^j\|_{\sigma,\omega_\alpha}^2 \\ &\quad + \max_{k\le n} \|e^k\|_N \left(C_T(u)h^2 + CN^{1-\sigma} \int_0^{t_n} \|u_t\|_{\sigma,\omega_\alpha}\, ds \right) \end{aligned} \tag{15}$$

Applying Lemma 4, we have

$$\begin{aligned} \|e^n\|_N &\le \left(C_T(u)h^2 + CN^{1-\sigma} \int_0^{t_n} \|u_t\|_{\sigma,\omega_\alpha}\, ds \right) + C\left(\sum_{k=1}^{n} h_k \|E(f^k)\|_{\omega_\alpha} \right) \\ &\quad + CN^{1-\sigma} \max_{k\le n} \|u^k\|_{\sigma,\omega_\alpha} \\ &\le C_T N^{1-\sigma} \left(\|u_0\|_{\sigma,\omega_\alpha} + \int_0^{t_n} \|u_t\|_{\sigma,\omega_\alpha}\, ds \right) + C_T \sum_{k=1}^{n} h_k \|E(f^k)\| + C_T(u)h^2 \end{aligned}$$

which completes the proof of the theorem. □

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DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE OF KONKUK UNIVERSITY, CHUNGJU, CHUNGBUK 380-701, KOREA
Email address: kimchang@kku.ac.kr