Numerical Life Prediction Method for Fatigue Failure of Rubber-Like Material Under Repeated Loading Condition

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Predicting fatigue life by numerical methods was almost impossible in the field of rubber materials. One of the reasons is that there is not obvious fracture criteria caused by non-standardization of material and excessively various way of mixing process. But, tearing energy as fracture factor can be applied to a rubber-like material regardless of different types of fillers, relative to other fracture factors and the crack growth process of rubber could be considered as the whole fatigue failure process by the existence of potential defects in industrial rubber components. This characteristic of fatigue failure could make it possible to predict the fatigue life of rubber components in theoretical way. FESEM photographs of the surface of industrial rubber components were analyzed for verifying the existence and distribution of potential defects. For the prediction of fatigue life, theoretical way of evaluating tearing energy for the general shape of test-piece was proposed. Also, algebraic expression for the prediction of fatigue life was derived from the rough cut growth rate equation and verified by comparing with experimental fatigue lives of dumbbell fatigue specimen in various loading condition.

Key Words : Rubber, Tearing Energy, Fatigue Life Prediction, Energy Release Rate, Latent Defects, Rough Cut Growth Rate

1. Introduction

Of the many possible causes for the failure of vulcanized rubbers, one particular process is fatigue failure. Fatigue failure of rubber is remarkably distinguished from metal's one in the view of crack generation and growth rate. Applied fatigue theory to rubber material should be somewhat different from so called metal fatigue theory. Though fatigue theory about rubber material has been studied and developed since 1950, application to design of rubber product have not been activated widely since its theoretical difficulties and theoretical researches which were concentrated on experimental phenomenon. Most of all, non-standardization of rubber material has made a key role.

Tearing energy as the key factor of fatigue failure for rubber material was proposed (Rivlin, 1952). The difference of tearing energy between elastic and viscoelastic was demonstrated using hysterisis loss factor (Andrew 1974). Hysterisis loss by viscoelasticity was evaluated quantitatively/qualitatively and it was proved that critical tearing energy could be calculated using C-C decomposition (Fukaori, 1998). The first step of fatigue life prediction for rubber material using tearing energy was proposed in the form of empirical equation (Thomas, 1958 ; Gent, 1964).

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But, tearing energy could not play a key role as fracture factor of rubber material by the absence of theoretical way of measuring crack length and calculating energy release rate in general 3 dimensional rubber components. VCE (Virtual Crack Extension) method which could evaluate energy release rate using crack tip special element was proposed (Blackburn, 1972). Whereas Rice's VCE method which calculates energy release rate from stiffness matrix had a weak point in analysis time, Hellen's method has a benefit with considering energy release rate directly from energy change of crack tip special element (Rice, 1968 ; Hellen, 1975). VCE method was applied with linear/non-linear elastic material mainly and its purpose is evaluation of stress intensity factor of test-piece which has designed crack, so that application to rubber material was almost impossible by the reason of complicated shape, severe non-linearity and un-predictable crack position.

Fatigue life could be estimated only by experiments using rubber product itself. Even though finite element method was applied to estimate fatigue life (Kims, 1996 ; Kim et al., 1998) recently, S-N data of fatigue specimen was needed. Kim et al. suggested strain and displacement as the factor of fatigue life and showed reasonable agreement between prediction and experiments. Tearing energy was mentioned only for crack growth of rubber-cord laminates but was only estimated experimentally (Lake, 2001).

In this paper, theoretical and simple method of evaluating tearing energy and fatigue life prediction derived from cut growth rate equation under repeated loading condition are proposed. Tearing energy is reconsidered as fatigue factor and fracture criteria of rubber material. The fracture source of rubber is reviewed by forgoing paper and experiments. Tearing energy equation being applicable to general shape of test piece is formulated from the level of potential energy. For the verification, fatigue life of 3 dimensional dumbbell fatigue specimens are compared with predicted result using developed code and static finite element analysis. Various average loading and amplitude condition is applied.

2. Fatigue Fracture of Rubber

2.1 Fracture criteria of rubber

Stress intensity factor, energy release rate, stress have relationship each other and can be used as fatigue criteria within the field of linear elastic. But, these factors could not be used for rubber material because stress intensity factor is very sensitive of shape and was derived from Irwin's local stress formulation assuming that behavior of material is in infinitesimal displacement and linear elastic whereas rubber shows large deformation behavior. Energy release rate could be applied to a rubber material from the fact that it is derived from the general phenomenon that energy is purely decreased by crack growth without restriction of material. Although J-integral has the same fundamental principle with energy release rate, it could not be used in rubber product because it needs the stress field of crack tip and depends on the shape of crack specimen. Rivlin's tearing energy T is as follows ;

$$T=-\frac{1}{t}\frac{\partial W}{\partial c} \text{ or } -\frac{\partial W}{\partial A} \tag{1}$$

In Eq. (1), W is total strain energy, c is crack length, A is crack area, t is the thickness of tear test-pieces which have a cut formed with a razor blade. Tearing energy of test-pieces which have different shape each other were measured experimentally. He showed that critical tearing energy at catastrophic and incipient tearing state is identical regardless of the shape of test-pieces. For a recipe of NR, 1.3 kgf/mm^2 at catastrophic tearing, 3.7 kgf/mm^2 at incipient tearing were estimated respectively. Also, tearing energy was evaluated mathematically for different recipes by using theoretical expression of pure shear test-piece and the energy stored elastically per unit volume. These two expressions are given by

$$T=w_0 l_0 \tag{2}$$

$$w_0=C_1\left(\lambda^2+\frac{1}{\lambda^2}-2\right) \tag{3}$$

The suffix 0 indicates undeformed state, w is strain energy density, C_1 is material constants and

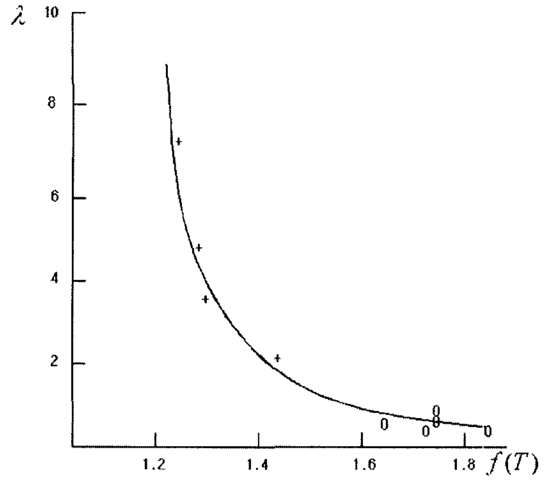

Fig. 1 Experimental and theoretical relation between tearing energy and extension ratio by Rivlin. Cross and circle denote different vulcanizate

λ is defined by extension ratio. Experimental result and theoretical function of tearing energy by extension ratio are shown in Fig. 1. The full line represents a theoretical result. The crosses and circles denote experimental results of different vulcanizate. In Rivlin's experiment, tearing energy was proved to be independent of the form of test-piece and applied loading condition. Also, tearing energy of NR which has different recipes could be estimated by theoretical expressions.

2.2 Appearance of ruptured rubber

In general, rubber has polymer structure in which carbon is combined with each other in zigzag form. If polymer structure is under tensional loading condition, polymer chain is going to be broken partially. The breakage of polymer chain results in forming micro cavity by chain reaction. If cavity grows into larger critical size, it becomes crack and tearing process is occurred.

Gent shows that carbon black could have function of initial crack from the experimental examination of the tensile strength of vulcanized rubber cylinders which was bonded to plane metal end-pieces (Gent, 1958). In his experiment, crack propagation from carbon black and wide crack distribution on cross-section of cylinder rubber was examined. Fig. 2 shows crack propagation

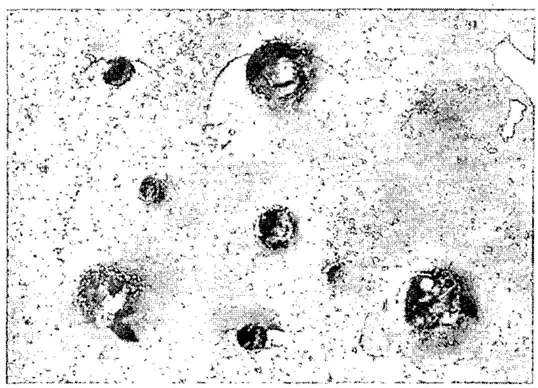

Fig. 2 Crack propagation from carbon black. Detailed section plot of cracked specimen by Gent (Magn. 13.)

from carbon black. There could be a number of potential defects larger than critical size as a result of rough manufacturing processes and various additions including carbon black. FESEM picture of the surface of engine mount and rubber bush showed in Fig. 3. These micro hole sized from 5 to 20 μm is distributed all around the surface. The other rubber product of door-seal, test-pieces and so on show the analogues result. (Larger and smaller micro hole than 10 μm were also existed, but not mentioned in this paper because the critical size of cavity referenced forgoing study was minimum 10 μm.) The gap between micro hole was measured about 20~500 μm for all test-pieces. If the critical size is effective as a crack, tearing could be occurred at any point in rubber product. SEM picture of the surface of engine mount right after fatigue test was showed in Fig. 4. Tearing was not occurred in this surface. It shows that cracks are generated from defects on all over the surface. Tearing can be regarded as occurring from these cracks generated by micro cavity at the point of highest stress. Thus, these defects function as a crack and the process of tearing could be considered as the growth process of defects with an exception of formation process.

2.3 Rough cut growth region

Tearing process with repeated loading and relaxation was studied (Thomas, 1958). A typical

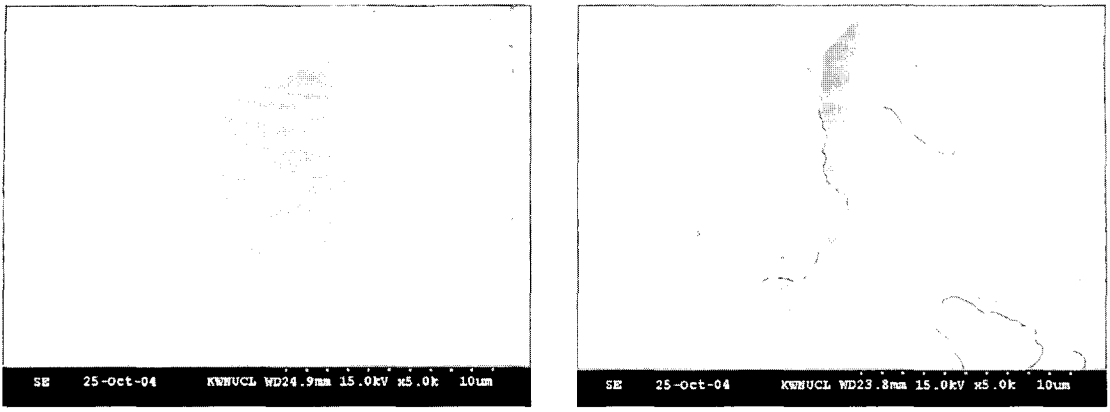

Fig. 3 FESEM picture of the surface of rubber product (left : engine mount, right : rubber bush)

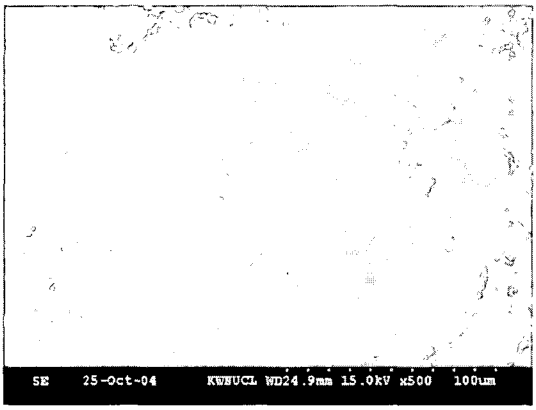

Fig. 4 FESEM picture of un-teared surface of engine mount right after fatigue test

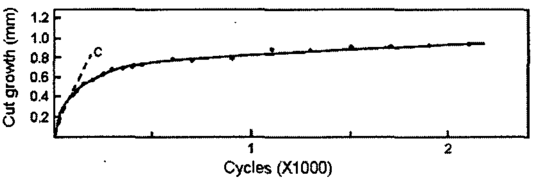

Fig. 5 Dynamic cut growth curve of rubber test-piece

cut growth curve is shown in Fig. 5, the cut growth rate versus cycles. Rapid cut growth rate of initial tearing region is changed gradually to slow down after a few thousand cycles, becomes substantially constant rate. The region of constant cut growth rate is called rough cut growth region and the rate equation in this region is,

$$\frac{dc}{dN}=\frac{T^2}{G_d} \tag{4}$$

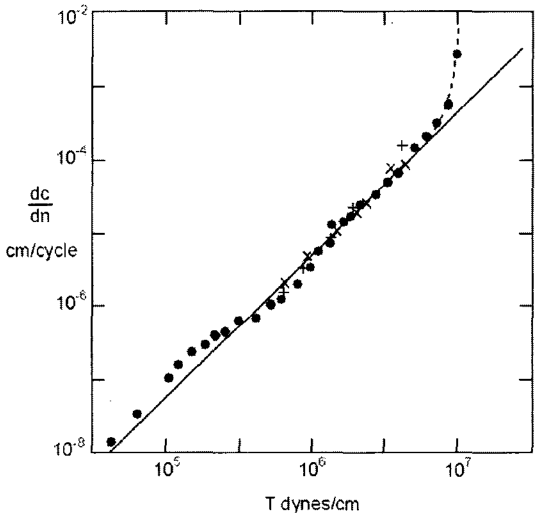

Fig. 6 Relationship between rate of cut growth dc/dN and the tearing energy T for various form of test-piece (Gent, 1964)

G_d is called dynamic cut growth rate constant, T is tearing energy, c is cut length and N means cycles. The cut growth rate right after rough cut growth region has much faster rate, so most of fatigue life in rubber material could be considered to be included in rough cut growth region. This fact is verified by the experiments which cut growth life was evaluated with various test-pieces of different shape and recipes. In these experiments, Evaluated life beyond rough cut growth region and calculated life by the equation of rough cut growth rate were shown nearly identical result as shown Fig. 6 and Table 1. The full

Table 1 Calculated and observed fatigue lives

λ	C_0 ($\times10^{-3}$)	N, kcycles	
		Observed	Calculated
2.36	36	3.0	4.4
2.08	46	6.5	7.2
1.47	43	78	88
1.205	70	700	760
1.098	78	8600	9500

line is theoretical result and symbols means the fatigue life of various test-pieces. Theoretical result shows constant rate of cut growth whereas a rapid change of experimental cut growth rate is showing in the vicinity of complete tearing point. The upward departure of the experimental points at high T values is probably due to the initial catastrophic tearing (Gent, 1964). λ is stretch ration and c_0 means initial cut length in Table 1. Therefore the rough cut growth rate equation could be used in predicting fatigue life for rubber-like material to have latent defects which is larger than critical crack length. It is obvious that the fatigue life of ideal rubber should be longer than the other, general rubber material.

3. Formulation of Tearing Energy

Tearing energy could be evaluated if and only if crack exists and growing because it was defined by the increment of crack length. There must be several assumption that larger crack than critical size is existed and the length of crack should increase as much as the proportional magnitude of load. The existence of latent defects which is large enough to be critical crack size and crack is propagated from latent defects was proved. Thus, the whole life of rubber product has intimate relation only with crack growth. Also, the rubber products which only have smaller defects or cracks must have longer fatigue life than the other rubber products which have critical size's crack. In that case, the predicted fatigue life by rough cut growth rate equation could be regarded as a minimum fatigue life among the same test-pieces. The critical size of crack was suggested as 10 μm for sharp shape, 100 μm for rounded shape. The assumption of the existence of critical size is sufficiently admissible because the defects larger than 10 μm is everywhere in rubber product.

First of all, we assume that defects larger than 10 μm is existed and assuming as initial crack, c_0. The criterion whether crack grows or not is to be incipient critical tearing energy. The potential energy release rate G is relating at a point on a crack profile the gradient of potential energy with respect to crack extension δc as

$$G=-\frac{d\Pi}{dC} \tag{5}$$

In finite element term, the potential energy Π for static behavior is defined as

$$\Pi=\frac{1}{2}\mathbf{u}^T\mathbf{K}\mathbf{u}-\mathbf{u}^T\mathbf{F} \tag{6}$$

in a deformed linear or non-linear material. $\mathbf{F}$ is now the vector of equivalent nodal loads, $\mathbf{u}$ is the corresponding vector of displacement and $\mathbf{K}$ is the structural stiffness matrix. The first term of Eq. (6) means internal energy. Accumulated strain energy W_j at each point j where data are measured within the domain of integration is given by

$$W_j=\sum_i w_{ij}v_j \tag{7}$$

where v_j is the volume represented by point j, i indicates accumulation over loading history and w is strain energy density. In this formulation, $W_j=W_j\,(u, c)$ is a function of displacements and crack length as appropriate for non-linear materials. Thus, from Eq. (6)

$$\Pi=\sum_j W_j-\mathbf{u}^T\mathbf{F} \tag{8}$$

therefore

$$\delta\Pi=\delta\mathbf{u}^T\frac{\partial\Pi}{\partial\mathbf{u}}\bigg|_c+\delta c^T\frac{\partial\Pi}{\partial c}\bigg|_{\mathbf{u}} \tag{9}$$

$$\begin{aligned}\delta\Pi=&\delta\mathbf{u}^T\left[\sum_j\frac{\partial W_j}{\partial\mathbf{u}}-\mathbf{F}\right]_c\\&+\delta c^T\left[\sum_j\frac{\partial W_j}{\partial c}-\frac{\partial\mathbf{F}^T}{\partial c}\right]_{\mathbf{u}}\end{aligned} \tag{10}$$

The first term is zero by equilibrium of the bracketed term. For extensions in the crack tip vicinity, $\partial\mathbf{F}$ is invariably zero, so the force term of second bracketed term can be ignored. Therefore,

$$\frac{\delta\Pi}{\delta c}=\sum_j\frac{\partial W_j}{\partial c} \tag{11}$$

The summation affects only elements containing the crack tip. Elsewhere, $\partial W/\partial c$ is zero, so the summation index j represents only the integration point in which the tearing energy is larger than incipient critical tearing energy. The incremental form of Eq. (11) is

$$\frac{\delta\Pi}{\delta c}=\sum_j\frac{\Delta W_j}{\Delta c} \tag{12}$$

By applying increments of crack area ΔA instead of crack length, tearing energy can be expressed by

$$T=\frac{\delta\Pi}{\delta A}=\sum_j\frac{\Delta W_j}{\Delta A} \tag{13}$$

where, the width of crack is assumed to be uniform or constant. If tearing is assumed to be occurred at one integration point k, an infinitesimal volume v_k is created as much as crack growth and strain energy should be released by the amount of $v_k w_k$. w_k is strain energy density stored at the integration point k. Thus

$$T=\sum_k\frac{v_k w_k}{\Delta A_k} \tag{14}$$

In Eq. (14) v_k and ΔA_k are the quadratic and cubic function of crack increments, Δc_k respectively. So, the fraction of v_k and ΔA_k is expressed by the first-degree function of crack increments in the simplest form of constants multiplication,

$$T=\sum_k\Delta c_k w_k \text{ or } \sum_k c_k w_k \tag{15}$$

Crack should be propagated only at the point of maximum tearing energy, crack length can be regarded as being uniform for overall maximum tearing energy point. Therefore

$$T=c\sum_k w_k \tag{16}$$

In this paper, FE modeling of tensional test-piece shown in Fig. 7 used for estimating tearing energy (Rivlin, 1952) was performed for verifying Eq. (16) and simple code was developed. Basically, evaluating tearing energy by this method requires one full finite element analysis to evaluate wi. Currently, the developed code is dealing with ABAQUS ASCII result file. The length of crack growth should be required to be determined before the step of second analysis. But, the Rivlin's tensional test-piece has already a cut of length by razor blade, so tearing energy could be easily evaluated using the result of 1'st step finite element analysis. Used rubber material is arbitrary natural rubber and identical boundary condition and loading condition was applied. Figure 8 shows applied boundary condition. Experimentally, 1.3 kgf/mm^2 of tearing energy was evaluated when the length of crack is increase by 5~10 mm. Maximum strain energy density, 0.276 kgf/mm^3 was detected at one integration point at crack tip and tearing energy 1.38~2.76 kgf/mm^2 was calculated by simple multiplication of maximum strain energy density and experimentally estimated cut length 5~10 mm.

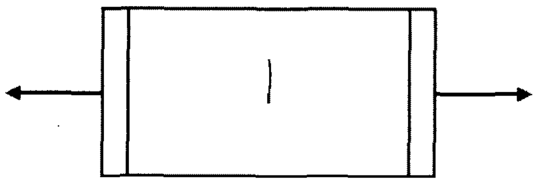

Fig. 7 Tensional test-piece with a cut by razor blade

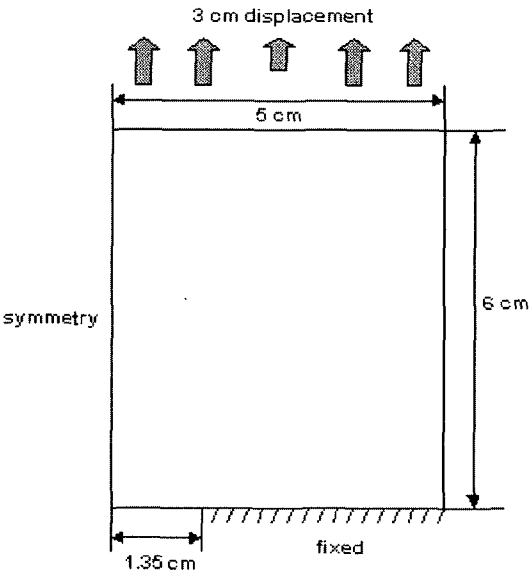

Fig. 8 1/4 test-piece and applied boundary condition

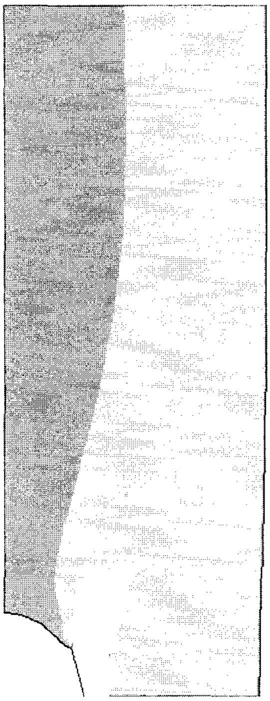

Fig. 9 Distribution of strain energy density (max= 0.276 kgf/mm²)

4. Fatigue Life Prediction of Rubber

4.1 Equation for fatigue life prediction

Using the simple tearing energy equation, Eq. (16), numerical method of fatigue life evaluation was developed. The number of cycles required to cause the crack to grow from an initial length c_0 to a length c_f obtained by integrating Eq. (4).

$$\int_0^{N_f} dN=\int_{c_0}^{c_f} \frac{G_d}{T^2}\, dc \tag{17}$$

Submitting Eq. (16)

$$N_f=\int_{c_0}^{c_f} \frac{G_d}{c^2\left(\sum_k w_k\right)^2}\, dc \tag{18}$$

The size and magnitude of maximum strain energy density is assumed not to be changed ideally.

$$N_f=\frac{G_d}{\left(\sum_k w_k\right)^2}\int_{c_0}^{c_f} \frac{1}{c^2}\, dc \tag{19}$$

$$N_f=\frac{G_d}{\left(\sum_k w_k\right)^2}\left[-\frac{1}{c}\right]_{c_0}^{c_f} \tag{20}$$

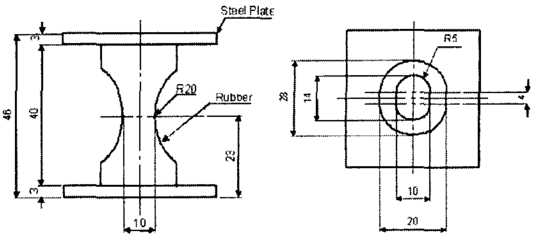

Fig. 10 Shape of rubber fatigue specimen

Therefore, equation for fatigue life prediction can be obtained as Eq. (21).

$$N_f=\frac{G_d}{\left(\sum_k w_k\right)^2}\left(\frac{1}{c_0}-\frac{1}{c_f}\right) \tag{21}$$

This equation shows that N_f becomes independent of c when the latter is much greater than c_0, and that a finite number of cycles is necessary to cause an indefinite increase in crack length. Thus, according to the above equation, the size of a fatigue specimen will have little effect on the number of cycles to failure if its dimensions are much greater than c_0. Ignoring the effect by latter crack size, the number of cycles to break or fatigue life will therefore given from Eq. (21), with $c_f \gg c_0$, by

$$N_f=\frac{G_d}{c_0\left(\sum_k w_k\right)^2} \tag{22}$$

4.2 Fatigue life prediction for dumbbell fatigue specimen

A three-dimensional dumbbell specimen in Fig. 10 was used for fatigue life evaluation of natural rubber. The 3D dumbbell specimen has an elliptical cross-section and parting lines are located on the minor axis of the specimen to avoid undesirable failure at the surface discontinuities. Used material is a carbon filled vulcanized natural rubber with a hardness of IRHD 60 (Kim et al., 2004). Kim conducted fatigue test in an ambient temperature of 20℃. Mean stress effects on the fatigue damage were taken into account by applying the zero and positive mean load of 49 and 98 N. Kim defined the fatigue life of 3D dumbbell specimen as the number of cycles to a complete separation.

Rubber can be considered as a hyper-elasitc material with incompressibility. Rubber material behavior is characterized by strain energy potential. In order to define the hyperelastic material behavior, material parameters in the Ogden (1972; 1976) strain energy potential (Ogden, 1972; 1976) of order 3 was determined using static uniaxial tension data identical with Kim's experiment. The 3D dumbbell specimen was modeled with 8-noded hexa hyperelastic element shown in Fig. 11. A nonlinear finite element analysis program of ABAQUS was used for the analysis of rubber specimen. Static finite element analysis was used as previous step which the information about strain energy density was acquired from. Figure 11 shows the strain energy density distribution of dumbbell specimen under a tensile loading of 150 N. Tearing energy was analyzed from static analysis result and fatigue life was calculated by developed code in which fatigue life equation, Eq. (22) was built in. Table 2 lists several loading condition. During fatigue life analysis, the maximum tearing energy was searched in whole FE model within the range of 1% error. 10 μm, critical crack length of rough cut growth and 200, dynamic cut growth constant of natural rubber were substituted (Gent, 1964; Thomas, 1958). Predicted result of fatigue life at each load condition is showed in Table 2 and compared with experimental result in Fig. 12. As shown in Table 2, identical tendency that fatigue life which is decreased as increasing amplitude can be observed in both of experimental and predicted result even though number of cycles predicted by theory is smaller than experimental result entirely. However, experimental result was determined by estimating lives to complete separation and the predicted result means the fatigue life previous of catastrophic tearing, considering only initial crack length. Therefore, predicted fatigue lives should be smaller than experimental lives. In the case of mean load 100 N and amplitude 150 N in which almost identical number of cycles with experiments is given may come from not considering about large tensile load enough to tear a rubber at once and the effect of compressional load. The difference between theory an experiment was generally 1.1~4.0 times, which can be occurred in experiments with identical fatigue specimen.

Table 2 Boundary condition & result of fatigue lives

Average load (N)	Amplitude (N)	Test data (cycle $\times 10^4$)	Predicted (cycle $\times 10^4$)
0	150	80	10.61
0	250	8	4.68
50	150	30	10.25
100	100	30	10.42
100	150	8	8.80

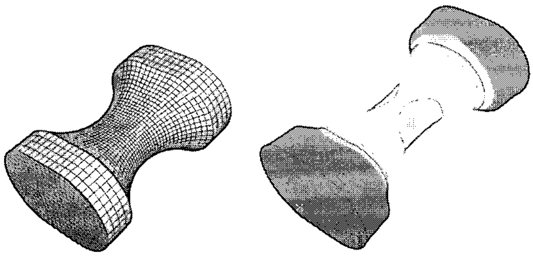

Fig. 11 3D FE model of rubber fatigue specimen

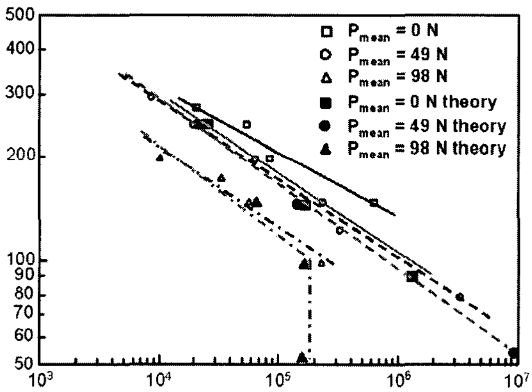

Fig. 12 Comparison experimental result with predicted fatigue lives

5. Conclusions

It has been seen from the foregoing experiments that tearing energy is independent with the form of test-piece, loading direction, different recipes of vulcanized natural rubber. These characters could bring a great advantage as fatigue factor for the rubber-like materials by considering existing fatigue test should be performed about each rubber product and recipes respectively because rubber's behavior is entirely dependent with shape and recipes. Formulation of tearing energy adjustable with finite element formulation was performed. The analysis result of tearing energy using developed code is compared with Rivlin's experimental of simple tensional test-piece and shows acceptable agreement. But, this verification is insufficient to determine how well the formulation is applicable to the rubber material. Algebraic expression of fatigue life evaluation for the rubber material has been developed from the rough cut growth equation and tearing energy formulation. Fatigue life of dumbbell specimen was predicted using developed equation incorporating static finite element analysis and verified fundamentally with the experimental result of KIMM. Identical tendency of fatigue lives according to various amplitudes and mean loads was showed. Predicted life was practically lower than the experimental result which was estimated at complete separation, but the difference between experiments and prediction was theoretically under expectation. However, predicted fatigue life in which compressional effect was apparent indicated inconsiderable difference with the experiments but still was overestimated. The effect of compressional load on the fatigue life of rubber needs to be studied in detail later on. A number of difficulties are existed to be solved for being practicable on real rubber product.

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