East Asian Math. J. 22 (2006), No. 1, pp. 111–130

COSYMPLECTIC SUBMERSIONS WITH VANISHING COSYMPLECTIC BOCHNER CURVATURE TENSOR

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Abstract. We study cosymplectic submersions with vanishing cosymplectic Bochner curvature tensor.

1. Introduction

Let M and B be C^∞ Riemannian manifolds. By a *Riemannian submersion* we mean a C^∞ mapping π from the total space M onto the base space B such that π is of maximal rank and the differential π_* of π preserves the lengths of vectors orthogonal to the fibre $\pi^{-1}(x)$ for all $x \in B$.

The theory of Riemannian submersions was initiated by O'Neill[8] and Gray[4]. A systematic exposition could be found in the A. Besse's book[1]. Presently, there is an extensive literature on the Riemannian submersions with different conditions imposed on the total space and on the fibres. In particular, B. Watson announced interesting results about almost Hermitian submersions[11] and two kinds of almost contact metric submersions[12].

On the other hand, B. H. Kim[6] and K. Takano[9] have investigated Riemannian submersion with Sasakian structure such that the contact Bochner curvature tensor of the total space vanishes identically. Also, K. Takano[10] has investigated Kählerian submersions with vanishing Bochner curvature tensor.

Received April 28, 2006.

2000 Mathematics Subject Classification: 53C40, 53C42, 53C50.

Key words and phrases: cosymplectic submersion, cosymplectic Bochner curvature tensor, Riemannian submersion.

This work was supported by University of Ulsan Reasearch Fund of 2004.

The odd-dimensional counterpart of Kählerian manifolds is cosymplectic manifolds. The canonical examples[3] of cosymplectic manifold are given by the product of a Kählerian manifold with $\mathbb{R}$ or with the circle S^1.

In this context, we study cosymplectic submersions with vanishing cosymplectic Bochner curvature tensor.

2. Riemannian Submersions

B.O'Neill[8] has characterized the geometry of a Riemannian submersion $\pi : M \to B$ in terms of the tensor fields T and A defined for vector fields E and F on M by

$$T_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F \text{ and } A_E F = \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F,$$

where ∇ is the Levi-Civita connection of metric g of M, the symbols $\mathcal{V}$ and $\mathcal{H}$ are the orthogonal projections of the tangent bundle $T(M)$ of M onto the vertical distribution $\mathcal{V}(M)$ and horizontal distribution $\mathcal{H}(M)$ in the tangent bundle $T(M)$, respectively. T is related to the second fundamental form of fibres, it is identically zero if and only if each fibre is totally geodesic. We call the Riemannian submersion with *totally geodesic fibre* if T vanishes identically. Also, since A is related to the integrability of $\mathcal{H}(M)$, it is identically zero if and only if $\mathcal{H}(M)$ is integrable. Moreover, if A and T vanish identically, then the total space is locally a product space of the base space and fibre.

We call a vector field X on M *projectable* if there exists a vector field X_* on B such that $\pi_*(X_p) = X_*\pi(p)$ for each $p \in M$, and say that X and X_* are *π-related*. Also, a vector field X on M is called *basic* if it is projectable and horizontal. Then we have[1,8]

LEMMA 2.1. *If X and Y are basic vector fields on M which are π-related to X_* and Y_* on B, then*

1. $\widehat{g}(X_*, Y_*) = g(X, Y) \circ \pi$, *where g is the metirc on M and $\widehat{g}$ the metric on B,*
2. $\mathcal{H}[X, Y]$ *is basic and is π-related to $[X_*, Y_*]$,*
3. $\mathcal{H}\nabla_X Y$ *is basic and π-related to $\widehat{\nabla}_{X_*}Y_*$, where $\widehat{\nabla}$ is the Riemannian connection of B.*

LEMMA 2.2. *Let X and Y be horizontal vector fields, U and V vertical vector fields. Then*

$$\nabla_U V = T_U V + \overline{\nabla}_U V,\ \nabla_U X = \mathcal{H}\nabla_U X + T_U X, \tag{2.1}$$

$$\nabla_X U = A_X U + \mathcal{V}\nabla_X U,\ \nabla_X Y = \mathcal{H}\nabla_X Y + A_X Y. \tag{2.2}$$

Furthermore, if X is basic, $\mathcal{H}\nabla_U X = A_X U$.

Next, we denote by R the curvature tensor of g, by $\overline{R}$ the collection of all curvature tensors of the Riemannian metric $\bar{g}$ on the fibre and by $\widehat{R}(X,Y)Z$ the horizontal vector field such that $\pi_*(\widehat{R}(X,Y)Z) = \widehat{R}(\pi_*X, \pi_*Y)\pi_*Z$ at each $p \in M$, where $\widehat{R}$ is the curvature tensor of $\acute{g}$ on B. Then we have[1,8]

LEMMA 2.3. *Let U, V, W, W' be vertical vector fields, and X, Y, Z, Z' horizontal vector fields, Then*

$$\begin{aligned} g(R(U,V)W,W') =& g(\overline{R}(U,V)W,W') + g(T_U W, T_V W') \\ &- g(T_V W, T_U W'), \end{aligned} \tag{2.3}$$

$$g(R(U,V)W,X) = g((\nabla_U T)_V W, X) - g((\nabla_V T)_U W, X), \tag{2.4}$$

$$\begin{aligned} g(R(X,U)Y,V) =& g((\nabla_X T)_U Y, V) - g((\nabla_U A)_X Y) \\ &+ g(T_U X, T_V Y) - g(A_X U, A_Y V), \end{aligned} \tag{2.5}$$

$$\begin{aligned} g(R(U,V)X,Y) =& g((\nabla_U A)_X V, Y) - g((\nabla_V A)_X U, Y) \\ &- g(A_X U, A_Y V) + g(A_X V, A_Y U) \\ &+ g(T_U X, T_V Y) - g(T_V X, T_U Y), \end{aligned} \tag{2.6}$$

$$\begin{aligned} g(R(X,Y)Z,U) =& - g((\nabla_Z A)_X Y, U) - g(A_X Y, T_U Z) \\ &+ g(A_Y Z, T_U X) + g(A_Z X, T_U Y), \end{aligned} \tag{2.7}$$

$$g(R(X,Y)Z,Z') = g(\widehat{R}(X,Y)Z,Z') + 2g(A_XY, A_ZZ') - g(A_YZ, A_XZ') + g(A_XZ, A_YZ'), \tag{2.8}$$

Also, we get[8]

$$g((\nabla_U A)_X Y, V) + g((\nabla_V A)_X Y, U) = g((\nabla_Y T)_U V, X) - g((\nabla_X T)_U V, Y). \tag{2.9}$$

For each $p \in M$, we denote by $\{X_1, X_2, \cdots, X_n\}$ and $\{U_1, U_2, \cdots, U_s\}$ local orthonormal basis of $\mathcal{H}(M)$ and $\mathcal{V}(M)$, respectively. Then we define[1]

$$g(A_X, A_Y) = \sum_{i=1}^{n} g(A_X X_i, A_Y X_i) = \sum_{\alpha=1}^{s} g(A_X U_\alpha, A_Y U_\alpha), \tag{2.10}$$

$$g(A_X, T_U) = \sum_{i=1}^{n} g(A_X X_i, T_U X_i) = \sum_{\alpha=1}^{s} g(A_X U_\alpha, T_U U_\alpha), \tag{2.11}$$

$$g(T_U, T_V) = \sum_{i=1}^{n} g(T_U X_i, T_V X_i) = \sum_{\alpha=1}^{s} g(T_U U_\alpha, T_V U_\alpha), \tag{2.12}$$

$$g(AU, AV) = \sum_{i=1}^{n} g(A_{X_i} U, A_{X_i} V) \tag{2.13}$$

$$g(TX, TY) = \sum_{i=1}^{s} g(T_{U_\alpha} X, T_{U_\alpha} Y) \tag{2.14}$$

$$g((\nabla_U T)_V, T_W) = \sum_{\alpha=1}^{s} g((\nabla_U T)_V U_\alpha, T_W U_\alpha), \tag{2.15}$$

$$(2.16) \qquad g((\nabla_X T)Y, TZ) = \sum_{\alpha=1}^{s} g((\nabla_U T)_{U_\alpha} Y, T_{U_\alpha} Z),$$

$$(2.17) \qquad \acute{\delta} A = -\sum_{i=1}^{n} (\nabla_{X_i} A)_{X_i}, \; \bar{\delta} T = -\sum_{\alpha=1}^{s} (\nabla_{U_\alpha} A)_{U_\alpha}.$$

Moreover, we define the symmetric tensor $\tilde{\delta} T$ by

$$(2.18) \qquad (\tilde{\delta} T)(U, V) = \sum_{i=1}^{n} g((\nabla_{X_i} T)_U V, X_i)$$

for vertical vector fields U and V. Also, the mean curvature vector along each fibre gives the horizontal vector field

$$(2.19) \qquad N = \sum_{\alpha=1}^{s} T_{U_\alpha} U_\alpha.$$

If N is identically zero, then each fibre is called a *minimal submanifold* of M.

Let Ric, $\widehat{\mathrm{Ric}}$ and $\overline{\mathrm{Ric}}$ be the Ricci tensors of the Riemannian metrics g, $\widehat{g}$ and $\overline{g}$, respectively. Then we have[1]

$$(2.20) \quad \mathrm{Ric}(U, V) = \overline{\mathrm{Ric}}(U, V) - g(N, T_U V) + g(AU, AV) + (\tilde{\delta} T)(U, V),$$

$$(2.21) \qquad \begin{aligned} \mathrm{Ric}(X, U) = & g((\bar{\delta} T)U, X) - g((\grave{\delta} A)X, U) \\ & - 2g(A_X, T_U) + g(\nabla_U N, X), \end{aligned}$$

$$(2.22) \qquad \begin{aligned} \mathrm{Ric}(X, Y) = & \widehat{\mathrm{Ric}}(X, Y) - 2g(A_X, A_Y) - g(TX, TY) \\ & + \frac{1}{2}\{g(\nabla_X N, Y) + g(\nabla_Y N, X)\}, \end{aligned}$$

where $\widehat{\mathrm{Ric}}$ is the horizontal symmetric 2-form on M such that $\widehat{\mathrm{Ric}}(X, Y) = \widehat{\mathrm{Ric}}(\pi_* X, \pi_* Y)$.

Moreover, if τ, $\hat{\tau}$, $\bar{\tau}$ are the scalar curvatures of the Riemannian metrics g, $\widehat{g}$ and $\overline{g}$, respectively, then

$$\tau = \hat{\tau} + \bar{\tau} - 2\hat{\delta}N - |N|^2 - |A|^2 - |T|^2, \tag{2.23}$$

where we denote $\hat{\tau} \circ \pi$ by $\hat{\tau}$ simply and we put $|N|^2 = g(N, N)$,

$$|A|^2 = \sum_{i=1}^{n} g(A_{X_i}, A_{X_i}) = \sum_{\alpha=1}^{s} g(AU_\alpha, AU_\alpha), \tag{2.24}$$

$$|T|^2 = \sum_{i=1}^{n} g(TX_i, TX_i) = \sum_{\alpha=1}^{s} g(T_{U_\alpha}, T_{U_\alpha}). \tag{2.25}$$

3. Characterizations of Cosymplectic Submersions

Let (M, ϕ, ξ, η, g) be an almost contact metric manifold. Then, we have

$$\phi^2 = -I + \eta \otimes \xi,\ \eta(\xi) = 1,\ g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{3.1}$$

for any vector fields X and Y on M. From (3.1), we deduce that

$$\phi \circ \xi = 0 \ \textit{and}\ \eta(X) = g(X, \xi), \tag{3.2}$$

for any vector field X on M. An almost contact metric manifold (M, ϕ, ξ, η, g) is said to be *integralbe* if $[\phi, \phi] + 2d\eta \oslash \xi = 0$ and *cosymplectic* if it is integrable, $d\eta = 0$ and $d\Phi = 0$, where the fundamental 2-form Φ of M is defined by $\Phi(X, Y) = g(X, \phi Y)$ for any vector fields X, Y on M. It can be shown that the cosymplectic structure is characterized by

$$\nabla_X \phi = 0 \text{ and } \nabla_X \eta = 0, \tag{3.3}$$

where ∇ is the connection of the metric g(for details, see[2,7,13]).

Let M and B be the almost contact metric manifolds with an almost contact metric structure (ϕ, ξ, η, g) and $(\widehat{\phi}, \widehat{\xi}, \widehat{\eta}, \widehat{g})$, respectively. Let $\pi : M \to B$ be a Riemannian submersion which satisfies

$$\pi_* \phi E = \widehat{\phi} \pi_* E, \quad \text{and} \quad \pi_* \xi = \widehat{\xi},$$

where E is a vector field on M. Then π is said to be a *cosymplectic submersion* if the total space M is cosymplectic.

It is clear from definition that the vertical and horizontal distributions determined by a cosymplectic submersion π are ϕ-invariant, that is, $\phi\{\mathcal{V}(M)\} \subseteq \mathcal{V}(M)$ and $\phi\{\mathcal{H}(M)\} \subseteq \mathcal{H}(M)$, and the base manifold B inherit a cosymplectic structure from the total space M. Moreover, B.Waston[12] proved that the horizontal distribution is integrable(i.e., $A = 0$), and each fibre is minimal(i.e., $N = 0$)(for details see[12])

3.1. Cosymplectic Submersions with Constant ϕ-Sectional Curvature

For a cosympletic manifold M of dimension $2n+1$, the Ricci tensor has the following properties.

$$\mathrm{Ric}(\phi E, \phi F) = \mathrm{Ric}(E, F), \ \mathrm{Ric}(E, \xi) = 0$$

for any vector fields E and F on M.

A plane section in T_pM is called a *ϕ-section* if there exists a unit vector E in T_pM orthogonal to ξ such that $\{E, \phi E\}$ is an orthonormal basis of the plane section. Then the sectional curvature $K(E, \phi E) = g(R(E, \phi E)\phi E, E)$ is called a *ϕ-sectional curvature*. It has been shown[2,7] that if a cosymplectic manifold M is of constant ϕ-sectional curvature c, then

$$\begin{aligned} g(\mathrm{R}(E_1, E_2)E_3, E_4) = -\frac{c}{4}\Big\{ & g(E_1, E_3)g(E_2, E_4) \\ & - g(E_1, E_4)g(E_2, E_3) + g(\phi E_1, E_3)g(\phi E_2, E_4) \\ & - g(\phi E_1, E_4)g(\phi E_2, E_3) + 2g(\phi E_1, E_2)g(\phi E_3, E_4) \\ & - \eta(E_1)\eta(E_3)g(E_2, E_4) + \eta(E_2)\eta(E_3)g(E_1, E_4) \\ & + \eta(E_1)\eta(E_4)g(E_2, E_3) - \eta(E_2)\eta(E_4)g(E_1, E_3)\Big\} \end{aligned} \tag{3.4}$$

for vector fields E_i $(i = 1, 2, 3, 4)$ on M.

Next we define[5] the so-called *cosymplectic Bochner curvature tensor* B^c and *η-Einstein tensor* Q on M, respectively, by

$$\begin{aligned}
g(B^c(E_1,E_2)E_3,E_4) &= g(R(E_1,E_2)E_3,E_4) \\
&+ \frac{1}{2(m+2)}\Big\{g(E_1,E_3)\mathrm{Ric}(E_2,E_4) - g(E_2,E_3)\mathrm{Ric}(E_1,E_4) \\
&+ g(E_2,E_4)\mathrm{Ric}(E_1,E_3) - g(E_1,E_4)\mathrm{Ric}(E_2,E_3) \\
&+ g(\phi E_1,E_3)S(E_2,E_4) - g(\phi E_2,E_3)S(E_1,E_4) \\
&+ g(\phi E_2,E_4)S(E_1,E_3) - g(\phi E_1,E_4)S(E_2,E_3) \\
&+ 2g(\phi E_1,E_2)S(E_3,E_4) + 2g(\phi E_3,E_4)S(E_1,E_2) \\
&+ \eta(E_1)\eta(E_4)\mathrm{Ric}(E_2,E_3) - \eta(E_2)\eta(E_4)\mathrm{Ric}(E_1,E_3) \\
&+ \eta(E_2)\eta(E_3)\mathrm{Ric}(E_1,E_4) - \eta(E_1)\eta(E_3)\mathrm{Ric}(E_2,E_4)\Big\} \\
&- \frac{\tau}{4(m+1)(m+2)}\Big\{g(E_1,E_3)g(E_2,E_4) - g(E_2,E_3)g(E_1,E_4) \\
&+ g(\phi E_1,E_3)g(\phi E_2,E_4) - g(\phi E_2,E_3)g(\phi E_1,E_4) \\
&+ 2g(\phi E_1,E_2)g(\phi E_3,E_4) \\
&+ \eta(E_1)\eta(E_4)g(E_2,E_3) - \eta(E_2)\eta(E_4)g(E_1,E_3) \\
&+ \eta(E_2)\eta(E_3)g(E_1,E_4) - \eta(E_1)\eta(E_3)g(E_2,E_4)\Big\}
\end{aligned}$$

and

$$Q(E,F) = \mathrm{Ric}(E,F) - \frac{\tau}{2m}g(E,F) + \frac{\tau}{2m}\eta(E)\eta(F)$$

for vector fields $E_i\,(i=1,2,3,4), E$ and F on M, where $S(E_i,E_j) = \mathrm{Ric}(\phi E_i,E_j)$ and τ denote the scalar curvature on M. A cosymplectic manifold M is said to be *cosymplectic Bochner flat*(*η-Einstein* resp.) if $B^c=0$($Q=0$ resp.). A cosymplectic manifod M is of constant ϕ-sectional curvature if and only if $B^c=0$ and $Q=0$ hold.

Next, we will seek fundamental equations of a cosymplectic submersion $\pi : M \to B$, where $\dim M = 2m+1$ and $\dim B = 2n+1$. Since the horizontal distribution is integrable ($A=0$) and each fiber is minimal ($N=0$), equations (2.3)-(2.9) are rewritten as follows:

$$g(\mathrm{R}(U,V)W,W') = g(\overline{R}(U,V)W,W') + g(T_UW,T_VW') - g(T_VW,T_UW'), \tag{3.5}$$

$$g(\mathrm{R}(U,V)W,X) = g((\nabla_UT)_VW,X) - g((\nabla_VT)_UW,X), \tag{3.6}$$

$$g(\mathrm{R}(X,U)Y,V) = g((\nabla_UT)_VW,X) + g(T_UX,T_VY), \tag{3.7}$$

$$g(\mathrm{R}(U,V)X,Y) = g(T_UX,T_VY) - g(T_VX,T_UY), \tag{3.8}$$

$$g(\mathrm{R}(X,Y)Z,U) = 0, \tag{3.9}$$

$$g(\mathrm{R}(X,Y)Z,Z') = g(\widehat{R}(X,Y)Z,Z'), \tag{3.10}$$

$$g((\nabla_YT)_UV,X) - g((\nabla_XT)_UV,Y) = 0 \tag{3.11}$$

for vertical vector fields U,V,W,W' and horizontal vector fields X,Y,Z,Z'.

Let $\{X_1,X_2,\cdots,X_{2n},U_1,U_2,\cdots,U_{2s},\xi\}$ be a local orthonormal frame on M such that $\{X_1,X_2,\cdots,X_{2n},X_{2n+1}=\xi\}$ and $\{U_1,U_2,\cdots,U_{2s}\}$, are local orthonormal bases of $\mathcal{H}$(M) and $\mathcal{V}$(M), respectively, where $X_{n+i}=\phi X_i\,(i=1,2,\cdots,n)$ and $U_{s+\alpha}=\bar{\phi}U_\alpha\,(\alpha=1,2,\cdots,s)$, where we have put $\bar{\phi}U=\mathcal{V}\phi U$ for any vertical vector field $U\in\mathcal{V}(M)$ and each fibre is of dimension $2s$.

By virtue of $A=0$ and $N=0$, we get from (2.20)-(2.23)

$$\mathrm{Ric}(U,V) = \overline{\mathrm{Ric}}(U,V) + (\tilde{\delta}T)(U,V), \tag{3.12}$$

$$\mathrm{Ric}(X,V) = g((\bar{\delta}T)U, X), \tag{3.13}$$

$$\mathrm{Ric}(X,Y) = \widehat{\mathrm{Ric}}(X,Y) - g(TX,TY), \tag{3.14}$$

$$\tau = \hat{\tau} + \bar{\tau} - |T|^2. \tag{3.15}$$

Also, we define skew-symmetric tensors S, $\widehat{S}$ and $\overline{S}$ by $\mathrm{S}(E,F) = \mathrm{Ric}(\phi E, F)$, $\widehat{\mathrm{S}}(X,Y) = \widehat{\mathrm{Ric}}(\phi X, Y)$ and $\overline{\mathrm{S}}(U,V) = \overline{\mathrm{Ric}}(\bar{\phi}U, V)$, respectively. Then we obtain from (3.12)-(3.14) that

$$\mathrm{S}(U,V) = \overline{\mathrm{S}}(U,V) + (\hat{\delta}T)(\bar{\phi}U, V), \tag{3.16}$$

$$\mathrm{S}(X,V) = g((\bar{\delta}T)U, \phi X), \tag{3.17}$$

$$\mathrm{S}(X,Y) = \widehat{\mathrm{S}}(X,Y) - g(T\phi X, TY). \tag{3.18}$$

From (3.12), we have

$$\begin{aligned} (\hat{\delta}T)(U,V) &= \mathrm{Ric}(U,V) - \overline{\mathrm{Ric}}(U,V) \\ &= \mathrm{Ric}(\bar{\phi}U, \bar{\phi}V) - \overline{\mathrm{Ric}}(\bar{\phi}U, \bar{\phi}V) \\ &= (\hat{\delta}T)(\bar{\phi}U, \bar{\phi}V). \end{aligned} \tag{3.19}$$

Since $\bar{\phi}^2 U = -U$, we also obtain

$$\begin{aligned} (\nabla_X T)_{\bar{\phi}U}\bar{\phi}V &= \nabla_X(T_{\bar{\phi}U}\bar{\phi}V) - T_{\nabla_X\bar{\phi}U}\bar{\phi}V - T_{\bar{\phi}U}(\nabla_X\bar{\phi}V) \\ &= \nabla_X(\phi^2 T_U V) - \phi^2 T_{\nabla_X U}V - \phi^2 T_U(\nabla_X V) \\ &= -(\nabla_X T)_U V. \end{aligned} \tag{3.20}$$

From this, we get immediately

$$(\tilde{\delta}T)(\bar{\phi}U, \bar{\phi}V) = -(\tilde{\delta}T)(U, V). \tag{3.21}$$

Using (3.19) and (3.21), we get

$$(\tilde{\delta}T)(U, V) = 0. \tag{3.22}$$

PROPOSITION 3.1. *Let $\pi : M \to B$ be a cosymplectic submersion. If the total space M is of constant ϕ-sectional curvature, then each fibre is an Einstein manifold.*

Proof. (3.12) and (3.22) yield $\mathrm{Ric}(U, V) = \overline{\mathrm{Ric}}(U, V)$. Thus we obtain from (3.4) that

$$\begin{aligned}\overline{\mathrm{Ric}}(U, V) &= \mathrm{Ric}(U, V) \\ &= \sum_{\alpha=1}^{2s} g(R(U_\alpha, U)V, U_\alpha) + \sum_{i=1}^{2n+1} g(R(X_i, U)V, X_i) \\ &= \frac{c(n+s+1)}{2} g(U, V),\end{aligned}$$

where we have used $\eta(U_\alpha) = \eta(X_i) = 0(\alpha = 1, \cdots, 2s,\ i = 1, \cdots, 2n)$ and $\eta(X_{2n+1}) = \eta(\xi) = 1$. □

3.2. Cosymplectic Submersions with Vanishing Cosymplectic Bochner Curvature Tensor

Now we shall consider $\pi : M \to B$ a cosymplectic submersion with vanishing cosymplectic Bochner curvature tensor. Then we see from (3.5)-(3.14), (3.17)-(3.19) and (3.22) that $g(B^c(E_1, E_2)E_3, E_4) = 0$ is equivalent to the following equations (3.23)-(3.28) for horizontal vector fields X, Y, Z, Z' and vertical vector fields U, V, W, W':

$$
\begin{aligned}
&g(\overline{R}(U,V)W,W') + g(T_UW,T_VW') - g(T_VW,T_UW') \\
&+ \frac{1}{2(m+2)}\Big\{g(U,W)\overline{\mathrm{Ric}}(V,W') - g(V,W)\overline{\mathrm{Ric}}(U,W') \\
&+ g(V,W')\overline{\mathrm{Ric}}(U,W) - g(U,W')\overline{\mathrm{Ric}}(V,W) \\
&+ g(\bar{\phi}V,W)\overline{\mathrm{S}}(U,W') - g(\bar{\phi}V,W')\overline{\mathrm{S}}(U,W) \\
&+ g(\bar{\phi}V,W')\overline{\mathrm{S}}(U,W) - g(\bar{\phi}U,W')\overline{\mathrm{S}}(V,W) \\
&+ 2g(\bar{\phi}U,V)\overline{\mathrm{S}}(W,W') + 2g(\bar{\phi}W,W')\overline{\mathrm{S}}(U,V)\Big\} \\
&- \frac{\tau}{4(m+1)(m+2)}\Big\{g(U,W)g(V,W') - g(V,W)g(U,W') \\
&+ g(\bar{\phi}U,W)g(\bar{\phi}V,W') - g(\bar{\phi}V,W)g(\bar{\phi}U,W') \\
&+ 2g(\bar{\phi}U,V)g(\bar{\phi}W,W')\Big\} = 0,
\end{aligned} \tag{3.23}
$$

$$
\begin{aligned}
&g((\nabla_UT)_VW,X) - g((\nabla_VT)_UW,X) \\
&+ \frac{1}{2(m+2)}\Big\{g(U,W)g((\bar{\delta}T)V,X) - g(V,W)g((\bar{\delta}T)U,X) \\
&+ g(\bar{\phi}U,W)g((\bar{\delta}T)V,\phi X) - g(\bar{\phi}V,W)g((\bar{\delta}T)U,\phi X) \\
&+ 2g(\bar{\phi}U,V)g((\bar{\delta}T)W,\phi X)\Big\} = 0,
\end{aligned} \tag{3.24}
$$

$$
\begin{aligned}
&g((\nabla_U T)_V W, X) + g(T_U X, T_V Y) + \frac{1}{2(m+2)} \\
&\times \Big[g(X,Y)\overline{\mathrm{Ric}}(U,V) + g(U,V)\Big\{\widehat{\mathrm{Ric}}(X,Y) - g(TX,TY)\Big\} \\
&+ g(\phi X, Y)\overline{\mathrm{S}}(U,V) + g(\bar{\phi}U,V)\Big\{\widehat{\mathrm{S}}(X,Y) - g(T\phi X, TY)\Big\} \\
&- \eta(X)\eta(Y)\overline{\mathrm{Ric}}(U,V)\Big] - \frac{\tau}{4(m+1)(m+2)}\Big\{g(X,Y) \\
&\times g(U,V) + g(\phi X,Y)g(\bar{\phi}U,V) - \eta(X)\eta(Y)g(U,V)\Big\} = 0,
\end{aligned}
\tag{3.25}
$$

$$
\begin{aligned}
&g(T_U X, T_V Y) - g(T_V X, T_U Y) + \frac{1}{(m+2)} \\
&\times \Big[g(\bar{\phi}U,V)\Big\{\widehat{\mathrm{S}}(X,Y) - g(T\phi X, TY)\Big\} + g(\phi X,Y) \\
&\times \overline{\mathrm{S}}(U,V)\Big] - \frac{\tau}{2(m+1)(m+2)}\Big\{g(\bar{\phi}U,V)g(\phi X,Y)\Big\} = 0,
\end{aligned}
\tag{3.26}
$$

$$
\begin{aligned}
&\frac{1}{2(m+2)}\Big\{g(X,Z)g((\bar{\delta}T)U,Y) - g(Y,Z)g((\bar{\delta}T)U,X) \\
&+ g(\phi X,Z)g((\bar{\delta}T)U,\phi Y) - g(\phi Y,Z)g((\bar{\delta}T)U,\phi X) \\
&+ 2g(\phi X,Y)g((\bar{\delta}T)U,\phi Z) + \eta(Y)\eta(Z)g((\bar{\delta}T)U,X) \\
&- \eta(X)\eta(Z)g((\bar{\delta}T)U,Y)\Big\} = 0,
\end{aligned}
\tag{3.27}
$$

$$
\begin{aligned}
&g(\widehat{R}(X,Y)Z,Z') + \frac{1}{2(m+2)} \\
&\times \Big[g(X,Z)\Big\{\widehat{\mathrm{Ric}}(Y,Z') - g(TY,TZ')\Big\} - g(Y,Z) \\
&\times \Big\{\widehat{\mathrm{Ric}}(X,Z') - g(TX,TZ')\Big\} + g(Y,Z')\Big\{\widehat{\mathrm{Ric}}(X,Z) \\
&-g(TX,TZ)\} - g(X,Z')\Big\{\widehat{\mathrm{Ric}}(Y,Z) - g(TY,TZ)\Big\} \\
&+ g(\phi X,Z)\Big\{\widehat{\mathrm{S}}(Y,Z') - g(T\phi Y,TZ')\Big\} - g(\phi Y,Z) \\
&\times \Big\{\widehat{\mathrm{S}}(X,Z') - g(T\phi X,TZ')\Big\} + g(\phi Y,Z')\Big\{\widehat{\mathrm{S}}(X,Z) \\
&-g(T\phi X,TZ)\} - g(\phi X,Z')\Big\{\widehat{\mathrm{S}}(Y,Z) - g(T\phi Y,TZ)\Big\} \\
&+ 2g(\phi X,Y)\Big\{\widehat{\mathrm{S}}(Z,Z') - g(T\phi Z,TZ')\Big\} + 2g(\phi Z,Z') \\
&\times \Big\{\widehat{\mathrm{S}}(X,Y) - g(T\phi X,TY)\Big\} + \eta(X)\eta(Z')\Big\{\widehat{\mathrm{Ric}}(Y,Z) \\
&-g(TY,TZ)\} - \eta(Y)\eta(Z')\Big\{\widehat{\mathrm{Ric}}(X,Z) - g(TX,TZ)\Big\} \\
&+ \eta(Y)\eta(Z)\Big\{\widehat{\mathrm{Ric}}(X,Z') - g(TX,TZ')\Big\} - \eta(X)\eta(Z) \\
&\times \Big\{\widehat{\mathrm{Ric}}(Y,Z') - g(TY,TZ')\Big\}\Big] - \frac{\tau}{4(m+1)(m+2)} \\
&\times \Big\{ g(X,Z)g(Y,Z') - g(Y,Z)g(X,Z') + g(\phi X,Z) \\
&\times g(\phi Y,Z') - g(\phi Y,Z)g(\phi X,Z') + 2g(\phi X,Y)g(\phi Z,Z') \\
&+ \eta(X)\eta(Z')g(Y,Z) - \eta(Y)\eta(Z')g(X,Z) \\
&+ \eta(Y)\eta(Z)g(X,Z') - \eta(X)\eta(Z)g(Y,Z')\Big\} = 0.
\end{aligned} \tag{3.28}
$$

Interchanging both X and Z with ϕY, we get from (3.27)

$$\begin{aligned}0 =&g(\phi Y,\phi Y)g((\bar{\delta}T)U,Y)-g(\phi Y,\phi Y)g((\bar{\delta}T)U,\phi^2 Y)\\&-2g(\phi Y,\phi Y)g((\bar{\delta}T)U,\phi^2 Y)\\=&4g(\phi Y,\phi Y)g((\bar{\delta}T)U,Y)-3\eta(Y)g(\phi Y,\phi Y)g((\bar{\delta}T)U,\xi)\\=&4g(\phi Y,\phi Y)g((\bar{\delta}T)U,Y)\end{aligned}$$

because of $g((\bar{\delta}T)U,\xi)=0$, which implies that

$$g((\bar{\delta}T)U,Y)=0. \tag{3.29}$$

Thus, equation (3.24) can be simplified as

$$g((\nabla_U T)_V W,X)-g((\nabla_V T)_U W,X)=0. \tag{3.30}$$

From (3.28), we have

$$\begin{aligned}0=&\sum_{i=1}^{2n+1} g(B^c(X_i,Y)Z,X_i)\\=&\widehat{\mathrm{Ric}}(Y,Z)-\frac{1}{2(m+2)}\Bigg[(2n+2)\left\{\widehat{\mathrm{Ric}}(Y,Z)\right.\\&\left.-g(TY,TZ)\right\}+(\hat{\tau}-|T|^2)\Big\{g(Y,Z)-\eta(Y)\eta(Z)\Big\}\Bigg]\\&+\frac{\tau}{2(m+1)(m+2)}(n+1)\Big\{g(Y,Z)-\eta(Y)\eta(Z)\Big\},\end{aligned} \tag{3.31}$$

that is,

$$\begin{aligned}&s\widehat{\mathrm{Ric}}(Y,Z)+(n+2)g(TY,TZ)-(\hat{\tau}-|T|^2)\\&\quad\times\Big\{g(Y,Z)-\eta(Y)\eta(Z)\Big\}+\frac{(n+1)\tau}{2(n+s+1)}\\&\quad\times\Big\{g(Y,Z)-\eta(Y)\eta(Z)\Big\}=0.\end{aligned} \tag{3.32}$$

Interchanging both Y and Z with X_i and taking a sum from 1 to $2n$, we get

$$\tau = \frac{n+s+1}{n(n+1)} \left\{ (n-s)\hat{\tau} - 2(n+1)|T|^2 \right\}, \tag{3.33}$$

which together with (3.32) implies

$$\begin{aligned} &s\widehat{\mathrm{Ric}}(Y,Z) + (n+2)g(TY,TZ) - (\hat{\tau} - |T|^2)\Big\{ g(Y,Z) \\ &\quad - \eta(Y)\eta(Z) \Big\} + \frac{1}{2n}\left\{ (n-s)\hat{\tau} - 2(n+1)|T|^2 \right\} \\ &\quad \times \Big\{ g(Y,Z) - \eta(Y)\eta(Z) \Big\} \\ &= s\widehat{\mathrm{Ric}}(Y,Z) + (n+2)g(TY,TZ) \\ &\quad - \frac{1}{2n}\left\{ s\hat{\tau} + (n+2)|T|^2 \right\} \Big\{ g(Y,Z) - \eta(Y)\eta(Z) \Big\} = 0. \end{aligned} \tag{3.34}$$

Thus, we have

$$\begin{aligned} g(TY,TZ) = &-\frac{s}{n+2}\widehat{\mathrm{Ric}}(Y,Z) \\ &+ \frac{s\hat{\tau} + (n+2)|T|^2}{2n(n+2)}\Big\{ g(Y,Z) - \eta(Y)\eta(Z) \Big\}. \end{aligned} \tag{3.35}$$

Substituting (3.35) to (3.28) yields

$$
\begin{aligned}
g(B^c(X,Y)Z,Z') &= g(\widehat{R}(X,Y)Z,Z')\\
&+\frac{1}{2(n+2)}\Big\{g(X,Z)\widehat{\mathrm{Ric}}(Y,Z')-g(Y,Z)\widehat{\mathrm{Ric}}(X,Z')\\
&+g(Y,Z')\widehat{\mathrm{Ric}}(X,Z)-g(X,Z')\widehat{\mathrm{Ric}}(Y,Z)\\
&+g(\phi X,Z)\widehat{S}(Y,Z')-g(\phi Y,Z)\widehat{S}(X,Z')\\
&+g(\phi Y,Z')\widehat{S}(X,Z)-g(\phi X,Z')\widehat{S}(Y,Z)\\
&+2g(\phi X,Y)\widehat{S}(Z,Z')+2g(\phi Z,Z')\widehat{S}(X,Y)\\
&+\eta(X)\eta(Z')\widehat{\mathrm{Ric}}(Y,Z)-\eta(Y)\eta(Z')\widehat{\mathrm{Ric}}(X,Z)\\
&+\eta(Y)\eta(Z)\widehat{\mathrm{Ric}}(X,Z')-\eta(X)\eta(Z)\widehat{\mathrm{Ric}}(Y,Z')\Big\}\\
&-\frac{1}{2(m+2)}\Bigg[\frac{s\hat{\tau}+(n+2)|T|^2}{n(n+2)}\Big\{g(X,Z)g(Y,Z')\\
&-g(Y,Z)g(X,Z')+g(\phi X,Z)g(\phi Y,Z')\\
&-g(\phi Y,Z)g(\phi X,Z')+2g(\phi X,Y)g(\phi Z,Z')\\
&+\eta(X)\eta(Z')g(Y,Z)-\eta(Y)\eta(Z')g(X,Z)\\
&+\eta(Y)\eta(Z)g(X,Z')-\eta(X)\eta(Z)g(Y,Z')\Big\}\\
&+\frac{\tau}{2(m+2)}\Big\{g(X,Z)g(Y,Z')-g(Y,Z)g(X,Z')\\
&+g(\phi X,Z)g(\phi Y,Z')-g(\phi Y,Z)g(\phi X,Z')\\
&+2g(\phi X,Y)g(\phi Z,Z')+\eta(X)\eta(Z')g(Y,Z)\\
&-\eta(Y)\eta(Z')g(X,Z)+\eta(Y)\eta(Z)g(X,Z')\\
&-\eta(X)\eta(Z)g(Y,Z')\Big\}\Bigg].
\end{aligned}
\tag{3.36}
$$

Here, calculating the second coefficient of this equation, we get from (3.33)

$$
\begin{aligned}
&\frac{1}{2(m+2)}\left\{\frac{s\hat{\tau}+(n+2)|T|^2}{n(n+2)}+\frac{\tau}{2(m+2)}\right\}\\
=&\frac{1}{2(m+2)}\left[\frac{s\hat{\tau}+(n+2)|T|^2}{n(n+2)}+\frac{\tau}{2n(n+1)}\right.\\
&\left.\times\left\{(n-s)\hat{\tau}-2(n+1)|T|^2\right\}\right]\\
=&\frac{1}{2(m+2)}\left[\frac{2s(n+1)+(n+2)(n-s)}{2n(n+1)(n+2)}\hat{\tau}+\frac{|T|^2}{n}-\frac{|T|^2}{n}\right\}\Bigg]\\
=&\frac{1}{2(m+2)}\left\{\frac{(m+2)\hat{\tau}}{2(n+1)(n+2)}\right\}\\
=&\frac{\hat{\tau}}{4(n+1)(n+2),}
\end{aligned}
\tag{3.37}
$$

which together with (3.36) leads to the following

THEOREM 3.2. *Let $\pi : M \to B$ be a cosymplectic submersion. If M is cosymplectic Bochner flat, then B is also cosymplectic Bochner flat.*

PROPOSITION 3.3. *If $\pi : M \to B$ is a cosymplectic submersion with vanishing cosymplectic Bochner curvature tensor, then we get $\frac{\hat{\tau}}{n(n+1)} + \frac{\bar{\tau}}{s(s+1)} \leq 0$, the equality holds if and only if the submersion has totally geodesic fibres.*

Proof. From (3.15) and (3.33) we get $\frac{\hat{\tau}}{n(n+1)} + \frac{\bar{\tau}}{s(s+1)} + \frac{n+2s+2}{ns(s+1)}|T|^2 = 0$, and so $\frac{\hat{\tau}}{n(n+1)} + \frac{\bar{\tau}}{s(s+1)} \leq 0$. It is clear from this that the equality holds if and only if $|T|^2 = 0$. □

PROPOSITION 3.4. *Let $\pi : M \to B$ be a cosymplectic submersion with vanishing cosymplectic Bochner curvature tensor. Each fibre is an Einstein manifold if and only if $g(T_V, T_W) = \frac{|T|^2}{2s} g(V,W)$ if $s > 1$.*

Proof. Suppose that each fibre is an Einstein manifold. Then we have $\overline{\mathrm{Ric}}(V,W) = cg(V,W)$. On the other hand, from (3.23) with $N = 0$, we get

$$
\begin{aligned}
&2n\overline{\mathrm{Ric}}(V,W) + 2(n+s+2)g(T_V,T_W) \\
&\qquad - \Big\{\bar{\tau} - \frac{(s+1)\tau}{(n+s+1)}\Big\}g(V,W) = 0.
\end{aligned} \tag{3.38}
$$

It follows from (3.32) and (3.36) that

$$
\begin{aligned}
(s+1)\tau = &-\frac{1}{s}\big\{(n^2+n-s^2-s)\bar{\tau} \\
&+ (n+s+1)(n+s+2)|T|^2\big\},
\end{aligned}
$$

Substituting this into (3.38) yields

$$
\begin{aligned}
&n\overline{\mathrm{Ric}}(V,W) + (n+s+2)g(T_V,T_W) \\
&\qquad - \frac{1}{2s}\big\{n\bar{\tau} + (n+s+2)|T|^2\big\}g(V,W) = 0,
\end{aligned} \tag{3.39}
$$

which implies that

$$
g(T_V,T_W) = \frac{|T|^2}{2s}\,g(V,W).
$$

Conversely, suppose that $g(T_V,T_W) = \frac{|T|^2}{2s}\,g(V,W)$. Then we get from (3.37)

$$
\overline{\mathrm{Ric}}(V,W) = \frac{\bar{\tau}}{2s}g(V,W),
$$

which completes the proof. □

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