East Asian Math. J. **22** (2006), No. 1, pp. 93–109

INTUITIONISTIC (S,T)-FUZZY h-IDEALS OF HEMIRINGS

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ABSTRACT. The concept of intuitionistic fuzzy set was first introduced by Atanassov in 1986. In this paper, we define the intuitionistic (S,T)-fuzzy left h-ideals of a hemiring by using an s-norm S and a t-norm T and study their properties. In particular, some results of fuzzy left h-ideals in hemirings recently obtained by Jun, Öztürk, Song, and others are extended and generalized to intuitionistic (S,T)-fuzzy ideals over hemirings.

1. Introduction.

Semirings and operators that preserve semiring matrix functions were studied by Beasley and Pullman in [5] and [6], in particular, they studied the matrices and their determinants over semirings. Although the ideals of semirings play a central role in the structure theory, the ideals of semirings do not in general coincide with the usual ring ideals and hence the usage of ideals in semirings is limited. In order to overcome the difficulty, Henriksen [11] defined a more restricted class of ideals in semirings, which is called the class of k-ideals, with the property that if the semiring X is a ring then a complex in X is a k-ideal if and only if it is a ring ideal. Another more restricted class of ideals in hemirings was given by Iizuka [12]. However, in an additively commutative semiring X, Iizuka [12] also noticed that an ideal of such semirings coincides with the ideal in a ring if the semiring is a hemiring. We call

Received March 17, 2006.

2000 Mathematics Subject Classification: 16Y60, 13E05, 03G25.

Key words and phrases: Hemiring; (imaginable) Intuitionistic fuzzy left h-ideal; Left h-ideal; intuitionistic (S,T)-fuzzy relation.

The research of the first author is supported by Natural Sciences Foundation of the Education Committee of Hubei Province(2004Z002,D200529001) and the research of the second author is partially supported by a UGC fund (HK) #2160210 (03/05) .

this ideal the h-ideal of the hemiring X. The properties of h-ideals and also the k-ideals in hemirings were thoroughly investigated by La Torre [17] in 1965 and by using the h-ideals and k-ideals, he established some analogues ring theorems for hemirings. Recently, Y.B.Jun and others [15] considered the fuzzification of h-ideals in hemirings and gave some properties. By considering the left h-ideals of a hemiring X, the fuzzy left h-ideals of X were investigated. Thus, some results of finite valued fuzzy left h-ideals and theirs characterization are given. In this paper, we introduce the intuitionistic (S,T)-fuzzy left h-ideals defined on intuitionistic fuzzy sets. By using these new fuzzy ideals and the new operations defined on the intuitionistic fuzzy sets , we are able to extend some results of fuzzy h-ideals in hemirings to intuitionistic (S,T)-fuzzy h-ideals of hemirings. We also give some properties of the intuitionistic (S,T)-fuzzy relations of a hemiring. In particular, we characterize the imaginable intuitionistic (S,T)-fuzzy left h-ideals of hemirings. For notations and terminology not given in this paper, the reader is referred to [2], [18],[19] and [20].

2. Preliminaries.

A *semiring* $(X,+,\cdot)$ is an algebraic system consisting of a non-empty set X together with two binary operations "+" and " $\cdot$ " on X such that $(X,+)$ is a semigroup reduct and $(X,\cdot)$ is also a semigroup reduct such that these two semigroups reducts are linked by distributive laws, that is , $a(b+c)=ab+ac$ and $(a+b)c=ac+bc$, for all $a,b,c \in X$. A semiring X is said to be *additively commutative* if $a+b=b+a$, for all $a,b \in X$. In other words, the additive reduct $(X,+)$ of the semiring $(X,+,\cdot)$ is a commutative semigroup. A zero element of a semiring $(X,+,\cdot)$ is an element $0 \in X$ such that $0 \cdot x = x \cdot 0 = 0$ and $0+x=x+0=x$ for all $x \in X$. By a *hemiring*, we mean an additively commutative semiring with zero. Ideals and fuzzy ideals of semirings have been recently obtained by many authors (see [1], [4],[8],[10],[12],[13-16],[19] and [20]).

We call a fuzzy set μ in a semiring X *fuzzy left ideal* if the following conditions are satisfied: (F1) $\mu(x+y) \geq \min\ \{\mu(x),\mu(y)\}$, for all $x,y \in X$; (F2) $\mu(xy) \geq \mu(y)$, for all $x,y \in X$. The concept of fuzzy right ideal of X can be defined dually. A fuzzy set μ in a semiring X is said to be an *anti fuzzy left ideal* of X if the following conditions are satisfied: (AF1) $\mu(x+y) \leq \max\{\mu(x), \mu(y)\}$, for all $x, y \in X$; (AF2) $\mu(xy) \leq \mu(y)$, for all $x, y \in X$. If μ is a fuzzy left ideal of a hemiring X, then it is trival to see that $\mu(0) \geq \mu(x)$, for all $x \in X$. A left ideal A is called a *left h-ideal* of a hemiring X if $x + a + z = b + z$ implies that $x \in A$, for any $x, z \in X$ and $a, b \in A$. Similarly, we can define the *right h-ideals* of a hemiring X. A fuzzy left ideal μ of a hemiring X is named *fuzzy* if $x+a+z = b+z$ implies that $\mu(x) \geq \min\{\mu(a), \mu(b)\}$, for all $a, b, x, z \in X$. *Fuzzy right h-ideals* can be similarly defined. The dual concept of fuzzy left h-ideal of X can be defined as follows: An anti-fuzzy left h-ideal μ of a hemiring X is an *anti fuzzy left ideal* of X if $x + a + z = b + z$ implies that $\mu(x) \leq \max\{\mu(a), \mu(b)\}$, for all $a, b, x, z \in X$. *Anti-fuzzy right h-ideals* can be similarly defined .

Recall that the complement of a fuzzy set μ is defined by $\overline{\mu}(x) = 1 - \mu(x)$. For $\alpha \in [0,1]$, the set $U(\mu; \alpha) = \{x \in X \mid \mu(x) \geq \alpha\}$ is called an upper α-level cut of μ and the set $L(\mu; \alpha) = \{x \in X \mid \mu(x) \leq \alpha\}$ is called a lower α-level cut of μ [18].

As it is well-known, any function $\delta : [0,1] \times [0,1] \to [0,1]$ such that $\delta(x,y) = \delta(y,x)$, $\delta(x,x) = x$, $\delta(\delta(x,y),z) = \delta(x, \delta(y,z))$ and $\delta(x,u) \leq \delta(x,w)$ for all $x, y, z, u, w \in [0,1]$, where $u \leq w$ is called an *imaginable t-norm* if $\delta(x,1) = x$, and an *imaginable s-norm* if $\delta(1,1) = 1$ and $\delta(x,0) = x$ for all $x \in [0,1]$.

Throughout this paper, we will denote the t-norm and s-norm by T and S, respectively.

As an important generalization of fuzzy sets in X, Atanassov (see [2,3]) introduced the concept of intuitionistic fuzzy sets defined on a non-empty set X which are the objects of the following form

$$A = \{(x, \alpha_A(x), \beta_A(x)) \mid x \in X\},$$

where the functions $\alpha_A : X \to [0,1]$ and $\beta_A : X \to [0,1]$ denote the degree of membership and the degree of non-membership, respectively, and also the following inequality $0 \leq \alpha_A(x) + \beta_A(x) \leq 1$, for all $x \in X$ holds.

The intuitionistic fuzzy set $A = (\alpha_A, \beta_A)$ in a hemiring X satisfying the following conditions is called an *intuitionistic fuzzy left h-ideal* of X :

(IF1) $\alpha_A(x+y) \geq \min\{\alpha_A(x), \alpha_A(y)\}$
and $\beta_A(x+y) \leq \max\{\beta_A(x), \beta_A(y)\}$, for all $x, y \in X$;

(IF2) $\alpha_A(xy) \geq \alpha_A(y)$ and $\beta_A(xy) \leq \beta_A(y)$, for all $x, y \in X$;
(IF3) For all $a, b, x, z \in X$, $x + a + z = b + z$ implies
$\alpha_A(x) \geq \min\{\alpha_A(a), \alpha_A(b)\}$ and $\beta_A(x) \leq \max\{\beta_A(a), \beta_A(b)\}$.

3. Intuitionistic (S, T)-fuzzy left h-ideals.

Definition 3.1. (i) An intuitionistic fuzzy set $A = (\alpha_A, \beta_A)$ in a hemiring X is called an *intuitionistic fuzzy left h-ideal* of X with respect to the t-norm T and the s-norm S (briefly, intuitionistic (S, T)-fuzzy left h-ideal of X) if it satisfies the following three conditions:
(ISTF1) $\alpha_A(x + y) \geq T(\alpha_A(x), \alpha_A(y))$
and $\beta_A(x + y) \leq S(\beta_A(x), \beta_A(y))$, for all $x, y \in X$;
(ISTF2) $\alpha_A(xy) \geq \alpha_A(y)$
and $\beta_A(xy) \leq \beta_A(y)$, for all $x, y \in X$;
(ISTF3) For all $a, b, x, z \in X$, $x + a + z = b + z$ implies
$\alpha_A(x) \geq T(\alpha_A(a), \alpha_A(b))$ and $\beta_A(x) \leq S(\beta_A(a), \beta_A(b))$.

(ii) An intuitionistic (S, T)-fuzzy left h-ideal $A = (\alpha_A, \beta_A)$ of a hemiring X is said to be *imaginable* if α_A and β_A satisfy the imaginable property.

Example 3.2. Consider a hemiring $S = \{0, 1, 2, 3\}$ with the following Cayley tables:

+	0	1	2	3
0	0	1	2	3
1	1	1	2	3
2	2	2	2	3
3	3	3	3	2

.	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	1	1	1
3	0	1	1	1

Define an intuitionistic fuzzy set $A = (\alpha_A, \beta_A)$ as follows:

$\alpha_A : X \to [0, 1]$ by letting $\alpha_A(0) = 0.6$ and $\alpha_A(x) = 0.3$ for all $x \neq 0$;

$\beta_A : X \to [0, 1]$ by letting $\beta_A(0) = 0.3$ and $\beta_A(x) = 0.5$ for all $x \neq 0$.

Let T be a t-norm which is defined by $T(\alpha, \beta)=\max\{\alpha + \beta - 1, 0\}$ and S an s-norm which is defined by $S(\alpha, \beta)=\min\{\alpha + \beta, 1\}$, for all $\alpha, \beta \in [0, 1]$. Then by routine computation, we see that $A = (\alpha_A, \beta_A)$ is an intuitionistic (S, T)-fuzzy left h-ideal of X.

Lemma 3.3. Every imaginable intuitionistic (S,T)-fuzzy left h-ideal of the hemiring X is an intuitionistic fuzzy left h-ideal.

Proof. Let $A=(\alpha_A,\beta_A)$ be an imaginable intuitionistic (S,T)-fuzzy left h-ideal of X. Then we can check the following conditions:

(IF1) $\alpha_A(x+y)\geq T(\alpha_A(x),\alpha_A(y))$ and $\beta_A(x+y)\leq S(\beta_A(x),\beta_A(y))$, for all $x,y\in X$. Since $A=(\alpha_A,\beta_A)$ is imaginable, we have

$\min\{\alpha_A(x),\alpha_A(y)\}=T(\min\{\alpha_A(x),\alpha_A(y)\},\min\{\alpha_A(x),\alpha_A(y)\})$
$\leq T(\alpha_A(x),\alpha_A(y))\leq\min\{\alpha_A(x),\alpha_A(y)\}$,

that is, $T(\alpha_A(x),\alpha_A(y))=\min\{\alpha_A(x),\alpha_A(y)\}$.

Similarly, we can show that

$\max\{\beta_A(x),\beta_A(y)\}=S(\max\{\beta_A(x),\beta_A(y)\},\max\{\beta_A(x),\beta_A(y)\})$
$\geq S(\beta_A(x),\beta_A(y))\geq\max\{\beta_A(x),\beta_A(y)\}$,

that is, $S(\beta_A(x),\beta_A(y))=\max\{\beta_A(x),\beta_A(y)\}$,

Hence, it follows that

$\alpha_A(x+y)\geq T(\alpha_A(x),\alpha_A(y))=\min\{\alpha_A(x),\alpha_A(y)\}$
and $\beta_A(x+y)\leq S(\beta_A(x),\beta_A(y))=\max\{\beta_A(x),\beta_A(y)\}$.

(IF3) Let $x+a+z=b+z$, $x,z,a,b\in X$. Then, we have $\alpha_A(x)\geq T(\alpha_A(a),\alpha_A(b))$ and $\beta_A(x)\leq S(\beta_A(a),\beta_A(b))$.

Since $A=(\alpha_A,\beta_A)$ is imaginable, we can easily see that

$T(\alpha_A(a),\beta_A(b))=\min\{\alpha_A(a),\alpha_A(b)\}$
and $S(\beta_A(a),\beta_A(b))=\max\{\beta_A(a),\beta_A(b)\}$.

Consequently, $\alpha_A(x)\geq T(\alpha_A(a),\alpha_A(b))=\min\{\alpha_A(a),\alpha_A(b)\}$
and $\beta_A(x)\geq S(\beta_A(a),\beta_A(b))=\max\{\beta_A(a),\beta_A(b)\}$.

Therefore, $A=(\alpha_A,\beta_A)$ is an intuitionistic fuzzy left h-ideal of X. □

Lemma 3.4.([15]). A fuzzy set μ in a hemiring X is a fuzzy left h-ideal of X if and only if the subset $U(\mu;\alpha)$, $\alpha\in[0,1]$ is a left h-ideal of X whenever $U(\mu;\alpha)\neq\emptyset$.

Lemma 3.5. A fuzzy set μ in X is an anti-fuzzy left h-ideal of X if and only if the subset $L(\mu;\alpha)$, $\alpha\in[0,1]$, of μ is a left h-ideal of X whenever $L(\mu;\alpha)\neq\emptyset$.

Proof. The proof is similar to the proof of Lemma 3.4. □

We now characterize the intuitionistic (S,T)-fuzzy left h-ideals of a hemiring X.

Theorem 3.6. If $A = (\alpha_A, \beta_A)$ is an imaginable intuitionistic fuzzy set in a hemiring X. Then $A = (\alpha_A, \beta_A)$ is an imaginable intuitionistic (S,T)-fuzzy left h-ideal of X if and only if the sets $U(\alpha_A;\alpha)$ and $L(\beta_A;\alpha)$ are left h-ideals of X, for every $\alpha \in [0,1]$, whenever $U(\alpha_A;\alpha) \neq \emptyset \neq L(\beta_A;\alpha)$.

Proof. let $A = (\alpha_A, \beta_A)$ be an intuitionistic (S,T)-fuzzy left h-ideal of a hemirng X. Then, by Lemma 3.3, $A = (\alpha_A, \beta_A)$ is an intuitionistic fuzzy left h-ideals of X. Hence, by Lemma 3.4 and 3.5, $U(\alpha_A;\alpha)$ and $L(\beta_A;\alpha)$ are left h-ideals of X.

Conversely, we know that $A = (\alpha_A, \beta_A)$ is an intuitionistic fuzzy left h-ideal of X by Lemma 3.4 and 3.5. Now, we verify the following conditions.

(ISTF1) $\alpha_A(x+y) \geq \min\{\alpha_A(x), \alpha_A(y)\} \geq T(\alpha_A(x), \alpha_A(y))$

and $\beta_A(x+y) \leq \max\{\beta_A(x), \beta_A(y)\} \leq S(\beta_A(x), \beta_A(y))$, for all $x, y \in X$;

(ISTF3) Let $x, z, a, b \in X$ such that $x+a+z = b+z$. Then $\alpha_A(x) \geq \min\{\alpha_A(a), \alpha_A(b)\} \geq T(\alpha_A(a), \alpha_A(b))$ and $\beta_A(x) \leq \max\{\beta_A(a), \beta_A(b)\}$ $\leq S(\beta_A(a), \beta_A(b))$. This shows that $A = (\alpha_A, \beta_A)$ is an imaginable intuitionistic (S,T)-fuzzy left h-ideal of X. □

Theorem 3.7. Let $A = (\alpha_A, \beta_A)$ be an intuitionistic (S,T)-fuzzy left h-ideal of the hemiring X. Then

$\alpha_A(x) = \sup\{\alpha \in [0,1] | x \in U(\alpha_A;\alpha)\}$

and $\beta_A(x) = \inf\{\alpha \in [0,1] | x \in L(\beta_A;\alpha)\}$, for all $x \in X$.

Proof. Let $\delta = sup\{\alpha \in [0,1] | x \in U(\alpha_A;\alpha)\}$ and $\epsilon > 0$. Then, we can easily see that $\delta - \epsilon < \alpha$, for some $\alpha \in [0,1]$ such that $x \in U(\alpha_A;\alpha)$. This leads to $\delta - \epsilon < \alpha_A(x)$ and hence $\delta < \alpha_A(x)$ because ϵ is arbitrary.

We now show that $\alpha_A(x) < \delta$. If $\alpha_A(x) = \beta$, then $x \in U(\alpha_A;\beta)$ and so $\beta \in \{\alpha \in [0,1] | x \in U(\alpha_A;\alpha)\}$. Thus, we have $\alpha_A(x) = \delta \leq sup\{\alpha \in [0,1] | x \in U(\alpha_A;\alpha)\} = \delta$. Consequently, we can deduce that $\alpha_A(x) = \delta = sup\{\alpha \in [0,1] | x \in U(\alpha_A;\alpha)\}$.

Now, let $\eta = inf\{\alpha \in [0,1] | x \in L(\beta_A;\alpha)\}$. Then, we have $inf\{\alpha \in [0,1] | x \in L(\beta_A;\alpha)\} < \eta + \epsilon$, for any $\epsilon > 0$, and so $\alpha < \eta + \epsilon$, for some $\alpha \in [0,1]$ with $x \in L(\beta_A;\alpha)$. Since $\beta_A(x) \leq \alpha$ and ϵ is arbitrary, we obtain that $\beta_A(x) \leq \eta$.

In order to prove that $\beta_A(x) \geq \eta$, we let $\beta_A(x) = \xi$. Then we have $x \in L(\beta_A;\xi)$ and so we obtain that $\xi \in \{\alpha \in [0,1] | x \in L(\beta_A;\alpha)\}$.

Hence $inf\{\alpha \in [0,1] | x \in L(\beta_A;\alpha)\} \leq \xi$, i.e., $\eta \leq \xi = \beta_A(x)$. Consequently, we see that $\beta_A(x) = \eta = \inf\{\alpha \in [0,1] | x \in L(\beta_A;\alpha)\}$. This completes the proof. □

Definition 3.8. Let f be a mapping which maps X to X'. If $A = (\alpha_A, \beta_A)$ is an intuitionistic fuzzy set in X', then the *inverse image* of A under f, denoted by $f^{-1}(A)$, is an intuitionistic fuzzy set in X, defined by $f^{-1}(A) = (f^{-1}(\alpha_A), f^{-1}(\beta_A))$.

Theorem 3.9. Let $f : X \to X'$ be a homomorphism of hemirings. If $A = (\alpha_A, \beta_A)$ is an intuitionistic (S,T)-fuzzy left left h-ideal of X'. Then the inverse image $f^{-1}(A) = (f^{-1}(\alpha_A), f^{-1}(\beta_A))$ of A under f is an intuitionistic (S,T)-fuzzy left h-ideal of X.

Proof. Let $A = (\alpha_A, \beta_A)$ be an intuitionistic (S,T)-fuzzy left h-ideal of X' and $x, y \in X$. We need only check the following conditions hold:

(ISTF1) $f^{-1}(\alpha_A)(x+y) = \alpha_A(f(x+y)) = \alpha_A(f(x)+f(y))$
$\geq T(\alpha_A(f(x)), \alpha_A(f(y))) = T(f^{-1}(\alpha_A)(x), f^{-1}(\alpha_A)(y))$
and $f^{-1}(\beta_A)(x+y) = \beta_A(f(x+y)) = \beta_A(f(x)+f(y))$
$\leq S(\beta_A)(f(x)), \beta_A(f(y))) = S(f^{-1}(\beta_A)(x), f^{-1}(\beta_A)(y))$;
(ISTF2) $f^{-1}(\alpha_A)(xy) = \alpha_A(f(xy)) = \alpha_A(f(x)f(y))$
$\geq \alpha_A(f(y)) = f^{-1}(\alpha_A)(y)$
and $f^{-1}(\beta_A)(xy) = \beta_A(f(xy)) = \beta_A(f(x)f(y))$
$\leq \beta_A(f(y)) = f^{-1}(\beta_A)(y)$;
(ISTF3) Let $x+a+z = b+z$ for $x, z, a, b \in X$. Then $f(x)+f(a)+f(z) = f(b)+f(z)$. Hence, it follows that
$f^{-1}(\alpha_A)(x) = \alpha_A(f(x))$
$\geq T(\alpha_A(f(a)), \alpha_A(f(b))) = T(f^{-1}(\alpha_A)(a), f^{-1}(\alpha_A)(b))$
and $f^{-1}(\beta_A(x)) = \beta_A(f(x))$
$\leq S(\beta_A(f(a)), \beta_A(f(b))) = S(f^{-1}(\beta_A(a)), f^{-1}(\beta_A(b)))$.

Thus $f^{-1}(A) = (f^{-1}(\alpha_A), f^{-1}(\beta_A)$ is an intuitionistic (S,T)-fuzzy left h-ideal of X. □

4. Intuitionistic (S,T)-fuzzy relations.

In this section, we study the intuitionistic(S,T)-fuzzy relations and extend some results of Jun. Ozturk and Song [15] from fuzzy h-ideals in hemirings to intuitionistic (S,T)-fuzzy h-ideals in hemirings.

Definition 4.1. An intuitionistic fuzzy set $A = (\alpha_A, \beta_A)$ is called an *intuitionistic fuzzy relation* on any set X if α_A and β_A are fuzzy relations on X.

Definition 4.2. Let $A = (\alpha_A, \beta_A)$ and $B = (\alpha_B, \beta_B)$ be intuitionstic fuzzy sets on a set X. If $A = (\alpha_A, \beta_A)$ is an intuitionistic fuzzy relation on a set X , then $A = (\alpha_A, \beta_A)$ is called an *intuitionistic (S,T)-fuzzy relation* on $B = (\alpha_B, \beta_B)$ if $\alpha_A(x, y) \leq T(\alpha_B(x), \alpha_B(y))$ and $\beta_A(x, y) \geq S(\beta_B(x), \beta_B(y))$ for all $x, y \in X$.

Definition 4.3. The *intuitionistic Cartesian product with (S,T)-norm* of A and B, denoted by $A \times B$, is an intuitionistic fuzzy set in X, which is defined by

$A \times B = (\alpha_A, \beta_A) \times (\alpha_B, \beta_B) = (\alpha_A \times \alpha_B, \beta_A \times \beta_B)$,

where

$(\alpha_A \times \alpha_B)(x, y) = T(\alpha_A(x), \alpha_B(y))$ and $(\beta_A \times \beta_B)(x, y) = S(\beta_A(x), \beta_B(y))$ hold for all $x, y \in X$.

Lemma 4.4. If $A = (\alpha_A, \beta_A)$ and $B = (\alpha_B, \beta_B)$ are intuitionistic fuzzy sets in a set X. Then we have

(i) $A \times B$ is an intuitionistic (S,T)-fuzzy relation on X;

(ii) $U(\alpha_A \times \alpha_B; \alpha) = U(\alpha_A; \alpha) \times U(\alpha_B; \alpha)$; and $L(\beta_A \times \beta_B; \alpha) = L(\beta_A; \alpha) \times L(\beta_B; \alpha)$ for all $\alpha \in [0, 1]$.

Definition 4.5. If $A = (\alpha_A, \beta_A)$ and $B = (\alpha_B, \beta_B)$ are intuitionistic fuzzy sets in a set X, then the strongest intuitionistic (S,T)-fuzzy relation on X that is an intuitionistic (S,T)-fuzzy relation on B is A_B, defined by

$A_B = (\alpha_{A_{\alpha_B}}, \beta_{A_{\beta_B}})$,

where $\alpha_{A_{\alpha_B}}(x, y) = T(\alpha_B(x), \alpha_B(y))$ and $\beta_{A_{\beta_B}}(x, y) = S(\beta_B(x), \beta_B(y))$, for all $x, y \in X$.

Lemma 4.6. For the intuitionistic fuzzy sets $A = (\alpha_A, \beta_A)$ and $B = (\alpha_B, \beta_B)$ in a set X, let A_B be the strongest intuitionistic (S,T)-fuzzy relation X. Then for any $\alpha \in [0, 1]$, we have

$U(\alpha_{A_{\alpha_B}}; \alpha) = U(\alpha_B; \alpha) \times U(\alpha_B; \alpha)$; $L(\beta_{A_{\beta_B}}; \alpha) = L(\beta_B; \alpha) \times L(\beta_B; \alpha)$.

Lemma 4.7. For the given intuitionistic fuzzy sets $A = (\alpha_A, \beta_A)$ and $B = (\alpha_B, \beta_B)$ in a hemiring X, let A_B be the strongest intuitionistic (S,T)-fuzzy relation on X. If A_B is an imaginable intuitionistic (S,T)-fuzzy left h-ideal of $X \times X$, then we have $\alpha_A(a) \leq \alpha_A(0)$ and $\beta_A(a) \geq \beta_A(0)$, for all $a \in X$.

Lemma 4.8. For all $\alpha, \beta, \delta, \gamma \in [0,1]$, we have

$$T(T(\alpha,\beta),T(\gamma,\delta)) = T(T(\alpha,\gamma),T(\beta,\delta));$$
$$S(S(\alpha,\beta),S(\gamma,\delta)) = S(S(\alpha,\gamma),S(\beta,\delta)).$$

By using the above lemmas we have the following theorem.

Theorem 4.9. If $A = (\alpha_A, \beta_A)$ and $B = (\alpha_B, \beta_B)$ are intuitionistic (S,T)-fuzzy left h-ideals of a hemiring X, then $A \times B$ is an intuitionistic (S,T)-fuzzy left h-ideal of $X \times X$.

Proof. Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be elements of $X \times X$. Then we can verify the following conditions:

(ISTF1) $(\alpha_A \times \alpha_B)(x+y) = (\alpha_A \times \alpha_B)((x_1,x_2)+(y_1,y_2))$
$= (\alpha_A \times \alpha_B)(x_1+y_1, x_2+y_2)$
$= T(\alpha_A(x_1+y_1), \alpha_B(x_2+y_2))$
$\geq T(T(\alpha_A(x_1),\alpha_A(y_1)), T(\alpha_B(x_2),\alpha_B(y_2)))$
$= T(T(\alpha_A(x_1),\alpha_B(x_2)), T(\alpha_A(y_1),\alpha_B(y_2)))$
$= T((\alpha_A \times \alpha_B)(x_1,x_2), (\alpha_A \times \alpha_B)(y_1,y_2))$
$= T((\alpha_A \times \alpha_B)(x), (\alpha_A \times \alpha_B)(y))$
and $(\beta_A \times \beta_B)(x+y) = (\beta_A \times \beta_B)((x_1,x_2)+(y_1,y_2))$
$= (\beta_A \times \beta_B)(x_1+y_1, x_2+y_2)$
$= S(\beta_A(x_1+y_1), \beta_B(x_2+y_2))$
$\leq S(S(\beta_A(x_1),\beta_A(y_1)), S(\beta_B(x_2),\beta_B(y_2)))$
$= S(S(\beta_A(x_1),\beta_B(x_2)), S(\beta_A(y_1),\beta_B(y_2)))$
$= S((\beta_A \times \beta_B)(x_1,x_2), (\beta_A \times \beta_B)(y_1,y_2))$
$= S((\beta_A \times \beta_B)(x), (\beta_A \times \beta_B)(y))$;
(IFST2) $(\alpha_A \times \alpha_B)(xy) = (\alpha_A \times \alpha_B)((x_1,x_2)(y_1,y_2))$
$= (\alpha_A \times \alpha_B)(x_1y_1, x_2y_2)$
$= T(\alpha_A(x_1y_1), \alpha_B(x_2y_2))$
$\geq T(\alpha_A(y_1), \alpha_B(y_2))$
$= (\alpha_A \times \alpha_B)(y_1,y_2)$
$= (\alpha_A \times \alpha_B)(y)$

and $(\beta_A \times \beta_B)(xy) = (\beta_A \times \beta_B)((x_1, x_2)(y_1, y_2))$
$= (\beta_A \times \beta_B)(x_1y_1, x_2y_2)$
$= S(\beta_A(x_1y_1), \beta_B(x_2y_2))$
$\leq S(\beta_A(y_1), \beta_B(y_2))$
$= (\beta_A \times \beta_B)(y_1, y_2)$
$= (\beta_A \times \beta_B)(y)$

(ISTF3) Let $x = (x_1, x_2), z = (z_1, z_2), a = (a_1, a_2)$ and $b = (b_1, b_2)$. Suppose that $x + a + z = b + z$. Then

$(x_1, x_2) + (a_1, a_2) + (z_1, z_2) = (b_1, b_2) + (z_1, z_2)$, and so $x_1 + a_1 + z_1 = b_1 + z_1$ and $x_2 + a_2 + z_2 = b_2 + z_2$. It follows that

$(\alpha_A \times \alpha_B)(x) = (\alpha_A \times \alpha_B)(x_1, x_2) = T(\alpha_A(x_1), \alpha_B(x_2))$
$\geq T(T(\alpha_A(a_1), \alpha_A(b_1)), T(\alpha_B(a_2), \alpha_B(b_2)))$
$= T(T(\alpha_A(a_1), \alpha_B(a_2)), T(\alpha_A(b_1), \alpha_B(b_2)))$
$= T((\alpha_A \times \alpha_B)(a_1, a_2), (\alpha_A \times \alpha_B)(b_1, b_2))$
$= T((\alpha_A \times \alpha_B)(a), (\alpha_A \times \alpha_B)(b))$
and $(\beta_A \times \beta_B)(x) = (\beta_A \times \beta_B)(x_1, x_2) = T(\beta_A(x_1), \beta_B(x_2))$
$\leq S(S(\beta_A(a_1), \beta_A(b_1)), S(\beta_B(a_2), \beta_B(b_2)))$
$= S(S(\beta_A(a_1), \beta_B(a_2)), S(\beta_A(b_1), \beta_B(b_2)))$
$= S((\beta_A \times \beta_B)(a_1, a_2), (\beta_A \times \beta_B)(b_1, b_2))$
$= S((\beta_A \times \beta_B)(a), (\beta_A \times \beta_B)(b))$.

This shows that $A \times A$ is an intuitionistic (S, T)-fuzzy left h-ideal of $X \times X$. $\square$

Corollary 4.10. If $A = (\alpha_A, \beta_A)$ and $B = (\alpha_B, \beta_B)$ are imaginable intuitionistic (S, T)-fuzzy left h-ideal of X. Then $A \times B$ is an imaginable intuitionistic (S, T)-fuzzy left h-ideal of $X \times X$.

Proof. By Theorem 4.9, $A \times B$ is an intuitionistic (S, T)-fuzzy left h-ideal of $X \times X$. Now, let $x = (x_1, x_2) \in X \times X$. Then we can easily verify that

$T((\alpha_A \times \alpha_B)(x), (\alpha_A \times \alpha_B)(x))$
$= T((\alpha_A \times \alpha_B)(x_1, x_2), (\alpha_A \times \alpha_B)(x_1, x_2))$
$= T(T(\alpha_A(x_1), \alpha_B(x_2)), T(\alpha_A(x_1), \alpha_B(x_2)))$
$= T(T(\alpha_A(x_1), \alpha_A(x_1)), T(\alpha_B(x_2), \alpha_B(x_2)))$
$= T(\alpha_A(x_1), \alpha_B(x_2))$
$= (\alpha_A \times \alpha_B)(x_1, x_2)$
$= (\alpha_A \times \alpha_B)(x)$;
and $S((\beta_A \times \beta_B)(x), (\beta_A \times \beta_B)(x))$

$= S((\beta_A \times \beta_B)(x_1,x_2),(\beta_A \times \beta_B)(x_1,x_2))$
$= S(S(\beta_A(x_1),\beta_B(x_2)),S(\beta_A(x_1),\beta_B(x_2)))$
$= S(S(\beta_A(x_1),\beta_A(x_1)),S(\beta_B(x_2),\beta_B(x_2)))$
$= S(\beta_A(x_1),\beta_B(x_2))$
$= (\beta_A \times \beta_B)(x_1,x_2)$
$= (\beta_A \times \beta_B)(x)$.

Hence, $A\times B$ is indeed an imaginable intuitionistic (S,T)-fuzzy left h-ideal of $X \times X$. ☐

The following therorem characterizes the imaginable intuitionistic (S,T)-fuzzy left h-ideals in a hemiring X.

Theorem 4.11. Let $A = (\alpha_A, \beta_A)$ and $B = (\alpha_B, \beta_B)$ be imaginable intuitionistic fuzzy left h-ideal of a hemiring X and A_B the strongest intuitionistic (S,T)-fuzzy relation on X. Then $B = (\alpha_B, \beta_B)$ is an imaginable intuitionistic (S,T)-fuzzy left h-ideal of X if and only if A_B is an imaginable intuitionistic (S,T)-fuzzy left h-ideal of $X \times X$.

Proof. Let $B = (\alpha_B, \beta_B)$ be an imaginable intuitionistic (S,T)-fuzzy left h-ideal of the hemiring X. Write $x = (x_1,x_2), y = (y_1,y_2) \in X \times X$. Then by using Lemma 4.7-4.8, we can verify the following conditions:

(ISTF1) $\alpha_{A_{\alpha_B}}(x+y)$
$= \alpha_{A_{\alpha_B}}((x_1,x_2)+(y_1,y_2)) = \alpha_{A_{\alpha_B}}(x_1+y_1,x_2+y_2)$
$=T(\alpha_B(x_1+y_1),\alpha_B(x_2+y_2))$
$\geq T(T(\alpha_B(x_1),\alpha_B(y_1)),T(\alpha_B(x_2),\alpha_B(y_2)))$
$= T(T(\alpha_B(x_1),\alpha_B(x_2)),T(\alpha_B(y_1),\alpha_B(y_2)))$
$= T(\alpha_{A_{\alpha_B}}(x_1,x_2),\alpha_{A_{\alpha_B}}(y_1,y_2))$
$= T(\alpha_{A_{\alpha_B}}(x),\alpha_{A_{\alpha_B}}(y))$;
and $\beta_{A_{\beta_B}}(x+y)$
$= \beta_{A_{\beta_B}}((x_1,x_2)+(y_1,y_2)) = \beta_{A_{\beta_B}}(x_1+y_1,x_2+y_2)$
$=S(\beta_B(x_1+y_1),\beta_B(x_2+y_2))$
$\leq S(S(\beta_B(x_1),\beta_B(y_1)),S(\beta_B(x_2),\beta_B(y_2)))$
$= S(S(\beta_B(x_1),\beta_B(x_2)),S(\beta_B(y_1),\beta_B(y_2)))$
$= S(\beta_{A_{\beta_B}}(x_1,x_2),\beta_{A_{\beta_B}}(y_1,y_2))$
$= S(\beta_{A_{\beta_B}}(x),\beta_{A_{\beta_B}}(y))$;
(ISTF2) $\alpha_{A_{\alpha_B}}(xy)$
$= \alpha_{A_{\alpha_B}}((x_1,x_2)(y_1,y_2)) = \alpha_{A_{\alpha_B}}(x_1y_1,x_2y_2)$

$T(\alpha_B(x_1y_1), \alpha_B(x_2y_2))$
$\geq T(\alpha_B(y_1), \alpha_B(y_2))$
$= \alpha_{A_{\alpha_B}}(y_1, y_2)$
$= \alpha_{A_{\alpha_B}}(y)$
and $\beta_{A_{\beta_B}}(xy)$
$= \beta_{A_{\beta_B}}((x_1, x_2)(y_1, y_2)) = \beta_{A_{\beta_B}}(x_1y_1, x_2y_2)$
$S(\beta_B(x_1y_1), \beta_B(x_2y_2))$
$\leq S(\beta_B(y_1), \beta_B(y_2))$
$= \beta_{A_{\beta_B}}(y_1, y_2)$
$= \beta_{A_{\beta_B}}(y)$;

(ISTF3) Let $a = (a_1, a_2), b = (b_1, b_2), x = (x_1, x_2)$ and $z = (z_1, z_2) \in X \times X$. If $x + a + z = b + z$,
then $x_1 + a_1 + z_1 = b_1 + z_1$ and $x_2 + a_2 + z_2 = b_2 + z_2$.
Hence, $\alpha_{A_{\alpha_B}}(x)$
$= \alpha_{A_{\alpha_B}}(x_1, x_2) = T(\alpha_B(x_1), \alpha_B(x_2))$
$\geq T(T(\alpha_B(a_1), \alpha_B(b_1)), T(\alpha_B(a_2), \alpha_B(b_2)))$
$= T(T(\alpha_B(a_1), \alpha_B(a_2)), T(\alpha_B(b_1), \alpha_B(b_2)))$
$= T(\alpha_{A_{\alpha_B}}(a_1, a_2), \alpha_{A_{\alpha_B}}(b_1, b_2))$
$= T(\alpha_{A_{\alpha_B}}(a), \alpha_{A_{\alpha_B}}(b))$
and $\beta_{A_{\beta_B}}(x)$
$= \beta_{A_{\beta_B}}(x_1, x_2) = S(\beta_B(x_1), \beta_B(x_2))$
$\leq S(S(\beta_B(a_1), \beta_B(b_1)), S(\beta_B(a_2), \beta_B(b_2)))$
$= S(S(\beta_B(a_1), \beta_B(a_2)), S(\beta_B(b_1), \beta_B(b_2)))$
$= S(\beta_{A_{\beta_B}}(a_1, a_2), \beta_{A_{\beta_B}}(b_1, b_2))$
$= S(\beta_{A_{\beta_B}}(a), \beta_{A_{\beta_B}}(b))$.

This shows that, A_B is an intuitionistic (S, T)-fuzzy left h-ideal of $X \times X$.

For any $x = (x_1, x_2) \in X \times X$, by using Lemma 4.7-4.8, we can show that
$T(\alpha_{A_{\alpha_B}}(x), \alpha_{A_{\alpha_B}}(x))$
$= T(\alpha_{A_{\alpha_B}}(x_1, x_2), \alpha_{A_{\alpha_B}}(x_1, x_2))$
$= T(T(\alpha_B(x_1), \alpha_B(x_2)), T(\alpha_B(x_1), \alpha_B(x_2)))$
$= T(T(\alpha_B(x_1), \alpha_B(x_1)), T(\alpha_B(x_2), \alpha_B(x_2)))$
$= T(\alpha_B(x_1), \alpha_B(x_2))$
$=\alpha_{A_{\alpha_B}}(x_1, x_2)$

$=\alpha_{A_{\alpha_B}}(x)$
and $S(\beta_{A_{\beta_B}}(x),\beta_{A_{\beta_B}}(x))$
$= S(\beta_{A_{\beta_B}}(x_1,x_2),\beta_{A_{\beta_B}}(x_1,x_2))$
$= S(S(\beta_B(x_1),\beta_B(x_2)),S(\beta_B(x_1),\beta_B(x_2)))$
$= S(S(\beta_B(x_1),\beta_B(x_1)),T(\beta_B(x_2),\beta_B(x_2)))$
$= S(\beta_B(x_1),\beta_B(x_2))$
$=\beta_{A_{\beta_B}}(x_1,x_2)$
$=\beta_{A_{\beta_B}}(x)$.

Hence, A_B is an imaginable intuitionistic (S,T)-fuzzy left h-ideal of $X\times X$.

To prove the converse of the theorem we need prove the conditions (ISTF1)-(ISTF3) hold. Let A_B be an imaginable intuitionistic (S,T)-fuzzy left h-ideal of $X\times X$. Then, for $x,y\in X$, we can show that:

(ISTF1) $\alpha_B(x+y) = T(\alpha_B(x+y),\alpha_B(x+y))$
$= \alpha_{A_{\alpha_B}}(x+y,x+y)$
$= \alpha_{A_{\alpha_B}}((x,x)+(y,y))$
$\geq T(\alpha_{A_{\alpha_B}}(x,x),\alpha_{A_{\alpha_B}}(y,y))$
$\geq T(T(\alpha_B(x),\alpha_B(x),T(\alpha_B(y),\alpha_B(y)))$
$= T(\alpha_B(x),\alpha_B(y))$
and $\beta_B(x+y) = S(\beta_B(x+y),\beta_B(x+y))$
$= \beta_{A_{\beta_B}}(x+y,x+y)$
$= \beta_{A_{\beta_B}}((x,x)+(y,y))$
$\leq S(\beta_{A_{\beta_B}}(x,x),\beta_{A_{\beta_B}}(y,y))$
$\leq S(S(\beta_B(x),\beta_B(x),S(\beta_B(y),\beta_B(y)))$
$= S(\beta_B(x),\beta_B(y))$;

(ISTF2) $\alpha_B(xy)$
$= T(\alpha_B(xy),\alpha_B(xy))$
$= \alpha_{A_{\alpha_B}}((x,x)(y,y))$
$\geq \alpha_{A_{\alpha_B}}(y,y)$
$= T(\alpha_B(y),\alpha_B(y))$
$= \alpha_B(y)$
and $\beta_B(xy)$
$= S(\beta_B(xy),\beta_B(xy))$
$= \beta_{A_{\beta_B}}((x,x)(y,y))$
$\leq \beta_{A_{\beta_B}}(y,y)$

$= S(\beta_B(y), \beta_B(y))$
$= \beta_B(y);$

(ISTF3) Suppose that $x + a + z = b + z$ $a, b, x, z \in X$. Then we have $(x, x) + (a, a) + (z, z) = (b, b) + (z, z)$. Since A_B is an imaginable intuitionistic (S, T)-fuzzy left h-ideal of $X \times X$, we have

$\alpha_B(x) = T(\alpha_B(x), \alpha_B(x))$
$= \alpha_{A_{\alpha_B}}(x, x)$
$\geq T(\alpha_{A_{\alpha_B}}(a, a), \alpha_{A_{\alpha_B}}(b, b))$
$= T(T(\alpha_B(a), \alpha_B(a)), T(\alpha_B(b), \alpha_B(b)))$
$=T(\alpha_B(a), \alpha_B(b))$
and $\beta_B(x) = S(\beta_B(x), \beta_B(x))$
$= \beta_{A_{\beta_B}}(x, x)$
$\leq S(\beta_{A_{\beta_B}}(a, a), \beta_{A_{\beta_B}}(b, b))$
$= S(S(\beta_B(a), \beta_B(a)), S(\beta_B(b), \beta_B(b)))$
$=S(\beta_B(a), \beta_B(b))$.

This shows that conditions (ISTF1)-(ISTF3) hold and hence $B = (\alpha_B, \beta_B)$ is an imaginable intuitionistic (S, T)-fuzzy left h-ideal of X. □

Definition 4.12. If $A = (\alpha_A, \beta_A)$ and $B = (\alpha_B, \beta_B)$ are imaginable intuitionistic fuzzy sets in any set X, then the intuitionistic (S, T)-product of A and B, denoted by $[A \cdot B]_{(S,T)}$, is defined by

$$\begin{aligned}[A \cdot B]_{(S,T)} &= [(\alpha_A, \beta_A) \cdot (\alpha_B, \beta_B)]_{(S,T)} \\ &= ([\alpha_A \cdot \alpha_B], [\beta_A \cdot \beta_B])_{(S,T)} \\ &= ([\alpha_A \cdot \alpha_B]_T, [\beta_A \cdot \beta_B]_S)\end{aligned}$$

where $[\alpha_A \cdot \alpha_B]_T(x) = T(\alpha_A(x), \alpha_B(x))$ and $[\beta_A \cdot \beta_B]_S(x) = S(\beta_A(x), \beta_B(x))$ for all $x \in X$.

Theorem 4.13. If $A = (\alpha_A, \beta_A)$ and $B = (\alpha_B, \beta_B)$ are imaginable intuitionistic (S, T)-fuzzy left h-ideals of a hemiring X. If T^*(resp. S^*) is a t-norm (resp. s-norm) which dominates T (resp. S) , that is,

$T^*(T(\alpha, \beta), T(\gamma, \delta)) \geq T(T^*(\alpha, \gamma), T^*(\beta, \delta))$
and $S^*(S(\alpha, \beta), S(\gamma, \delta)) \leq S(S^*(\alpha, \gamma), S^*(\beta, \delta))$
for all $\alpha, \beta, \gamma, \delta \in [0, 1]$.

Then for the intuitionistic (S^*, T^*)-product of A and B, $[A \cdot B]_{(S*,T*)}$ is an intuitionistic (S, T)-fuzzy left h-ideal of X.

Proof. Let $x, y \in X$, then we have

(ISTF1) $[\alpha_A \cdot \alpha_B]_T * (x+y) = T^*(\alpha_A(x+y), \alpha_B(x+y))$
$\geq T^*(T(\alpha_A(x), \alpha_A(y)), T(\alpha_B(x), \alpha_B(y)))$
$\geq T(T^*(\alpha_A(x), \alpha_B(x)), T^*(\alpha_A(y), \alpha_B(y)))$
$= T([\alpha_A \cdot \alpha_B]_T * (x), [\alpha_A \cdot \alpha_B]_T * (y))$
and $[\beta_A \cdot \beta_B]_S * (x+y) = S^*(\beta_A(x+y), \beta_B(x+y))$
$\leq S^*(S(\beta_A(x), \beta_A(y)), S(\beta_B(x), \beta_B(y)))$
$\leq S(S^*(\beta_A(x), \beta_B(x)), S^*(\beta_A(y), \beta_B(y)))$
$= S([\beta_A \cdot \beta_B]_S * (x), [\beta_A \cdot \beta_B]_S * (y))$;
(ISTF2) $[\alpha_A \cdot \alpha_B]_T * (xy) = T^*(\alpha_A(xy), \alpha_B(xy))$
$\geq T^*(\alpha_A(y), \alpha_B(y)) = [\alpha_A \cdot \alpha_B]_T * (y)$
and $[\beta_A \cdot \beta_B]_S * (xy) = S^*(\beta_A(xy), \beta_B(xy))$
$\leq S^*(\beta_A(y), \beta_B(y)) = [\beta_A \cdot \beta_B]_S * (y)$;
(ISTF3) Let $x, z, a, b \in X$ be such that $x + a + z = b + z$. Then
$[\alpha_A \cdot \alpha_B]_T * (x) = T^*(\alpha_A(x), \alpha_B(x))$
$\geq T^*(T(\alpha_A(a), \alpha_A(b)), T(\alpha_B(a), \alpha_B(b)))$
$\geq T(T^*(\alpha_A(a), \alpha_B(a)), T^*(\alpha_A(b), \alpha_B(b)))$
$= T([\alpha_A \cdot \alpha_B]_T * (a), [\alpha_A \cdot \alpha_B]_T * (b))$
and $[\beta_A \cdot \beta_B]_S * (x) = S^*(\beta_A(x), \beta_B(x))$
$\leq S^*(S(\beta_A(a), \beta_A(b)), S(\beta_B(a), \beta_B(b)))$
$\leq S(S^*(\beta_A(a), \beta_B(a)), S^*(\beta_A(b), \beta_B(b)))$
$= S([\beta_A \cdot \beta_B]_S * (a), [\beta_A \cdot \beta_B]_S * (b))$.

Therefore $[A \cdot B]_{(S*,T*)}$ is an intuitionistic (S, T)-fuzzy left h-ideal of X. □

Let $f : X \to X'$ be an onto homomorphism of hemirings. Let T (resp. S) and T^* (resp. S^*) be the t-norms (resp. s-norms) such that T^* (resp. S^*) dominates T (resp. S). If $A = (\alpha_A, \beta_A)$ and $B = (\alpha_B, \beta_B)$ are imaginable intuitionistic fuzzy left h-ideal of X', then for the intuitionistic (S^*, T^*)-product of A and B, $[A \cdot B]_{(S*,T*)}$ is an intuitionistic (S, T)-fuzzy left h-ideal of X'. Since every onto homomorphic inverse image of an intuitionistic (S, T)-fuzzy left h-ideal is an intuitionistic (S, T)-fuzzy left h-ideal, the inverse images

$f^{-1}(A), f^{-1}(B)$, and $f^{-1}([A\cdot B]_{(S*,T*)})$ are also intuitionistic (S,T)-fuzzy left h-ideals of X. In the final theorem we describe the relations between $f^{-1}([A\cdot B]_{(S*,T*)})$ and the intuitionistic (S^*,T^*)-product $[f^{-1}(A)\cdot f^{-1}(B)]_{(S*,T*)}$ of $f^{-1}(A)$ and $f^{-1}(B)$.

Theorem 4.14. Let $f: X \to X'$ be an onto homomorphism of hemirings. Let T^* (resp. S^*) be a $t-$norm (resp. s-norm) such that T^* (resp. S^*) dominates T (resp. S). If $A=(\alpha_A,\beta_A)$ and $B=(\alpha_B,\beta_B)$ are intuitionistic (S,T)-fuzzy left h-ideals of X', then for the intuitionistic (S^*,T^*)-product $[A\cdot B]_{(S*,T*)}$ of A and B and the intuitionistic (S^*,T^*)-product $[f^{-1}(A)\cdot f^{-1}(B)]_{(S*,T*)}$ of $f^{-1}(A)$ and $f^{-1}(B)$, we have

$f^{-1}([A\cdot B]_{(S*,T*)}) = [f^{-1}(A)\cdot f^{-1}(B)]_{(S*,T*)}$

Proof. Let $x\in X$. Then, by computation, we have

$f^{-1}([\alpha_A\cdot\alpha_B]_T*)(x) =([\alpha_A\cdot\alpha_B]_T*)(f(x)) =T^*(\alpha_A(f(x)),\alpha_B(f(x)))$
$=T^*(f^{-1}(\alpha_A)(x), f^{-1}(\alpha_B)(x)) =[f^{-1}(\alpha_A)\cdot f^{-1}(\alpha_B)]_T*(x)$;
and $f^{-1}([\beta_A\cdot\beta_B]_S*)(x) =([\beta_A\cdot\beta_B]_S*)(f(x)) =S^*(\beta_A(f(x)),\beta_B(f(x)))$
$=S^*(f^{-1}(\beta_A)(x), f^{-1}(\beta_B)(x)) =[f^{-1}(\beta_A)\cdot f^{-1}(\beta_B)]_S*(x)$.
Hence, $f^{-1}([A\cdot B]_{(S*,T*)}) = [f^{-1}(A)\cdot f^{-1}(B)]_{(S*,T*)}$. $\square$

Acknowledgements

The authors are highly grateful to referees and Professor W.A. Dudek for their valuable comments and suggestions for improving the paper.

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