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COMMON FIXED POINTS OF A LIMIT WEAK COMPATIBLE PAIR OF A MULTIVALUED MAPPING AND A SINGLE VALUED MAPPING IN D-METRIC SPACES

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ABSTRACT. This paper deals with a general contraction. Two fixed-point theorems for a limit weak-compatible pair of a multi-valued map and a self map on a D -metric space have been established. These results improve significantly, the main results of Dhage, Jennifer and Kang [5] by reducing its assumption and generalizing its contraction simultaneously. At the same time some results of Singh, Jain and Jain [12] are generalized from a self map to a pair of a set-valued and a self map. Theorems of Veerapandi and Rao [16] get generalized and improved by these results. All the results of this paper are new.

1. Introduction

The study of common fixed points for set-valued mappings has been at the center of rigorous research activities and a number of significant results have been obtained by several researchers. Most of these results are extensions and generalizations of the celebrated fixed-point theorem for set -valued maps first established by Nadler [10] in metric spaces. The common fixed point theorems for the pairs of self map and a set-valued map have been studied by Fisher [6, 7], Garegnani

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and Zanco [8] etc. under weaker versions of the commutativity condition.

Generalizing the notion of metric space, Dhage [2] introduced D-metric space and proved the existence of unique fixed point of a self-map satisfying a contractive condition. Dealing with D-metric space Ahmad et. al. [1], Dhage [2, 3], Dhage et. al [4], Rhoades [11], B. Singh, Shishir Jain and Shobha Jain [12], and others made a significant contribution in fixed point theory of D-metric spaces while Veerapandi et. al. [16] established some fixed point theorems for set-valued maps in D-metric spaces.

Recently Dhage, Jennifer and Kang [5] deal with some results for fixed points of a pair of limit coincidentally commuting set-valued map and a self map in a D-metric space.While studying [5] it was strongly felt that the domains of x, y can be restricted properly i.e. to some orbit, also that the number of factors in the contractive condition can be increased. The present paper is an effort in the same direction. Among others an improvement in a fixed point theorem of Veerapandi and Rao [16] has been derived by dropping the assumption of continuity of multivalued map T there.

Main result of this paper proves the existence of the unique common fixed point for a limit weak compatible pair of a set-valued map and a self map satisfying a general contractive condition using orbital concept for the domains of variables x, y, completeness and boundedness. The results of the said references of D-metric spaces are also generalized significantly by this paper.

2. Preliminaries

DEFINITION 1. [2] Let X be a non-empty set. A generalized metric (or D-metric) on X is a function from $X \times X \times X \to R^+$ (the set of non-negative real numbers) satisfying:

(D-1) $\rho(x, y, z) = 0$ if and only if $x = y = z$.

(D-2) $\rho(x, y, z) = \rho\{P(y, x, z)\} = \cdots$, where $P\{x, y, z\}$,is a permutation of x, y, z and

(D-3) $\rho(x, y, z) \le \rho(x, y, a) + \rho(x, a, z) + \rho(a, y, z), \forall x, y, z, a \in X$.

The pair (X, ρ) is called a D-metric space.

DEFINITION 2. [2]A sequence $\{x_n\}$ of points in a D-metric space (X,ρ) is said to be D-convergent to a point $x \in X$ if for $\epsilon > 0, \exists n_0 \in N$ such that, $\forall m, n \geq n_0$, $\rho(x_m, x_n, x) \leq \epsilon$. This sequence is said to be D-Cauchy sequence if for $\epsilon > 0, \exists n_0 \in N$ such that $\forall m > n, p > m, n \geq n_0, \rho(x_n, x_m, x_p) \leq \epsilon.(X,\rho)$ is said to be complete if every D-Cauchy sequence in it converges to some point of X.

DEFINITION 3. Let F be a multivalued map on D-metric space (X,ρ). Let $x_0 \in X$ be arbitrary. A sequence $\{x_n\}$ in X is said to be an orbit of F at x_0 denoted by $O(F, x_0)$ if $x_n \in F^n(x_0), \forall n \in N$. If F is a single-valued self map on X then for $x_0 \in X$, let $x_1 = Fx_0, x_2 = Fx_1 = F^2x_0, \cdots, x_{n-1} = F^{n-1}x_0, \cdots$. Then the sequence $\{x_n\}$ is called the orbit of F at the point x_0 and is denoted by $O(F, x_0)$.

DEFINITION 4. Let F be a multivalued map on D-metric space (X,ρ). An orbit $O(F, x_0)$ is said to be complete if every D-Cauchy sequence in it converges to an element of X.

DEFINITION 5. A subset A of a D-metric space (X,ρ) is said to be bounded if there exists $M > 0$ such that $\rho(u, v, w) \leq M, \forall u, v, w \in A$ and M is said to be a bound of it.

DEFINITION 6. [16] Let $CB(X)$ be the collection of all non-empty bounded and closed subsets of a D-metric space (X,ρ) and $A, B, C \in CB(X)$. Let
$\delta(A, B, C) = Sup\{\rho(a, b, c) : a \in A, b \in B, c \in C\}$,
Then $(CB(X), \delta)$ is a D -metric space.

DEFINITION 7. Let F be a multivalued map on D-metric space (X,ρ) . A point $u \in X$ is said to be a fixed point of F if $u \in Fu$. Also for a sequence $\{x_n\} \in X$, we say $\{Fx_n\} \to z$, if $Lim_{m,n\to\infty}\delta(Fx_m, Fx_n, z) = 0$.

DEFINITION 8. [5] Let F be a multivalued map and g be a self map on D-metric space (X,ρ) . The pair (F, g) is said to be weak compatible if $Fy = \{gy\}$, for some $y \in X$ implies $Fgy = \{gFy\}$.

DEFINITION 9. Let $F : X \to CB(X)$ and $g : X \to X$. The pair (F, g) is said to be limit weak compatible or limit coincidently commuting if $\lim_{n\to\infty} Fx_n = \{\lim_{n\to\infty} gx_n\}$, for some x_n in X implies $\lim_{n\to\infty} Fgx_n = \lim_{n\to\infty} gFx_n$.

For set-valued maps and weak compatible maps in D-metric spaces, we refer to [13] and [14] respectively.

Definition 10. Let (X, ρ) be a D-metric space, and $\{A_n\}$ be a sequence in $B(X)$. A map $F : X \to A \in CB(X)$ is said to be continuous with respect to $\{A_n\}$, if $A_{k_n} \to CB(X)$ implies $FA_{k_n} \to F(A)$ where $\{A_{k_n}\}$ is some subsequence of $\{A_n\}$.

A self map g on X is said to be continuous with respect to$\{x_n\} \subseteq X$, if for any subsequence $\{x_{k_n}\}, x_{k_n} \to x \in X$ implies $gx_{k_n} \to gx$.

Proposition 1. *In a D-metric space (X, ρ), if ρ is continuous in two variables, then limit of a sequence in X is unique, if it exists.*

Proof. Let $\{x_n\} \to x$ and $\{x_n\} \to y$ in (X, ρ).
Now

$$lim_{n\to\infty}\rho(x_n, x_{n+p}, x) = 0,$$

implies

$$\rho(x, y, x) = 0,$$

as ρ is continuous in two variables. Thus, $x = y$. Hence limit of a sequence is unique in a D-metric space, if it exists. □

Proposition 2. *Let (X, ρ) be a D-metric space, F be a multivalued map and g be a self map on X. If pair (F, g) is limit weak compatible then (F, g) is weak compatible.*

Proof. Let (F, g) is limit weak-compatible. Let for some $x \in X, Fx = \{gx\}$. Taking $x_n = x$ we get, $gFx = Fgx$. □

Let Φ denote the class of functions $\phi : R^+ \to R^+$ such that ϕ is upper semi-continuous, ϕ is non-decreasing, $\phi(t) < t$, for $t > 0$.

Proposition 3. *Let g be a self map in a D-metric space (X, ρ) and $F : X \to CB(X)$ such that $F(X) \subseteq g(X)$. For some $x_0 \in X$, define sequences $\{x_n\}$ and $\{y_n\}$ in X by $y_n = gx_n \in Fx_{n-1}, \forall n \in N$. Then*

$$\{x_o, x_1, x_2, \cdots\} = \{x_n\} \in O(g^{-1}F, x_0),$$
$$\{y_1, y_2, y_3, \cdots\} = \{y_n\} \in O(Fg^{-1}, y_1), \quad \text{where } y_1 = Fx_0.$$

Proof. As $gx_1 \in Fx_0$ we have $x_1 \in g^{-1}Fx_0$. Also $gx_2 \in Fx_1$ gives $x_2 \in g^{-1}Fx_1 = (g^{-1}F)^2x_0$. Similarly, $gx_n \in Fx_{n-1}$ gives $x_n \in g^{-1}Fx_{n-1} = (g^{-1}F)^n x_0$.
Again $y_1 = gx_1 \in Fx_0, y_2 = gx_2 \in Fx_1 \in F(g^{-1}Fx_0) = (Fg^{-1})Fx_0, y_3 = gx_3 \in Fx_2 \in F(g^{-1}F)^2x_0 = (Fg^{-1})^2Fx_0$. Similarly, $y_n \in (Fg^{-1})^{n-1}Fx_0$. □

Note: $\{y_n\} = \{y_1, y_2, y_3, \cdots\} = O(Fg^{-1}, y_1)$, where $y_1 = Fx_0$, is said to be an $(F/g)-$ orbit at x_0 .It is also written as $O(Fg^{-1}, Fx_0)$.
Before proving the main result we need the following lemma.

LEMMA 1. *Let g be a self map in a D -metric space (X, ρ) and $F : X \to CB(X)$ be such that $F(X) \subseteq g(X)$. such that for some $x_0 \in X$, and for some $\phi \in \Phi$,*

$$(2.141)\quad \delta(Fx, Fy, Fz) \leq \phi \;\; Max \begin{pmatrix} \rho(gx, gy, gz), \delta(Fx, Fy, gz), \\ \delta(gx, Fx, gz), \delta(gy, Fy, gz), \\ \delta(gx, Fy, gz), \delta(gy, Fx, gz), \\ \delta(gx, gy, Fz), \delta(gx, Fx, Fz), \\ \delta(gy, Fy, Fz), \delta(gx, Fy, Fz), \\ \delta(gy, Fx, Fz) \end{pmatrix},$$

$\forall x, y, z \in O(g^{-1}F, x_0)$.

Let $\{x_n\}$ and $\{y_n\}$ be sequences defined in X as above. Let $\{X_n\}$ be a sequence in $CB(X)$ given by

$$y_n = gx_n \in Fx_{n-1} = X_n, \forall n \in N.$$

If $F(\{x_n\}) = \bigcup_{i \in N} X_i$ is bounded then
(I) $\{y_n\}$ is a $D-$ Cauchy sequence in $O(Fg^{-1}, Fx_0)$.
(II) If δ is continuous in one variable then, $gx_n \to u$ implies $Fx_n \to \{u\}$.

Proof. **of (I)**: Define a positive real sequence $\{\gamma_n\}$ in R^+ by
$$\gamma_i = Sup_{j,k \in N} \delta(X_i, X_{i+j}, X_{i+j+k}), \forall i \in N.$$
Then $\gamma_i \geq 0$ and γ_i is a non-increasing sequence for all i. Each γ_i is finite as $\bigcup_{i \in N} X_i$ is bounded. Hence it tends to a limit say γ . In the following we show that $\gamma = 0$. We have, using (2.141) for $m > n$,

$$
\begin{aligned}
\delta(X_n, X_{n+p}, X_m) &= \delta(Fx_{n-1}, Fx_{n+p-1}, Fx_{m-1}) \\
&\leq \phi \quad Max \begin{pmatrix} \rho(y_{n-1}, y_{n+p-1}, y_{m-1}), \delta(X_n, X_{n+p}, y_{m-1}), \\ \delta(y_{n-1}, X_n, y_{m-1}), \delta(y_{n+p-1}, X_{n+p}, y_{m-1}), \\ \delta(y_{n-1}, X_{n+p}, y_{m-1}), \delta(X_n, y_{n+p-1}, y_{m-1}), \\ \delta(y_{n-1}, y_{n+p-1}, X_m), \delta(y_{n-1}, X_n, X_m), \\ \delta(y_{n+p-1}, X_{n+p}, X_m), \delta(y_{n-1}, X_{n+p}, X_m), \\ \delta(X_n, y_{n+p-1}, X_m) \end{pmatrix} \\
&\leq \phi \quad Max \begin{pmatrix} \delta(X_{n-1}, X_{n+p-1}, X_{m-1}), \delta(X_n, X_{n+p}, X_{m-1}), \\ \delta(X_{n-1}, X_n, X_{m-1}), \delta(X_{n+p-1}, X_{n+p}, X_{m-1}), \\ \delta(X_{n-1}, X_{n+p}, X_{m-1}), \delta(X_n, X_{n+p-1}, X_{m-1}), \\ \delta(X_{n-1}, X_{n+p-1}, X_m), \delta(X_{n-1}, X_n, X_m), \\ \delta(X_{n+p-1}, X_{n+p}, X_m), \delta(X_{n-1}, X_{n+p}, X_m), \\ \delta(X_n, X_{n+p-1}, X_m) \end{pmatrix} \\
&\leq \phi \quad Max(\gamma_{n-1}, \gamma_n, \gamma_{n+p-1}) \\
&= \phi \ (\gamma_{n-1}).
\end{aligned}
$$

Thus,

$$\delta(X_n, X_{n+p}, X_{n+p+t}) \leq \phi(\gamma_{n-1}) \tag{2.1}$$

Taking supremum over p and t we get,

$$\gamma_n \leq \phi(\gamma_{n-1})$$

Letting $n \to \infty$ we get,

$$\gamma \leq \phi(\gamma) < \gamma, \text{ if } \gamma > 0,$$

which is a contradiction. Hence $\gamma = 0$ i.e. $\gamma_n \to 0$, as $n \to \infty$.
Again,

$$
\begin{aligned}
\rho(y_n, y_{n+p}, y_{n+p+t}) &\leq \delta(X_n, X_{n+p}, X_m) \\
&\leq \phi(\gamma_{n-1}) \qquad \text{using}(2.1),
\end{aligned}
$$

Letting $n \to \infty$ we get,

$$Lim_{n\to\infty} \ \rho(y_n, y_{n+p}, y_{n+p+t}) \leq Lim_{n\to\infty} \phi(\gamma_{n-1}) = 0.$$

Hence $\{y_n\}$ is a D-Cauchy sequence in $O(Fg^{-1}, Fx_0)$.

Proof of (II): Let $gx_n \to u$.

$$\begin{aligned} Lim_{n\to\infty}\ \delta(Fx_n, Fx_{n+p}, u) &= Lim_{n\to\infty}\ \delta(Fx_n, Fx_{n+p}, gx_m) \\ &\leq Lim_{n\to\infty}\delta(Fx_n, Fx_{n+p}, X_m), \\ &\leq Lim_{n\to\infty}\delta(X_{n+1}, X_{n+p+1}, X_m), \\ &\leq Lim_{n\to\infty}\phi(\gamma_n), \qquad \text{using}(2.1). \\ &= 0. \end{aligned}$$

□

Therefore, $Lim_{n\to\infty}\delta(Fx_n, Fx_{n+p}, u) = 0$ and we get, $Fx_n \to \{u\}$, in the D-metric space $(CB(X), \delta)$,

In [5] Dhage, Jennifer and Kang have proved the following:

Theorem 3.1 [5] Let X be a D-metric space and $F : X \to CB(X)$ and $g : X \to X$ be two mappings satisfying, for some positive number r ,

$$\delta^r(Fx, Fy, Fz) \leq \phi \quad Max \begin{pmatrix} \rho^r(gx, gy, gz), \delta^r(Fx, Fy, gz), \\ \delta^r(gx, Fx, gz), \delta^r(gy, Fy, gz), \\ \delta^r(gx, Fy, gz), \delta^r(gy, Fx, gz) \end{pmatrix}$$

for all $x, y, z \in X$, and where $\phi : R^+ \to R^+$, ϕ is non-decreasing, $\phi(t) < t, t > 0$, and $\sum \phi^n(t) < \infty$ for each $t \in R^+$. Further suppose that

(a) $F(X) \subseteq g(X)$ and g(X) is bounded.

(b) $\{F, g\}$ is limit weak compatible.

(c) F or g is (F/g)- orbitally continuous.

Further if $X$is (F/g)-orbitally complete D-metric space, then $F$and g have a unique fixed point $u \in X$ such that $Fu = \{u\} = gu$.

The following is a unique common fixed point theorem for a limit weak-compatible pair of multivalued map and a self map, both non-continuous, on an unbounded and incomplete D-metric space. It generalizes the result of [5] significantly. .

Note: In the following (X, ρ) will stand for a D-metric space and δ will be the D-metric for the associated D-metric $(CB(X), \delta)$, where

$CB(X)$ is the set of all bounded subsets of X.

3. MAIN RESULTS

THEOREM 1. *Let g be a self map in a D -metric space (X, ρ) and $F : X \to CB(X)$, with δ continuous in three variables, satisfying*

(3.11) $F(X) \subseteq g(X)$.
(3.12) *For some $x_0 \in X$, orbit $O(Fg^{-1}, y_1)$ is complete.*
(3.13) *g is continuous with respect to $\{X_n\}$, where the sequences $\{x_n\}$ in X and $\{X_n\}$ is in $CB(X)$ are defined by $y_n = gx_n \in Fx_{n-1} = X_n, \forall n \in N$.*
(3.14) *$F(\{x_n\}) = \bigcup_{i \in N} X_i$ is bounded,*
(3.15) *the pair (F, g) is limit weak compatible.*
(3.16) *For some $x_0 \in X$, and for some $\phi \in \Phi$ and $\forall x, y \in O(g^{-1}F, x_0)$ and $z \in X$*

$$\delta(Fx, Fy, Fz) \leq \phi \quad Max \begin{pmatrix} \rho(gx, gy, gz), \delta(Fx, Fy, gz), \\ \delta(gx, Fx, gz), \delta(gy, Fy, gz), \\ \delta(gx, Fy, gz), \delta(gy, Fx, gz), \\ \delta(gx, gy, Fz), \delta(gx, Fx, Fz), \\ \delta(gy, Fy, Fz), \delta(gx, Fy, Fz), \\ \delta(gy, Fx, Fz) \end{pmatrix}.$$

Then F and g have the unique common fixed point in X

Proof. For some $x_0 \in X$, construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that $y_n = gx_n \in Fx_{n-1}, \forall n \in N$. Therefore by lemma 1 ,$\{y_n\} = \{gx_n\}$ is a D-Cauchy sequence in $O(Fg^{-1}, Fx_0)$, which is complete.

$$y_n \to u \in X \tag{3.1}$$

As (F, g) is limit weak compatible, $\lim_{n\to\infty} Fgx_n = \lim_{n\to\infty} gFx_n$.
As g is continuous with respect to X_n and $gx_n \in X_n$. Now

$$Fx_n \to u \Rightarrow X_n \to u \Rightarrow gX_n \to gu \Rightarrow ggx_n \to gu. \tag{3.2}$$

Also

$$\lim_{n\to\infty} Fgx_n = \lim_{n\to\infty} gFx_n = g \lim_{n\to\infty} Fx_n = gu. \tag{3.3}$$

Step1: Putting $x = x_n, y = x_n, z = gx_n$ in (3.16) we get,

$$\delta(Fx_n,Fx_n,Fgx_n) \leq \phi \ \ Max \begin{pmatrix} \rho(gx_n, gx_n, ggx_n), \delta(Fx_n, Fx_n, ggx_n), \\ \delta(gx_n, Fx_n, ggx_n), \delta(gx_n, Fx_n, ggx_n), \\ \delta(gx_n, Fx_n, ggx_n), \delta(gx_n, Fx_n, ggx_n), \\ \delta(gx_n, gx_n, Fgx_n), \delta(gx_n, Fx_n, Fgx_n), \\ \delta(gx_n, Fx_n, Fgx_n), \delta(gx_n, Fx_n, Fgx_n), \\ \delta(gx_n, Fx_n, Fgx_n) \end{pmatrix}$$

Letting $n \to \infty$, using (3.1), (3.2) and (3.3) we get,

$$\begin{aligned} \delta(u,u,gu) &\leq \phi\ (\delta(u,u,gu)), \\ &< \delta(u,u,gu), \quad \text{if } \delta(u,u,gu) > 0, \end{aligned}$$

which is not possible. Thus, $\delta(u,u,gu) = 0$ which gives

$$u = gu. \tag{3.4}$$

Step 2: Putting $x = x_n, y = x_n$ and $z = u$ in (3.16) we get,

$$\delta(Fx_n,Fx_n,Fu) \leq \phi \ \ Max \begin{pmatrix} \rho(gx_n, gx_n, gu), \delta(Fx_n, Fx_n, gu), \\ \delta(gx_n, Fx_n, gu), \delta(gx_n, Fx_n, gu), \\ \delta(gx_n, Fx_n, gu), \delta(gx_n, Fx_n, gu), \\ \delta(gx_n, gx_n, Fu), \delta(gx_n, Fx_n, Fu), \\ \delta(gx_n, Fx_n, Fu), \delta(gx_n, Fx_n, Fu), \\ \delta(gx_n, Fx_n, Fu) \end{pmatrix}.$$

Letting $n \to \infty$, using (3.1) and (3.5) we get,

$$\begin{aligned} \delta(u,u,Fu) &\leq \phi\left(\ \delta(u,u,Fu)\ \right), \\ &< \delta(u,u,Fu), \qquad \text{if } \delta(u,u,Fu) > 0, \end{aligned}$$

which is a contradiction. Thus, $\delta(u, u, Fu) = 0$ which gives $u = Fu$. Hence $u = gu = Fu$. Therefore, u is a common fixed point of F and g .

Step 3: (Uniqueness) Let w be another common fixed point of F and g , then

$$w = Fw = gw. \tag{3.5}$$

Taking $x = x_n, y = x_n$ and $z = w$ in (3.16) we get,

$$\delta(Fx_n, Fx_n, Fw) \leq \phi \ \ Max \begin{pmatrix} \rho(gx_n, gx_n, gw), \delta(Fx_n, Fx_n, gw), \\ \delta(gx_n, Fx_n, gw), \delta(gx_n, Fx_n, gw), \\ \delta(gx_n, Fx_n, gw), \delta(gx_n, Fx_n, gw), \\ \delta(gx_n, gx_n, Fw), \delta(gx_n, Fx_n, Fw), \\ \delta(gx_n, Fx_n, Fw), \delta(gx_n, Fx_n, Fw), \\ \delta(gx_n, Fx_n, Fw) \end{pmatrix}.$$

Letting $n \to \infty$, using (3.1) and (3.5) we get,

$$\begin{aligned} \delta(u, u, w) &\leq \phi \left(\ \delta(u, u, w) \ \right), \\ &< \delta(u, u, w), \qquad \text{if } \delta(u, u, w) > 0, \end{aligned}$$

which is a contradiction. Thus, $\delta(u, u, w) = 0$ which gives $u = w$. Therefore, u is the unique common fixed point of F and g .

□

Theorem 2. *Let g be a self map in a D-metric space (X, ρ) and $F : X \to CB(X)$, with δ continuous in two variables, satisfying (3.11), (3.12), (3.14), (3.15) and*

(3.21) *One of g or F is continuous with respect to $\{X_n\}$, where the sequences $\{x_n\}$ in X and $\{X_n\}$ in $CB(X)$ are defined by $y_n = gx_n \in Fx_{n-1} = X_n, \forall n \in N$.*

(3.22) *For some $\phi \in \Phi$ and for all $x, y \in O(g^{-1}F, x_0)$ and $z \in X$ and $z \in X_n, \forall n \in N$*

$$\delta(Fx,Fy,Fz) \leq \phi \;\; Max\begin{pmatrix} \rho(gx,gy,gz),\delta(Fx,Fy,gz), \\ \delta(gx,Fx,gz),\delta(gy,Fy,gz), \\ \delta(gx,Fy,gz),\delta(gy,Fx,gz), \\ \delta(gx,gy,Fz),\delta(gx,Fx,Fz), \\ \delta(gy,Fy,Fz),\delta(gx,Fy,Fz), \\ \delta(gy,Fx,Fz) \end{pmatrix},$$

$$\forall x,y,z \in O(g^{-1}F,x_0)$$

Then F and g have the unique common fixed point in X.

Proof. For some $x_0 \in X$, construct sequences $\{x_n\}$ and $\{y_n\}$ in Xas in theorem 1. Therefore, (3.1) holds. The case when g is continuous has been proved in theorem 1. Hence we give below the proof when F is continuous with respect to $\{X_n\}$.
As F is continuous,

$$\lim_{n\to\infty} FFx_n = F\lim_{n\to\infty} Fx_n = Fu. \tag{3.6}$$

Also as $gx_n \in Fx_{n-1}(= X_n)$ and (F,g) is limit weak compatible we have,

$$\lim_{n\to\infty} gFx_n = \lim_{n\to\infty} Fgx_n = F\lim_{n\to\infty} gx_n = Fu. \tag{3.7}$$

Step 4: Putting $x = x_n, y = x_n$ and $z = Fx_n$ in (3.22) we get,

$$\delta(Fx_n,Fx_n,FFx_n) \leq \phi \;\; Max\begin{pmatrix} \rho(gx_n,gx_n,gFx_n),\delta(Fx_n,Fx_n,gFx_n), \\ \delta(gx_n,Fx_n,gFx_n),\delta(gx_n,Fx_n,gFx_n), \\ \delta(gx_n,Fx_n,gFx_n),\delta(gx_n,Fx_n,gFx_n), \\ \delta(gx_n,gx_n,FFx_n),\delta(gx_n,Fx_n,FFx_n), \\ \delta(gx_n,Fx_n,FFx_n),\delta(gx_n,Fx_n,FFx_n), \\ \delta(gx_n,Fx_n,FFx_n) \end{pmatrix}.$$

Letting $n \to \infty$, using (3.1), (3.6) and (3.7) we get,

$$\begin{aligned} \delta(u,u,Fu) &\leq \phi(\delta(u,u,Fu)), \\ &< \delta(u,u,Fu), \qquad \text{if } \delta(u,u,Fu) > 0 \end{aligned}$$

which is not possible. Thus, $\rho(u, u, Fu) = 0$which gives

$$u = Fu. \tag{3.8}$$

As $F(X) \subseteq g(X)$, there exists $v \in X$ such that

$$u = gv. \tag{3.9}$$

Step 5: Putting $x = x_n, y = x_n$ and $z = v$ in (3.22) we get,

$$\delta(Fx_n,Fx_n, Fv) \leq \phi \quad Max \begin{pmatrix} \rho(gx_n, gx_n, gv), \delta(Fx_n, Fx_n, gv), \\ \delta(gx_n, Fx_n, gv), \delta(gx_n, Fx_n, gv), \\ \delta(gx_n, Fx_n, gv), \delta(gx_n, Fx_n, gv), \\ \delta(gx_n, gx_n, Fv), \delta(gx_n, Fx_n, Fv), \\ \delta(gx_n, Fx_n, Fv), \delta(gx_n, Fx_n, Fv), \\ \delta(gx_n, Fx_n, Fv) \end{pmatrix}.$$

Letting $n \to \infty$, using (3.1) and (3.9) we get,

$$\begin{aligned} \delta(u, u, Fv) &\leq \phi(\delta(u, u, Fv)), \\ &< \delta(u, u, Fv), \qquad \text{if } \delta(u, u, Fv) > 0 \end{aligned}$$

which is not possible. Thus, $\delta(u, u, Fv) = 0$ which gives $u = Fv$. Therefore $u = gv = Fv$. As (F, g) is limit weak compatible and so is weak compatible. Hence $Fu = gu$. Thus, from (3.8) it follows $Fu = gu = u$. Uniqueness follows from step 3 of theorem 1. □

REMARK 1. Theorem 2 generalizes the result of [5] in the following sense:

• The contractive condition of the above theorem is more general than that of [5] as it contains eleven factors on the right.

• The function ϕ taken in above theorem is less restrictive than that of [5] as $\sum_{\infty}^{n=1} \phi^n(t)$need not to be summable here.

• In above theorem, $\bigcup_i X_i = \bigcup_i F(x_{i-1}) = \bigcup_{n \in N} Fx_n = F(X) \subset g(X)$ is assumed to be bounded. Hence the domain $g(X)$ of boundedness of [5] is larger than that of above corollary.

• Domain of z in the theorem must include $z = \{X_n\}, n \in N$. This has not been mentioned in [5] and it has been used by the authors in the proof of their theorem on page 2528 of [5] to arrive at (3.21) there.

Note: The power of r in [5] applied on δ and ρ gets cancelled throughout. Hence the mention of power r in [5] is insignificant.

In [12] B. Singh, Shshir Jain and Shobha Jain have proved the following:

Corollary 3.4 [12]: Let S be a self map on a D-metric space (X, D) such that for $x_0 \in X$ the orbit $O(S, x_0)$ is bounded and complete and $\forall x, y \in O(S, x_0), \forall z \in X$, there exists $\lambda \in [0, 1)$ such that:

$$D(Sx, Sy, Sz) \leq \lambda Max \left(\begin{array}{l} D(x,y,z), D(Sx,x,z), D(Sy,y,z), \\ D(Sx,y,z), D(Sy,x,z), D(x,y,Sz), \\ D(Sx,x,Sz), D(Sy,y,Sz), D(Sx,y,Sz), \\ D(Sy,x,Sz) \end{array} \right).$$

Then S has a unique fixed point.

Taking $g = I$, the identity map on X in theorem 2, we have the following generalization of said corollary of [12] for set-valued mappings in D-metric spaces:

COROLLARY 1. *Let F be a multivalued map on a D-metric space (X, ρ), with ρ continuous in three variables, satisfying:*
(i) For some $x_0 \in X$, orbit $O(F, x_0)$ is complete, defining the sequences $\{x_n\}$ in X and $\{X_n\}$ in $CB(X)$ by $x_n \in Fx_{n-1} = X_n, \forall n \in N$.
$(ii) F(\{x_n\}) = \bigcup_{i \in N} X_i$ is bounded.
(iii) For some $\phi \in \Phi$, for all $x, y \in O(F, x_0), z \in X$ and $z \in X_n, \forall n \in N$,

$$\delta(Fx, Fy, Fz) \leq \phi Max \left(\begin{array}{l} \rho(x,y,z), \delta(Fx,Fy,z), \delta(x,Fx,z), \\ \delta(y,Fy,z), \delta(x,Fy,z), \delta(y,Fx,z), \\ \delta(x,y,Fz), \delta(x,Fx,Fz), \delta(y,Fy,Fz), \\ \delta(x,Fy,Fz), \delta(y,Fx,Fz) \end{array} \right).$$

Then F has the unique fixed point in X.

Proof. Result follows from Theorem 2, by taking $g = I$, the identity map on X. Here $O(Fg^{-1}, y_1) = O(F, x_1) \subset O(F, x_0)$.

□

REMARK 2. Above corollary generalizes the result of [12] for set-valued map F. Even the contraction here, is more general. Further it represents a generalization of of result of Rhoades [11] theorem 1 as well.

In [16] Veerapandi and Chandrasekhara Rao have proved the following:
Theorem 1 [16]: Let (X, D) be a complete and bounded D-metric space (X, D). If $T : X \rightarrow CB(X)$ is a multivalued contractive mapping, then T has a fixed point. T is said to be a multivalued contraction if for some $q \in [0, 1)$
$H(Tx_1, Tx_2, Tx_3) \leq qD(x_1, x_2 x_3)$, for all $x_1, x_2, x_3 \in X$.
Following corollary generalizes this result by taking $g = I$, the identity map and $\phi = q \in [0, 1)$ in theorem 1 we have,

COROLLARY 2. *Let F be a multivalued map on a D-metric space (X, ρ), with ρ continuous in three variables. If for some $x_0 \in X$, $\{x_n\} = O(F, x_0)$ is complete, $\bigcup_i X_i = \bigcup_i F(x_i)$ is bounded and*
(iv) For some $\phi \in \Phi$, for all $x, y \in O(F, x_0), z \in X$

$$\delta(Fx,Fy,Fz)$$
$$\leq q \quad Max \begin{pmatrix} \rho(x,y,z), \delta(Fx,Fy,z), \delta(x,Fx,z), \\ \delta(y,Fy,z), \delta(x,Fy,z), \delta(y,Fx,z), \\ \delta(x,y,Fz), \delta(x,Fx,Fz), \delta(y,Fy,Fz), \\ \delta(x,Fy,Fz), \delta(y,Fx,Fz) \end{pmatrix}.$$

Then F has a unique fixed point in X .

REMARK 3. Above corollary generalizes the quoted result of [14] in the following sense:
• Restriction of the corollary to the first factor only gives above result of [16]. • Domains of variables x, y and that of boundedness and completeness have been restricted.
• The above corollary shows that the existing fixed point is unique.

Again in [16] Veerapandi and Chandrasekhara Rao have proved the following other:
Theorem 2 [16]: Let (X, D) be a complete and bounded D-metric space (X, D). If $T : X \rightarrow CB(X)$ is a multivalued orbitally continuous map satisfying:

$H(Tx, Ty, Tz) \leq a_1 D^*(x, y, z) + a_2 D^*(x, Tx, z) + a_3 D^*(y, Ty, Tz)$, $\forall x, y, z \in X$, where $a_1, a_2, a_3 \in R^+$ with $a_1 + a_2 + a_3 < 1$. Here $D^*(A, B, C) = \inf\{D(a, b, c) : \forall a \in A, \forall b \in B, \forall c \in C\}$.
Then T has a fixed point.
The following corollary is a generalization of it:

COROLLARY 3. *Let F be a multivalued map on a D-metric space (X, ρ), with δ continuous in three variables, satisfying $(i), (ii)$ and (v) Let $a_i, 1 \leq i \leq 11$, be constants such that $a_i \geq 0, \forall i, \sum_{i=1}^{11} a_i < 1, \forall x, y \in O(F, x_0), z \in X$,*

$$\begin{aligned}\delta(Fx, Fy, Fz) \leq\ & a_1 \rho(x, y, z) + a_2 \delta(Fx, Fy, z) + a_3 \delta(x, Fx, z) \\ & + a_4 \delta(y, Fy, z) + a_5 \delta(x, Fy, z) + a_6 \delta(y, Fx, z) \\ & + a_7 \delta(x, y, Fz) + a_8 \delta(x, Fx, Fz) + a_9 \delta(y, Fy, Fz) \\ & + a_{10} \delta(x, Fy, Fz) + a_{11} \delta(y, Fx, Fz).\end{aligned}$$

Then F has the unique fixed point in X.

Proof. Let $\sum_{i=1}^{11} a_i = q (< 1)$. Let

$$M(x, y, z) = \quad Max \begin{pmatrix} \rho(x, y, z), \delta(Fx, Fy, z), \delta(x, Fx, z), \\ \delta(y, Fy, z), \delta(x, Fy, z), \delta(y, Fx, z), \\ \delta(x, y, Fz), \delta(x, Fx, Fz), \delta(y, Fy, Fz), \\ \delta(x, Fy, Fz), \delta(y, Fx, Fz) \end{pmatrix}.$$

Then from condition (v) we have,

$$\begin{aligned}\delta(Fx, Fy, Fz) &\leq \sum_{i=1}^{11} a_i M(x, y, z) \\ &\leq M(x, y, z) \sum_{i=1}^{11} a_i \\ &= q M(x, y, z) \quad (q < 1),\end{aligned}$$

$\forall x, y \in O(F, x_0), z \in X$ and $z = X_n, \forall n \in N$, rest follows from corollary 2.

□

REMARK 4. Taking $a_2 = a_4 = a_5 = a_6 = a_7 = a_8 = a_{10} = a_{11} = 0$, and noting that $D^*(A, B, C) \leq \delta(A, B, C), \forall A, B, C \in CB(X)$, above quoted theorem of [14] follows from corollary 3.

Corollary 3, reduces the domains of x and y, completeness and that of boundedness. Further it does not need $F(orT)$ to be continuous in any way as assumed in [16].

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