East Asian Math. J. **22** (2006), No. 1, pp. 1–15

FURTHER REVERSES OF THE SCHWARZ INEQUALITY IN INNER PRODUCT SPACES

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Abstract. Some new reverses of the Schwarz inequality in inner product spaces and applications for vector-valued sequnces and integrals are given.

1. Introduction

Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$. One of the most important inequalities in inner product spaces with numerous applications, is the *Schwarz inequality*

$$|\langle x,y\rangle|^2 \le \|x\|^2\|y\|^2, \quad x,y\in H, \tag{1.1}$$

with equality if and only if there exists a scalar $\alpha\in\mathbb{K}$ with $x=\alpha y$.

If $\bar{B}(y,r) := \{z\in H | \|z-y\|\le r\}$ is the closed ball of radius $r>0$ centered in y and $x\in\bar{B}(y,r)$, then the following *reverse* of the Schwarz inequality holds [1]

$$(0\le)\ \|x\|\|y\| - |\langle x,y\rangle| \le \|x\|\|y\| - |\mathrm{Re}\langle x,y\rangle| \tag{1.2}$$

$$\le \|x\|\|y\| - \mathrm{Re}\langle x,y\rangle \le \frac{1}{2}r^2. \tag{1.3}$$

The constant $\frac{1}{2}$ is best possible in (1.2).

If we assume that $\|x-y\|\le r<\|y\|$, then the following quadratic reverse of the Schwarz inequality also holds [2]

Received September 14, 2004.
2000 Mathematics Subject Classification: Primary 46C05; Secondary 26D15.
Key words and phrases: Schwarz inequality, Triangle inequality, Bessel inequality.

$$(0 \leq) \|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 \leq \|x\|^2 \|y\|^2 - [\operatorname{Re} \langle x, y\rangle]^2 \leq r^2 \|x\|^2 . \tag{1.4}$$

The multiplicative constant 1 in front of r^2 is best possible in the sense that it cannot be replaced by a smaller one.

If $\|x - y\| \leq r < \|y\|$, then the following multiplicative version of the Schwarz inequality is valid (see [2])

$$(1 \leq) \frac{\|x\| \|y\|}{\operatorname{Re} \langle x, y\rangle} \leq \frac{\|y\|}{\sqrt{\|y\|^2 - r^2}}. \tag{1.5}$$

After simple calculation, (1.5) is clearly equivalent to:

$$(0 \leq) \|x\| \|y\| - \operatorname{Re} \langle x, y\rangle \leq \frac{r^2}{\sqrt{\|y\|^2 - r^2}\left(\sqrt{\|y\|^2 - r^2} + \|y\|^2\right)} \operatorname{Re} \langle x, y\rangle . \tag{1.6}$$

Other types of reverse results of the Schwarz inequality are presented in the following.

Let $A, a \in \mathbb{K}$ ($\mathbb{K} = \mathbb{C}, \mathbb{R}$) and $x, y \in H$. If

$$\operatorname{Re} \langle Ay - x, x - ay\rangle \geq 0, \tag{1.7}$$

or, equivalently,

$$\left\| x - \frac{a + A}{2} \cdot y \right\| \leq \frac{1}{2} |A - a| \|y\| , \tag{1.8}$$

holds, then one has the inequality [3]

$$(0 \leq) \|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 \leq \frac{1}{4} |A - a|^2 \|y\|^4 . \tag{1.9}$$

The constant $\frac{1}{4}$ is sharp in (1.9).

This inequality has been refined in [4] as follows

$$(0 \leq) \|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 \leq \frac{1}{4}|A-a|^2 \|y\|^4 - \left|\frac{A+a}{2}\|y\|^2 - \langle x, y\rangle\right|^2, \tag{1.10}$$

provided (1.7) or (1.8) holds. A different refinement of (1.9) has been obtained in [5]:

$$(0 \leq) \|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 \leq \frac{1}{4}|A-a|^2 \|y\|^4 - \|y\|^2 \operatorname{Re}\langle Ay - x, x - ay\rangle. \tag{1.11}$$

It has been shown in [4] that the bounds provided by (1.10) and (1.11) for the Schwarz difference $\|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2$ cannot be compared in general, meaning that, sometimes one bound is better than the other.

If $A, a \in \mathbb{K}$ satisfy either (1.7) or (1.8) and $\operatorname{Re}(A\bar{a}) > 0$, then the following inequality also holds (see [6] and [7])

$$\|x\| \|y\| \leq \frac{1}{2} \cdot \frac{\operatorname{Re}\left[\left(\bar{A}+\bar{a}\right)\langle x, y\rangle\right]}{\left[\operatorname{Re}\left(\bar{A}a\right)\right]^{\frac{1}{2}}} \leq \frac{1}{2} \cdot \frac{|A+a|}{\left[\operatorname{Re}\left(\bar{A}a\right)\right]^{\frac{1}{2}}} |\langle x, y\rangle|. \tag{1.12}$$

The constant $\frac{1}{2}$ is best possible in both inequalities.

In the particular case where $A = M \geq m = a > 0$, the inequality (1.12) becomes

$$\|x\| \|y\| \leq \frac{1}{2} \cdot \frac{(M+m)}{\sqrt{Mm}} \operatorname{Re}\langle x, y\rangle, \tag{1.13}$$

provided

$$\operatorname{Re}\langle My - x, x - my\rangle \geq 0 \tag{1.14}$$

or, equivalently,

$$\left\|x - \frac{M+m}{2} y\right\| \leq \frac{1}{2}(M-m)\|y\|. \tag{1.15}$$

The constant $\frac{1}{2}$ is sharp in (1.13).

An equivalent additive version of (1.13) is the following result

$$(0 \leq) \|x\| \|y\| - \operatorname{Re}\langle x, y\rangle \leq \frac{1}{2} \cdot \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{\sqrt{Mm}} \operatorname{Re}\langle x, y\rangle . \tag{1.16}$$

In this inequality $\frac{1}{2}$ is also sharp.

As pointed out in [2], the quadratic version in (1.12), i.e.,

$$\|x\|^2 \|y\|^2 \leq \frac{1}{4} \cdot \frac{|A - a|^2}{\operatorname{Re}(A\bar{a})} |\langle x, y\rangle|^2 \tag{1.17}$$

has the following equivalent additive version [2]

$$(0 \leq) \|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 \leq \frac{1}{4} \cdot \frac{|A - a|^2}{\operatorname{Re}(A\bar{a})} |\langle x, y\rangle|^2 , \tag{1.18}$$

in which $\frac{1}{4}$ is the best possible multiplicative constant.

Finally, if $A, a \in \mathbb{K}$ and $x, y \in H$ satisfy either (1.7) or (1.8) and $A \neq -a$, then we have the inequality [1]

$$\begin{aligned}
&(0 \leq) \|x\| \|y\| - |\langle x, y\rangle| \\
&\leq \|x\| \|y\| - \left| \operatorname{Re}\left[\frac{\bar{A} + \bar{a}}{|A + a|} \langle x, y\rangle\right]\right| \\
&\leq \|x\| \|y\| - \operatorname{Re}\left[\frac{\bar{A} + \bar{a}}{|A + a|} \langle x, y\rangle\right] \\
&\leq \frac{1}{4} \cdot \frac{|A - a|^2}{|A + a|} \|y\|^2 .
\end{aligned} \tag{1.19}$$

The constant $\frac{1}{4}$ is best possible in (1.19).

The main aim of this paper is to provide new reverse inequalities for the Schwarz result. Some applications in relation to the triangle and Bessel inequalities are also provided.

2. Some New Reverses

The following result holds.

THEOREM 1. *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$ and $x,y\in H$, $\rho_1,\rho_2>0$. If*

$$\rho_1 \le \left|\|x\|-\|y\|\right| (\le) \|x-y\| \le \rho_2, \tag{2.1}$$

then we have the inequalities

$$\begin{aligned}(0\le)\ \|x\|\,\|y\|-|\langle x,y\rangle| &\le \|x\|\,\|y\|-|\mathrm{Re}\,\langle x,y\rangle| \\ &\le \|x\|\,\|y\|-\mathrm{Re}\,\langle x,y\rangle \\ &\le \frac{1}{2}\left(\rho_2^2-\rho_1^2\right).\end{aligned} \tag{2.2}$$

The constant $\frac{1}{2}$ is best possible in the sense that it cannot be replaced by a smaller quantity.

Proof. We need only to prove that the last inequality holds and the sharpness of the constant occurs.

Taking the square in the second inequality in (2.1), we have

$$\|x\|^2-2\mathrm{Re}\,\langle x,y\rangle+\|y\|^2\le\rho_2^2.$$

This is equivalent to

$$2\left(\|x\|\,\|y\|-\mathrm{Re}\,\langle x,y\rangle\right)+\left(\|x\|-\|y\|\right)^2\le\rho_2^2. \tag{2.3}$$

Utilising the first inequality in (2.1), we have

$$\rho_1^2\le\left(\|x\|-\|y\|\right)^2. \tag{2.4}$$

Now, making use of (2.3) and (2.4), we deduce (2.2).

To prove the sharpness of the constant $\frac{1}{2}$, let us assume that there exists a constant $C>0$ such that

$$\|x\|\,\|y\|-\mathrm{Re}\,\langle x,y\rangle\le C\left(\rho_2^2-\rho_1^2\right) \tag{2.5}$$

provided the condition (2.1) is satisfied.

Let $e\in H$ with $\|e\|=1$ and for $\rho_2>\rho_1>0$, define

$$x=\frac{\rho_2+\rho_1}{2}\cdot e,\qquad y=\frac{\rho_1-\rho_2}{2}\cdot e. \tag{2.6}$$

Then

$$\|x-y\|=\rho_2 \text{ and } \left|\|x\|-\|y\|\right|=\rho_1,$$

showing that the condition (2.1) is satisfied with equality on both sides.

If we replace in (2.5) the vectors x and y as defined by (2.6), we deduce

$$\frac{\rho_2^2 - \rho_1^2}{2} \geq C\left(\rho_2^2 - \rho_1^2\right),$$

which implies $C \geq \frac{1}{2}$, and the theorem is completely proved. □

REMARK 1. If $\rho_1 = 0$, then (2.2) will produce the known inequality (1.2).

The following corollary holds.

COROLLARY 1. *With the assumptions of Theorem 1, we have the inequality:*

$$(0 \leq)\ \|x\| + \|y\| - \|x+y\| \leq \sqrt{\rho_2^2 - \rho_1^2}. \tag{2.7}$$

Proof. We have, by (2.2), that

$$(\|x\| + \|y\|)^2 - \|x+y\|^2 = 2\left(\|x\|\,\|y\| - \operatorname{Re}\langle x, y\rangle\right) \leq \rho_2^2 - \rho_1^2$$

giving

$$(\|x\| + \|y\|)^2 \leq \|x+y\|^2 + \rho_2^2 - \rho_1^2. \tag{2.8}$$

Taking the square root in (2.8) and taking into account that

$$\sqrt{\alpha + \beta} \leq \sqrt{\alpha} + \sqrt{\beta} \quad \text{for} \quad \alpha, \beta \geq 0,$$

we deduce the desired inequality (2.7). □

If $(H; \langle\cdot,\cdot\rangle)$ is a Hilbert space and $\{e_i\}_{i\in I}$ is an orthornormal family in H, i.e., we recall that $\langle e_i, e_j\rangle = \delta_{ij}$ for any $i, j \in I$, where δ_{ij} is Kronecker's delta, then we have the following inequality which is known widely in the literature as *Bessel's inequality*

$$\sum_{i\in I} |\langle x, e_i\rangle|^2 \leq \|x\|^2 \quad \text{for each} \quad x \in H. \tag{2.9}$$

Here, the meaning of the sum is

$$\sum_{i\in I} |\langle x, e_i\rangle|^2 = \sup_{F\subset I}\left\{\sum_{i\in F} |\langle x, e_i\rangle|^2, \quad F \text{ is a finite part of } I\right\}.$$

The following result provides a reverse of Bessel's inequality.

THEOREM 2. *Let $(H;\langle\cdot,\cdot\rangle)$ be a Hilbert space and $\{e_i\}_{i\in I}$ an orthornormal family in H. If $x\in H\backslash\{0\}$ and for $\rho_2,\rho_1>0$ we have*

$$(2.10)\qquad \rho_1\le\|x\|-\left(\sum_{i\in I}|\langle x,e_i\rangle|^2\right)^{\frac{1}{2}}(\le)\left\|x-\sum_{i\in I}\langle x,e_i\rangle e_i\right\|\le\rho_2,$$

then we have the inequality:

$$(2.11)\qquad (0\le)\,\|x\|-\left(\sum_{i\in I}|\langle x,e_i\rangle|^2\right)^{\frac{1}{2}}\le\frac{1}{2}\cdot\frac{\rho_2^2-\rho_1^2}{\left(\sum_{i\in I}|\langle x,e_i\rangle|^2\right)^{\frac{1}{2}}}.$$

The constant $\frac{1}{2}$ is best possible in (2.11).

Proof. Consider $y:=\sum_{i\in I}\langle x,e_i\rangle e_i$. Obviously, since H is a Hilbert space, $y\in H$. We also note that

$$\|y\|=\left\|\sum_{i\in I}\langle x,e_i\rangle e_i\right\|=\sqrt{\left\|\sum_{i\in I}\langle x,e_i\rangle e_i\right\|^2}=\sqrt{\sum_{i\in I}|\langle x,e_i\rangle|^2},$$

and thus (2.10) is in fact (2.1) of Theorem 1.

Since

$$\begin{aligned}\|x\|\,\|y\|-\operatorname{Re}\langle x,y\rangle&=\|x\|\left(\sum_{i\in I}|\langle x,e_i\rangle|^2\right)^{\frac{1}{2}}-\operatorname{Re}\left\langle x,\sum_{i\in I}\langle x,e_i\rangle e_i\right\rangle\\&=\left(\sum_{i\in I}|\langle x,e_i\rangle|^2\right)^{\frac{1}{2}}\left[\|x\|-\left(\sum_{i\in I}|\langle x,e_i\rangle|^2\right)^{\frac{1}{2}}\right],\end{aligned}$$

then, by (2.2), we deduce the desired result (2.11).

We will prove the sharpness on the constant for the case of one element, i.e., $I=\{1\}$, $e_1=e\in H$, $\|e\|=1$. For this, assume that there exists a constant $D>0$ such that

$$(2.12)\qquad \|x\|-|\langle x,e\rangle|\le D\cdot\frac{\rho_2^2-\rho_1^2}{|\langle x,e\rangle|},$$

provided $x\in H\backslash\{0\}$ satisfies the condition

$$\rho_1\le\|x\|-|\langle x,e\rangle|\,(\le)\,\|x-\langle x,e\rangle e\|\le\rho_2.$$

Assume that $x=\lambda e+\mu f$ with $e,f\in H$, $\|e\|=\|f\|=1$ and $e\perp f$.

We look for positive numbers λ, μ such that

$$\|x - \langle x, e\rangle e\| = \rho_2 > \rho_1 = \|x\| - |\langle x, e\rangle| . \tag{2.13}$$

Since (for $\lambda, \mu > 0$)

$$\|x - \langle x, e\rangle e\| = \mu$$

and

$$\|x\| - |\langle x, e\rangle| = \sqrt{\lambda^2 + \mu^2} - \lambda$$

hence, by (2.13), we get $\mu = \rho_2$ and

$$\sqrt{\lambda^2 + \rho_2^2} - \lambda = \rho_1$$

giving

$$\lambda^2 + \rho_2^2 = (\lambda + \rho_1)^2 = \lambda^2 + 2\lambda\rho_1 + \rho_1^2$$

from where we deduce

$$\lambda = \frac{\rho_2^2 - \rho_1^2}{2\rho_1} > 0.$$

With these values for λ and μ, we have

$$\|x\| - |\langle x, e\rangle| = \rho_1, \qquad |\langle x, e\rangle| = \frac{\rho_2^2 - \rho_1^2}{2\rho_1},$$

and thus, from (2.12), we deduce

$$\rho_1 \leq D \cdot \frac{\rho_2^2 - \rho_1^2}{\frac{\rho_2^2 - \rho_1^2}{2\rho_1}},$$

giving $D \geq \frac{1}{2}$, and the theorem is completely proved. □

The following corollary follows immediately.

COROLLARY 2. *Let $x, y \in H$ with $\langle x, y\rangle \neq 0$ and $\rho_2 \geq \rho_1 > 0$ such that*

$$\|y\| \rho_1 \leq \|x\| \|y\| - |\langle x, y\rangle| (\leq) \left\| \|y\| x - \frac{\langle x, y\rangle}{\|y\|} \cdot y \right\| \leq \rho_2 \|y\| . \tag{2.14}$$

Then the following reverse of Schwarz's inequality is valid:

$$(0 \leq) \|x\| \|y\| - |\langle x, y\rangle| \leq \frac{1}{2} \left(\rho_2^2 - \rho_1^2\right) \frac{\|y\|^2}{|\langle x, y\rangle|}. \tag{2.15}$$

The constant $\frac{1}{2}$ is best possible.

3. Other Reverses Related to the Triangle Inequality

The following result holds.

THEOREM 3. *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space and $R\geq 1$. For $x,y\in H$, the following statements are equivalent:*

(i) *The ensuing reverse of the triangle inequality holds*

$$\|x\|+\|y\|\leq R\|x+y\|; \tag{3.1}$$

(ii) *The subsequent reverse of Schwarz's inequality holds*

$$(0\leq)\ \|x\|\|y\|-\operatorname{Re}\langle x,y\rangle\leq\frac{1}{2}\left(R^2-1\right)\|x+y\|^2. \tag{3.2}$$

The constant $\frac{1}{2}$ is best possible in (3.2).

Proof. Taking the square in (3.1), we have, after some calculation,

$$2\|x\|\|y\|\leq\left(R^2-1\right)\|x\|^2+2R^2\operatorname{Re}\langle x,y\rangle+\left(R^2-1\right)\|y\|^2. \tag{3.3}$$

Subtracting from both sides of (3.3) the quantity $2\operatorname{Re}\langle x,y\rangle$, we get

$$\begin{aligned}2\left(\|x\|\|y\|-\operatorname{Re}\langle x,y\rangle\right)&\leq\left(R^2-1\right)\left[\|x\|^2+2\operatorname{Re}\langle x,y\rangle+\|y\|^2\right]\\&=\left(R^2-1\right)\|x+y\|^2\end{aligned}$$

which is equivalent to (3.2).

To prove the best constant, let us assume that there exists a positive number $E>0$ such that

$$0\leq\|x\|\|y\|-\operatorname{Re}\langle x,y\rangle\leq E\left(R^2-1\right), \tag{3.4}$$

provided x,y,R satisfy (3.1).

Assume $R>1$ and choose $x=\frac{1-R}{2}e$, $y=\frac{1+R}{2}e$ where $e\in H$ with $\|e\|=1$. Then

$$x+y=e,\qquad\frac{\|x\|+\|y\|}{R}=1$$

and thus (3.1) holds with equality.

From (3.4), for the above choices for x and y, we have $\frac{1}{2}\left(R^2-1\right)\leq D\left(R^2-1\right)$, which shows that $D\geq\frac{1}{2}$. □

The following corollary holds.

COROLLARY 3. *Let $x, y, e \in H$ with $\|e\| = 1$ and $R \geq 1$ such that*

$$\|x\| \leq R\mathrm{Re}\langle x, e\rangle \quad \text{and} \quad \|y\| \leq R\mathrm{Re}\langle y, e\rangle . \tag{3.5}$$

Then we have the inequality:

$$(0 \leq) \|x\| \|y\| - \mathrm{Re}\langle x, y\rangle \leq \frac{1}{2}\left(R^2 - 1\right)\|x + y\|^2 . \tag{3.6}$$

Proof. Adding the two inequalities in (3.5), we get

$$\begin{aligned} \|x\| + \|y\| &\leq R\mathrm{Re}\langle x + y, e\rangle \leq R\left|\mathrm{Re}\langle x + y, e\rangle\right| \\ &\leq R\left|\langle x + y, e\rangle\right| \leq R\|x + y\|\|e\| = R\|x + y\| . \end{aligned}$$

Utilising the implication "(i) $\Longrightarrow$ (ii)" of Theorem 3, we deduce the desired inequality (3.6). □

REMARK 2. It is an open problem whether $\frac{1}{2}$ is the best possible constant in (3.6). Preliminary investigations show that $\frac{1}{4}$ may be the candidate for the best constant, but we do not have any analytic proof of this fact.

The following result also holds.

THEOREM 4. *Let $(H; \langle\cdot,\cdot\rangle)$ be an inner product space and $r \in (0, 1]$, $x, y \in H$. The following statements are equivalent*

(i) *We have the reverse of the norm continuity inequality*

$$\left|\|x\| - \|y\|\right| \geq r\|x - y\| ; \tag{3.7}$$

(ii) *We have the subsequent reverse of Schwarz's inequality*

$$0 \leq \|x\| \|y\| - \mathrm{Re}\langle x, y\rangle \leq \frac{1}{2}\left(1 - r^2\right)\|x - y\|^2 . \tag{3.8}$$

The constant $\frac{1}{2}$ is best possible in (3.8).

Proof. Taking the square in (3.7), we have

$$\|x\|^2 - 2\|x\| \|y\| + \|y\|^2 \geq r^2\left[\|x\|^2 - 2\mathrm{Re}\langle x, y\rangle + \|y\|^2\right]$$

which is clearly equivalent to

$$\left(1 - r^2\right)\left[\|x\|^2 - 2\mathrm{Re}\langle x, y\rangle + \|y\|^2\right] \geq 2\left(\|x\| \|y\| - \mathrm{Re}\langle x, y\rangle\right),$$

or with (3.8).

To prove the sharpness of the constant, assume that there exists a positive number F such that

$$0 \leq \|x\| \|y\| - \operatorname{Re}\langle x, y\rangle \leq F\left(1-r^2\right)\|x-y\|^2, \tag{3.9}$$

provided x, y and r satisfy (3.7).

For $r \in (0,1)$, consider the vectors

$$x = \frac{1+r}{2}e, \qquad y = \frac{1-r}{2}e, \quad \text{where } \ e \in H, \ \|e\| = 1.$$

Then

$$|\|x\| - \|y\|| = r, \quad \|x-y\| = 1,$$

and by (3.9), we get

$$\frac{1-r^2}{4} - \frac{r^2-1}{4} \leq F\left(1-r^2\right),$$

from where we deduce $F \geq \frac{1}{2}$. □

4. Discrete Inequalities

Assume that $(K; (\cdot,\cdot))$ is a Hilbert space over the real or complex number field. Assume also that $p_i \geq 0$, $i \in \mathbb{N}$ with $\sum_{i=1}^{\infty} p_i = 1$ and define

$$\ell_p^2(K) := \left\{ \mathbf{x} := (x_i)_{i\in\mathbb{N}} \middle| \, x_i \in \mathbb{K}, \ i \in \mathbb{N} \ \text{ and } \ \sum_{i=1}^{\infty} p_i \|x_i\|^2 < \infty \right\}.$$

It is well known that $\ell_p^2(K)$ endowed with the inner product $\langle\cdot,\cdot\rangle_p$ defined by

$$\langle \mathbf{x}, \mathbf{y}\rangle_p := \sum_{i=1}^{\infty} p_i (x_i, y_i)$$

and generating the norm

$$\|\mathbf{x}\|_p := \left(\sum_{i=1}^{\infty} p_i \|x_i\|^2\right)^{\frac{1}{2}}$$

is a Hilbert space over $\mathbb{K}$.

We may state the following discrete inequality providing a reverse of the Cauchy-Bunyakovsky-Schwarz result.

PROPOSITION 1. *Assume that* $\mathbf{x},\mathbf{y}\in\ell_p^2(K)$ *and* $\rho_2>\rho_1\geq 0$ *such that*

$$\|x_i-y_i\|\leq\rho_2 \quad \text{for each} \quad i\in\mathbb{N} \tag{4.1}$$

and

$$\left|\left(\sum_{i=1}^{\infty}p_i\|x_i\|^2\right)^{\frac{1}{2}}-\left(\sum_{i=1}^{\infty}p_i\|y_i\|^2\right)^{\frac{1}{2}}\right|\geq\rho_1. \tag{4.2}$$

Then we have the inequalities

$$\begin{aligned}(0\leq)&\left(\sum_{i=1}^{\infty}p_i\|x_i\|^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^{\infty}p_i\|y_i\|^2\right)^{\frac{1}{2}}-\left|\sum_{i=1}^{\infty}p_i(x_i,y_i)\right|\\ &\leq\left(\sum_{i=1}^{\infty}p_i\|x_i\|^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^{\infty}p_i\|y_i\|^2\right)^{\frac{1}{2}}-\left|\sum_{i=1}^{\infty}p_i\mathrm{Re}(x_i,y_i)\right|\\ &\leq\left(\sum_{i=1}^{\infty}p_i\|x_i\|^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^{\infty}p_i\|y_i\|^2\right)^{\frac{1}{2}}-\sum_{i=1}^{\infty}p_i\mathrm{Re}(x_i,y_i)\\ &\leq\frac{1}{2}\left(\rho_2^2-\rho_1^2\right).\end{aligned} \tag{4.3}$$

The constant $\frac{1}{2}$ *is best possible in (5.3).*

Proof. From (4.1) we have

$$\|\mathbf{x}-\mathbf{y}\|_p=\left(\sum_{i=1}^{\infty}p_i\|x_i-y_i\|^2\right)^{\frac{1}{2}}\leq\left(\sum_{i=1}^{\infty}p_i\rho_2^2\right)^{\frac{1}{2}}=\rho_2.$$

Since (4.2) is in fact the following inequality

$$\left|\|\mathbf{x}\|_p-\|\mathbf{y}\|_p\right|\geq\rho_1$$

then, on applying Theorem 1 for $\ell_p^2(K)$ and $\langle\cdot,\cdot\rangle_p$, we deduce the desired inequality (4.3). □

COROLLARY 4. *Under the assumptions of Proposition 1, we have*

$$\begin{aligned}(0 \leq) \left(\sum_{i=1}^{\infty} p_i \|x_i\|^2\right)^{\frac{1}{2}} + \left(\sum_{i=1}^{\infty} p_i \|y_i\|^2\right)^{\frac{1}{2}} \\ - \left(\sum_{i=1}^{\infty} p_i \|x_i + y_i\|^2\right)^{\frac{1}{2}} \\ \leq \sqrt{\rho_2^2 - \rho_1^2}.\end{aligned}$$

REMARK 3. Similar results may be stated if one uses the other inequalities obtained in Sections 2 and 3, but we omit the details.

5. Integral Inequalities

Assume that $(K; (\cdot,\cdot))$ is a Hilbert space over the real or complex number field $\mathbb{K}$. If $\rho : [a,b] \subset \mathbb{R} \to [0,\infty)$ is a Lebesgue integrable function with $\int_a^b \rho(t)\,dt = 1$, then we may consider the space $L_\rho^2([a,b];K)$ of all functions $f : [a,b] \to K$, that are Bochner measurable and $\int_a^b \rho(t)\|f(t)\|^2\,dt < \infty$. It is well known that $L_\rho^2([a,b];K)$ endowed with the inner product $\langle\cdot,\cdot\rangle_\rho$ defined by

$$\langle f, g\rangle_\rho := \int_a^b \rho(t)\,(f(t), g(t))\,dt$$

and generating the norm

$$\|f\|_\rho := \left(\int_a^b \rho(t)\|f(t)\|^2\,dt\right)^{\frac{1}{2}}$$

is a Hilbert space over $\mathbb{K}$.

The following reverse of the vector-valued Cauchy-Bunyakovsky-Schwarz integral inequalities holds.

PROPOSITION 2. *Assume that* $f, g \in L_\rho^2([a,b];K)$ *and* $\rho_2 > \rho_1 \geq 0$ *such that*

$$\|f(t) - g(t)\| \leq \rho_2 \quad \textit{for a.e. } t \in [a,b] \tag{5.1}$$

and

$$\left|\left(\int_a^b \rho(t)\|f(t)\|^2 dt\right)^{\frac{1}{2}} - \left(\int_a^b \rho(t)\|g(t)\|^2 dt\right)^{\frac{1}{2}}\right| \geq \rho_1. \tag{5.2}$$

Then we have the inequality

$$\begin{aligned} 0 &\leq \left(\int_a^b \rho(t)\|f(t)\|^2 dt\right)^{\frac{1}{2}}\left(\int_a^b \rho(t)\|g(t)\|^2 dt\right)^{\frac{1}{2}} \\ &\qquad - \left|\int_a^b \rho(t)(f(t), g(t))\, dt\right| \\ &\leq \left(\int_a^b \rho(t)\|f(t)\|^2 dt\right)^{\frac{1}{2}}\left(\int_a^b \rho(t)\|g(t)\|^2 dt\right)^{\frac{1}{2}} \\ &\qquad - \left|\int_a^b \rho(t)\operatorname{Re}(f(t), g(t))\, dt\right| \\ &\leq \left(\int_a^b \rho(t)\|f(t)\|^2 dt\right)^{\frac{1}{2}}\left(\int_a^b \rho(t)\|g(t)\|^2 dt\right)^{\frac{1}{2}} \\ &\qquad - \int_a^b \rho(t)\operatorname{Re}(f(t), g(t))\, dt \\ &\leq \frac{1}{2}\left(\rho_2^2 - \rho_1^2\right). \end{aligned} \tag{5.3}$$

The constant $\frac{1}{2}$ is best possible.

Proof. From (5.1), we have

$$\|f-g\|_\rho = \left(\int_a^b \rho(t)(\|f(t)-g(t)\|)^2 dt\right)^{\frac{1}{2}} \leq \left(\int_a^b \rho(t)\rho_2^2 dt\right)^{\frac{1}{2}} = \rho_2.$$

Since (5.2) is in fact the inequality

$$\left|\|f\|_\rho - \|g\|_\rho\right| \geq \rho_1,$$

then, on applying Theorem 1 for $\ell_p^2(K)$ on $\langle\cdot,\cdot\rangle_\rho$, we deduce the desired inequality (5.3). □

COROLLARY 5. *Under the assumptions of Proposition 2, we have:*

$$(0 \leq) \left(\int_a^b \rho(t) \|f(t)\|^2 dt \right)^{\frac{1}{2}} + \left(\int_a^b \rho(t) \|g(t)\|^2 dt \right)^{\frac{1}{2}} - \left(\int_a^b \rho(t) \|f(t) + g(t)\|^2 dt \right)^{\frac{1}{2}} \leq \sqrt{\rho_2^2 - \rho_1^2}.$$

REMARK 4. Similar results may be stated if one uses the other inequalities obtained in Sections 2 and 3, but we omit the details.

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