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COMMON FIXED POINTS FOR MAPS IN D-METRIC SPACE USING STRONGLY TANGENTIAL AND ORBITALLY LOWER SEMI-CONTINUITY CONDITIONS

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Abstract. In this paper, we obtain some unique common fixed point theorems for four self-maps and pair of maps in D-metric space using orbitally lower semi continuity conditions.

1. Introduction

Generalizing the notion of metric space, Dhage [1] introduced D-metric space and claimed that D-metric is sequentially continuous in all the three variables. Based on this claim so many authors obtained fixed and common fixed point theorems in D-metric space.

Naidu et.al. [2, 3, 4] observed that there are D-metrics which are not continuous even in a single variable and D-convergent sequences may have more than one limit point. Naidu et.al. [4] obtained several modifications of existing theorems in D-metric spaces. Naidu et.al. [5] introduced the concept of orbitally lower semi-continuity in D-metric space for a single self map and obtained some fixed point theorems. In this paper we obtain some unique common fixed point theorems for two and four self maps using orbitally lower semi continuity conditions for strongly tangential maps.

2. Preliminaries

First we give some known definitions.

Definition 2.1. ([1]) Let X be a non empty set and D: $X \times X \times X \to IR^+$ a function satisfying

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1. $D(x,y,z)=0$ if and only if $x=y=z$,
2. $D(x,y,z)=D(p(x,y,z))$ where $p(x,y,z)$is a permutation function of x,y, and z,
3. $D(x,y,z)\leq D(a,y,z)+D(x,a,z)+D(x,y,a)$ $\forall x,y,z,a\in X$.

   Then D is called a D-metric or generalized metric on X and (X,D) is called a D-metric space.

DEFINITION 2.2. ([1]) A sequence $\{x_n\}$ in a D-metric space (X,D) is said to be D-convergent to a point $x\in X$ if

$$\lim_{m,n\to\infty} D(x_m,x_n,x)=0.$$

A sequence $\{x_n\}$ in (X,D) is said to be D-Cauchy if

$$\lim_{m,n,p\to\infty} D(x_m,x_n,x_p)=0.$$

Clearly every D-convergent sequence is a D-Cauchy sequence.

In this paper we obtain unique common fixed point theorems for four self-maps and a pair of Jungck type maps using the following notions in D-metric space.

DEFINITION 2.3. ([6]) Let (X,d) be a metric space and f, g self-maps on X. f and g are said to be tangential if there exits a sequence $\{x_n\}$ in X and $u\in X$ such that $fx_n\to u$, $gx_n\to u$.

DEFINITION 2.4. Let (X,D) be a D-metric space and f, g be self maps on X. We say that f and g be Jungck type strongly tangential if there exists a sequence $\{x_n\}$ in X and $u\in X$ such that $fx_n=gx_{n+1}$, $n=0,1,2,...$ and $\{fx_n\}$ is D-convergent to u.

DEFINITION 2.5. Let f, g, S and T be four self-maps on a D-metric space (X,D). We say that f, g, S and T are strongly tangential if there exist sequences $\{x_n\}$ and $\{y_n\}$ and $u\in X$ such that

$$y_0=Sx_0,\ y_{2n+1}=fx_{2n}=Tx_{2n+1},\ y_{2n+2}=gx_{2n+1}=Sx_{2n+2}$$

for $n=0,1,2,\cdots$ and $\{y_n\}$ is D-convergent to u.

DEFINITION 2.6. ([5]) Let f be a self-map on a D-metric space X,

$$\begin{aligned} G(x) &= \min\{D(x,x,fx),D(x,fx,fx)\},\\ H(x) &= \max\{D(x,x,fx),D(x,fx,fx)\}. \end{aligned}$$

The ordered pair (G,H) is said to be f-orbitally lower semi continuous at $u\in X$ if

$$G(u) \leq \lim_{n\to\infty} H(f^n x)$$

whenever $x \in X$ is such that $\{f^n x\}$ is a D-convergent sequence with u as a limit.

Definition 2.7. Let f and g be self maps on a D-metric space X,

$$\begin{aligned} G(x) &= \min\{D(f_x, fx, gx), D(fx, gx, gx)\}, \\ H(x) &= \max\{D(fx, fx, gx), D(fx, gx, gx)\}. \end{aligned}$$

We say that the ordered pair (G, H) is said to be Jungck type(f, g)-orbitally lower semi continuous at $u \in X$ if

$$G(u) \leq \lim_{n\to\infty} H(x_n)$$

whenever $x \in X$ is such that $fx_n = gx_{n+1}$, $n = 0, 1, 2, ...$and $\{fx_n\}$ is a D-convergent sequence with u as a limit.

Definition 2.8. Let f, g, S and T be four self-maps on a D-metric space X,

$$\begin{aligned} G(x) &= \min\{D(fx, fx, Sx), D(fx, Sx, Sx), D(gx, gx, Tx), D(gx, Tx, Tx)\}, \\ G^*(x) &= \max\{D(fx, fx, Sx), D(fx, Sx, Sx), D(gx, gx, Tx), D(gx, Tx, Tx)\}, \\ H_1(x) &= \max\{D(fx, fx, Sx), D(fx, Sx, Sx)\}, \\ H_2(x) &= \max\{D(gx, gx, Tx), D(gx, Tx, Tx)\}. \end{aligned}$$

We say that (G, H_1, H_2) or (G^*, H_1, H_2) is said to be (f, g, S, T)-orbitally lower semi continuous at $u \in X$ if

$$G(u) \leq \lim_{n\to\infty} \max\{H_1(x_{2n}), H_2(x_{2n+1})\}$$

or

$$G^*(u) \leq \lim_{n\to\infty} \max\{H_1(x_{2n}), H_2(x_{2n+1})\}$$

respectively, whenever there exist sequences $\{x_n\}$and $\{y_n\}$ in X such that

$$y_0 = Sx_0,\ y_{2n+1} = fx_{2n} = Tx_{2n+1},\ y_{2n+2} = gx_{2n+1} = Sx_{2n+2}$$

for $n = 0, 1, 2, \cdots$ and $\{y_n\}$ is D-convergent sequence with u as a limit.

When $S = T = I$ we similarly define (G, H_1, H_2) is (f, g)-orbitally lower semi-continuous.

Definition 2.9. Let f and g be two self-maps on a D-metric space X. f and g are said be partially commuting or coincidentally commuting if $fgu = gfu$ whenever there exists $u \in X$ such that $fu = gu$.

3. Main theorems

Theorem 3.1. *Let f, g, S and T be four self maps on a D-metric space X such that*

1. *f, g, S and T are strongly tangential ,*
2. *(G, H_1, H_2) is (f, g, S, T) - orbitally lower semi continuous on X ,*
3. *$f(X) \subseteq T(X)$, $g(X) \subseteq S(X)$,*
4. *f and S ; g and T are coincidentally commuting ,*
5. *$D(fx, gy, z) < max\{D(Sx, Ty, z), D(fx, Sx, z), D(gy, Ty, z), D(fx, Ty, z), D(gy, Sx, z)\}$ $\forall x, y, z \in X$ with $z \neq fx$ or gy.*

Then f, g, S and T have a unique common fixed point.

Proof. From 3.1 (1) , there exist sequences $\{x_n\}$, $\{y_n\}$ and $u \in X$ such that

$$y_0 = Sx_0,\ y_{2n+1} = fx_{2n} = Tx_{2n+1},\ y_{2n+2} = gx_{2n+1} = Sx_{2n+2}$$

for $n = 0, 1, 2, \cdots$ and $\{y_n\}$is D-convergent to u. Hence $\{y_n\}$ is a D-Cauchy sequence.

From 3.1 (2), we have

$$\begin{aligned}
&\min\{D(fu, fu, Su), D(fu, Su, Su), D(gu, gu, Tu), D(gu, Tu, Tu)\} \\
\leq\ &\lim_{n\to\infty} \max\{max\{D(fx_{2n}, fx_{2n}, Sx_{2n}), D(fx_{2n}, Sx_{2n}, Sx_{2n})\}, \\
&\max\{D(gx_{2n+1}, gx_{2n+1}, Tx_{2n+1}), D(gx_{2n+1}, Tx_{2n+1}, Tx_{2n+1})\} \\
=\ &\lim_{n\to\infty} \max\{\max\{D(y_{2n+1}, y_{2n+1}, y_{2n}), D(y_{2n+1}, y_{2n}, y_{2n})\}, \\
&\max\{D(y_{2n+2}, y_{2n+2}, y_{2n+1}), D(y_{2n+2}, y_{2n+1}, y_{2n+1})\}\} \\
=\ &0 \text{ since } \{y_n\} \text{ is } D-\text{Cauchy}.
\end{aligned}$$

Therefore, $fu = Su$ or $gu = Tu$.

Suppose $fu = Su$. Since $f(X) \subseteq T(X)$, there exists $v \in X$ such that $fu = Tv$.

Suppose $fu \neq gv$. Then from 3.1 (5) we have

$$\begin{aligned}
D(fu, gv, fu)\ <\ &\max\{D(Su, Tv, fu), D(fu, Su, fu), D(gv, Tv, fu), \\
&D(fu, Tv, fu), D(gv, Su, fu)\} = D(fu, gv, fu),
\end{aligned}$$

which is a contradiction. Hence $fu = gv$. Thus $fu = Su = gv = Tv$.

Since the pair f and S is coincidentally commuting, we have $ffu = fSu = Sfu = SSu$. ......(I)

Since the pair g and T is coincidentally commuting, we have $ggv = gTv = Tgv = TTv$. ......(II)

Suppose $fu \neq f^2u$.

$$\begin{aligned} D(fu, fu, f^2u) &= D(fv, gv, f^2u) \\ &< \max\{D(Su, Tv, f^2u), D(fu, Su, f^2u), \\ &\quad D(gv, Tv, f^2u), D(fu, Tv, f^2u), D(gv, fu, f^2u)\} \\ &= D(fu, fu, f^2u), \end{aligned}$$

which is a contradiction. Hence $f^2u = fu$.

Now from (I) fu is a common fixed point of f and S.

Suppose $gv \neq g^2v$. Then

$$\begin{aligned} D(gv, gv, g^2v) &= D(fu, gv, g^2v) \\ &< max\{D(Su, Tv, g^2v), D(fu, Su, g^2v), D(gv, Tv, g^2v), \\ &\quad D(fu, Tv, g^2u), D(gv, fu, g^2v)\} \\ &= D(gv, gv, g^2v), \end{aligned}$$

which is a contradiction. Hence $g^2v = gv$.

Now from (II) gv is a common fixed point of g and T.

Since $fu = gv$, it follows that fu or gv is a common fixed point of f, g, S and T.

Uniqueness of common fixed point follows easily by applying 3.1(5) two times.

Similarly we can prove if $gu = Tu$. □

THEOREM 3.2. *Let f, g, S and T be four self-maps on a D-metric space X such that*

1. *f, g, S and T are strongly tangential ,*
2. *(G^*, H_1, H_2) is (f, g, S, T)- orbitally lower semi continuous on X ,*
3. *$D(fx, gy, z) < \max\{D(Sx, Ty, z), D(fx, Sx, z), D(gy, Ty, z), D(fx, Ty, z), D(gy, Sx, z)\}$ $\forall x, y, z \in X$ with $z \neq fx$ or gy.*

Then f, g, S and T have a unique common fixed point.

Proof. From 3.2 (1) and 3.2 (2) there exists $u \in X$ such that$fu = Su$ and

$gu = Tu$.

Write $w_1 = fu = Su$, $w_2 = gu = Tu$.

Suppose $w_1 \neq w_2$.

$$D(w_1, w_2, w_1) = D(fu, gu, w_1)$$

$$\begin{aligned}
&< \max\{D(Su,Tu,w_1),D(fu,Su,w_1),D(gu,Tu,w_1),\\
&\quad D(fu,Tu,w_1),D(gu,Su,w_1)\}\\
&= \max\{D(w_1,w_2,w_1),D(w_2,w_2,w_1)\}
\end{aligned}$$

Also

$$\begin{aligned}
D(w_1,w_2,w_2) &= D(fu,gu,w_2)\\
&< \max\{D(Su,Tu,w_2),D(fu,Su,w_2),D(gu,Tu,w_2),\\
&\quad D(fu,Tu,w_2),D(gu,Su,w_2)\} = D(w_1,w_1,w_2).
\end{aligned}$$

Hence

$$D(w_1,w_2,w_1) < D(w_2,w_2,w_1) < D(w_1,w_1,w_1).$$

Therefore, $w_1 = w_2$. Thus $fu = Su = gu = Tu$.
Suppose $fu \neq u$.

$$\begin{aligned}
D(fu,fu,u) &= D(fu,gu,u)\\
&< \max\{D(Su,Tu,u),D(fu,Su,u),D(gu,Tu,u),\\
&\quad D(fu,Tu,u),D(gu,Su,u)\}\\
&= D(fu,fu,u),
\end{aligned}$$

which is a contradiction. Hence $fu = u$. Thus u is a common fixed point of f, g, S and T.

Uniqueness of common fixed point follows easily from 3.2(3). □

COROLLARY 3.3. *Let f and g be two self maps on a D-metric space X such that*

1. *f and g are strongly tangential,*
2. *(G,H_1,H_2) is (f,g) - orbitally lower semi continuous on X,*
3. *$D(fx,gy,z) < \max\{D(x,y,z),D(x,fx,z),D(y,gy,z), D(x,gy,z),D(y,fx,z)\}$ $\forall x,\ y,\ z \in X$ with $z \neq fx$ or gy.*

Then f and g have a unique common fixed point.

Proof. It follows from Theorem 3.1 with $S = T = I$. □

COROLLARY 3.4. *Let f and S be two self maps on aD-metric space X such that*

1. *f and S are Jungck type strongly tangential,*
2. *The order pair (G,H) is Jungck type (f,S)-orbitally lower semi-continuous on X,*
3. *$D^r(fx,fy,z) < max\{D^r(Sx,Sy,z),D^r(fx,Sx,z), D^r(fy,Sy,z),D^r(fx,Sy,z),D^r(fy,Sx,z)\}$ $\forall x,\ y,\ z \in X$ with $z \neq fx$or fy , where r is a positive integer.*

Then f and S have a unique common fixed point.

Proof. From 3.4(1), 3.4(2) there exists $u \in X$ such that $fu = Su$.
Suppose $fu \neq u$.

$$\begin{aligned} D^r(fu, fu, u) \; &< \; \max\{D^r(Su, Su, u), D^r(fu, Su, u), D^r(fu, Su, u), \\ &\quad D^r(fu, Su, u), D^r(fu, Su, u)\} = D^r(fu, fu, u), \end{aligned}$$

which is a contradiction. Hence fu $= u$. Thus u is a common fixed point of f and S.

Suppose z_1 , z_2 are two common fixed points of f and S such that $z_1 \neq z_2$.

$$\begin{aligned} D^r(z_1, z_1, z_2) \; &= \; D^r(fz_1, fz_1, z_2) \\ &< \; \max\{D^r(Sz_1, Sz_1, z_2), D^r(fz_1, Sz_1, z_2), D^r(fz_1, Sz_1, z_2), \\ &\quad D^r(fz_1, Sz_1, z_2), D^r(fz_1, Sz_1, z_2)\} \\ &= \; D^r(z_1, z_1, z_2), \end{aligned}$$

which is a contradiction. Hence $z_1 = z_2$. Thus f and S have a unique common fixed point. □

Finally, we prove the following.

THEOREM 3.5. *Let f and S be two self maps on a D-metric space X such that*

1. *f and S are Jungck type strongly tangential ,*
2. *The order pair (G, H) is Jungck type (f, S)-orbitally lower semi continuous on X,*
3. *f and S are coincidentally commuting ,*
4.

$$\begin{aligned} D^r(fx, fy, fz) < \max\{&D^r(Sx, Sy, Sz), D^r(Sx, fx, Sz), D^r(Sy, fy, Sz), \\ &D^r(Sx, fy, Sz), D^r(Sy, fx, Sz)\} \; \forall x, y, z \in X with fz \neq f x or f y. \end{aligned}$$

Then f and S have a unique common fixed point.

Proof. From 3.5(1) and 3.5(2) there exists $u \in X$ such that $fu = Su$.
From 3.5(3), we have $f(fu) = fSu = Sfu = S(Su)$ ——- (i)
Now from 3.5(4) we have

$$\begin{aligned} D^r(f^2u, f^2u, fu) < max\{&D^r(Sfu, Sfu, Su), D^r(Sfu, f^2u, Su), \\ &D^r(Sfu, ffu, Su), D^r(Sfu, f^2u, Su), D^r(Sfu, f^2u, Su)\} \\ &= D^r(f^2u, f^2u, fu), \end{aligned}$$

which is a contradiction. Hence $f^2u = fu$.

From (i), $S(fu) = f^2u = fu$. Thus fu is a common fixed point of f and S.

Uniqueness of common fixed point of f and S follows easily from 3.5(4). $\square$

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