JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **19**, No.3, September 2006

ON STABILITY OF THE ORTHOGONALLY CUBIC TYPE FUNCTIONAL EQUATION

Ick-Soon Chang*

Abstract. In this article, we establish the stability of the orthogonally cubic type functional equation $2f(x+2y)+2f(x-2y)+2f(2x)+7[f(x)+f(-x)]=4f(x)+8[f(x+y)+f(x-y)],\ x\perp y$ in which $\perp$ is the orthogonality in the sense in the Rätz.

1. Introduction

In 1940, S. M. Ulam [11] proposed the following question concerning the stability of group homomorphisms: *Under what condtion does there is an additive mapping near an approximately additive mapping between a group and a metric group?*

In next year, D. H. Hyers [5] answers the problem of Ulam under the assumption that the groups are Banach spaces. A generalized version of the theorem of Hyers for approximately linear mappings was given by Th. M. Rassias [8]. Since then, the stability problems of various functional equation have been extensively investigated by a number of authors.

The cubic function $f(x)=ax^3$ satisfies the functional equation

$$(1)\quad f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x).$$

Hence, throughout this paper, we promise that the equation (1) is called a cubic functional equation and every solution of the equation (1) is said to be a cubic function. The functional equation (1) was solved by Jun and Kim [6]. Moreover, they investigated the Hyers-Ulam-Rassias stability for the functional equation (1).

Received July 24, 2006.
2000 Mathematics Subject Classification: Primary 39B72, 39B52.
Key words and phrases: cubic function, orthogonally cubic function.

Now we introduced the cubic type functional equation as follows:

$$2f(x+2y)+2f(x-2y)+2f(2x)+7[f(x)+f(-x)] \\ =4f(x)+8[f(x+y)+f(x-y)]. \tag{2}$$

It is easy to see that the function $f(x) = ax^3 + b$ is a solution of the functional equation (2). The main goal of this note is to offer the stability of the orthogonally cubic type functional equation (2) for all x, y with $x \perp y$, where $\perp$ is the orthogonality in the sense of Rätz.

2. Stability of Eq. (2)

Let us recall the orthogonality in the sense of J. Rätz [9]; Suppose that X is a real vector space with dim $X \geq 2$ and $\perp$ is a binary relation on X with the following properties:

(O1) totality of $\perp$ for zero: $x \perp 0$, $0 \perp x$ for all $x \in X$;
(O2) independence: if $x \in X - \{0\}$, $x \perp y$, then x, y are linearly independent;
(O3) homogeneity: if $x \in X$, $x \perp y$, then $\alpha x \perp \beta y$ for all $\alpha, \beta \in \mathbb{R}$;
(O4) the Thalesian property: if P is a 2-dimensional subspace of X, $x \in P$ and $\lambda \in \mathbb{R}_+$, then there exists $y_0 \in P$ such that $x \perp y_0$ and $x + y_0 \perp \lambda x - y_0$.

The pair $(X, \perp)$ is called an orthogonality space. By an orthogonality normed space we mean an orthogonality space having a normed structure.

Definition 2.1. Let X and Y be an orthogonality and a real vector space. A mapping $f : X \to Y$ is said to orthogonally cubic if it satisfies the so-called orthogonally cubic functional equation (1) for all $x, y \in X$ with $x \perp y$.

Lemma 2.1. *Let X and Y be an orthogonality and a real vector space. If a function $f : X \to Y$ satisfies the functional equation (2) for all $x \in X$ with $x \perp y$, if and only if C is orthogonally cubic, where $C : X \to Y$ is a function defined by $C(x) = f(x) - f(0)$ for all $x \in X$.*

Proof. (*Necessity.*) From the assumption, it follows that

$$C(x+2y)+C(x-2y)+C(2x)+\frac{7[C(x)+C(-x)]}{2} \\ =2C(x)+4[C(x+y)+C(x-y)]. \tag{3}$$

for all $x, y \in X$ with $x \perp y$. In particular, $C(0) = 0$. Observe that $x \perp 0$ for all $x \in X$. Putting $x = 0$ in (3), we arrive at

$$C(2y) + C(-2y) = 4[C(y) + C(-y)]. \tag{4}$$

Letting $y = 0$ in (3) gives the equation

$$C(2x) = 8C(x) - \frac{7[C(x) + C(-x)]}{2}. \tag{5}$$

Let us replace x by $-x$ in (5) and then we get

$$C(-2x) = 8C(-x) - \frac{7[C(x) + C(-x)]}{2}. \tag{6}$$

By adding (5) and (6), we find that

$$C(2x) + C(-2x) = C(x) + C(-x)$$

and by comparing with (4), $C(-x) = -C(x)$. Therefore (3) now becomes

$$C(x + 2y) + C(x - 2y) + C(2x) = 2C(x) + 4[C(x + y) + C(x - y)]. \tag{7}$$

for all $x, y \in X$ with $x \perp y$. Setting $y = 0$ in (7) leads to the identity $C(2x) = 8C(x)$. If $y \perp x$, then by (O3) $y \perp 2x$. By replacing x by $2x$ in (7), then we see that C is orthogonally cubic.

(*Sufficiency.*) Suppose that C is orthogonally cubic, i.e.,

$$C(2x + y) + C(2x - y) = 2C(x + y) + 2C(x - y) + 12C(x) \tag{8}$$

for all $x, y \in X$ with $x \perp y$. Note that that $x \perp 0$ for all $x \in X$. If we take $x = y = 0$ in (8), then it is clear that $C(0) = 0$. Setting $x = 0$ in (8) yields to $C(-y) = -C(y)$ and letting $y = 0$ in (8), we obtain that $C(2x) = 8C(x)$. If $x \perp y$, then by (O3) $x \perp 2y$. Replacing y by $2y$ in (8), we have

$$C(x + 2y) + C(x - 2y) + C(2x) = 2C(x) + 4[C(x + y) + C(x - y)]$$

Since C is an odd function, (3) holds for all $x, y \in X$ with $x \perp y$. So we see that a function f satisfies the functional equation (2) for all $x, y \in X$ with $x \perp y$. The proof of Lemma is complete. □

From now on, let X be an orthogonality normed space and Y be a Banach space. Given a mapping $f : X \to Y$, we set

$$Df(x, y) := 2f(x + 2y) + 2f(x - 2y) + 2f(2x) + 7[f(x) + f(-x)] -4f(x) - 8[f(x + y) + f(x - y)].$$

THEOREM 2.2. *Suppose that $f : X \to Y$ is a mapping for which there exists a function $\phi : X^2 \to [0, \infty)$ such that $\sum_{i=0}^{\infty} \frac{\phi(2^i x, 2^i y)}{8^i}$ converges and*

$$\|Df(x,y)\| \leq \delta + \phi(x,y) \tag{9}$$

for all $x, y \in X$ with $x \perp y$, where $\delta \geq 0$. Then there exists a unique orthogonally cubic function $C : X \to Y$ satisfying the inequality

$$\|f(x) - C(x)\| \leq \frac{1}{8}\Big[\sum_{i=0}^{\infty} \frac{1}{8^i}\Big(\frac{9\delta}{4} + \frac{13\phi(2^i x, 0)}{12} + \frac{7[\phi(-2^i x, 0) + \phi(0, 2^i x)]}{12}\Big)\Big] + \|f(0)\| \tag{10}$$

for all $x \in X$.

Proof. Let F be a function on X defined by $F(x) = f(x) - f(0)$ for all $x \in X$. Then $F(0) = 0$. Note that $x \perp 0$ for all $x \in X$. Substitution of $x = 0$ in (9) yields

$$\|F(2y) + F(-2y) - 4[F(y) + F(-y)]\| \leq \frac{\delta + \phi(0,y)}{2}. \tag{11}$$

Next, we let $y = 0$ in (9) to obtain

$$\Big\|F(2x) + \frac{7[F(x) + F(-x)]}{2} - 8F(x)\Big\| \leq \frac{\delta + \phi(x,0)}{2}. \tag{12}$$

Interchanging x with $-x$ in (12), we get

$$\Big\|F(-2x) + \frac{7[F(x) + F(-x)]}{2} - 8F(-x)\Big\| \leq \frac{\delta + \phi(-x,0)}{2}. \tag{13}$$

It follows from (12) and (13) that

$$\|F(2x) + F(-2x) - [F(x) + F(-x)]\| \leq \delta + \frac{\phi(x,0) + \phi(-x,0)}{2}. \tag{14}$$

Combining (11) and (14), we see that

$$\|F(x) + F(-x)\| \leq \frac{\delta}{2} + \frac{\phi(x,0) + \phi(-x,0) + \phi(0,x)}{6}. \tag{15}$$

Thus, using (12) and (15), we find that

$$\Big\|F(x) - \frac{F(2x)}{8}\Big\| \leq \frac{1}{8}\Big[\frac{9\delta}{4} + \frac{13\phi(x,0)}{12} + \frac{7[\phi(-x,0) + \phi(0,x)]}{12}\Big]. \tag{16}$$

By replacing x by $2x$ in (16) and dividing 8 and summing the resulting inequality with (16), then we get

$$(17)\quad \Big\|F(x)-\frac{F(2^2x)}{8^2}\Big\| \leq \frac{1}{8}\Big[\frac{9\delta}{4}+\frac{13\phi(x,0)}{12}+\frac{7[\phi(-x,0)+\phi(0,x)]}{12}\Big] +\frac{1}{8^2}\Big[\frac{9\delta}{4}+\frac{13\phi(2x,0)}{12}+\frac{7[\phi(-2x,0)+\phi(0,2x)]}{12}\Big]$$

An induction implies that

$$(18)\quad \Big\|F(x)-\frac{F(2^nx)}{8^n}\Big\| \leq \frac{1}{8}\sum_{i=0}^{n-1}\frac{1}{8^i}\Big[\frac{9\delta}{4}+\frac{13\phi(2^ix,0)}{12} +\frac{7[\phi(-2^ix,0)+\phi(0,2^ix)]}{12}\Big].$$

In order to prove convergence of the sequence $\{\frac{F(2^nx)}{8^n}\}$, we divide inequality (18) by 8^m and also replace x by 2^mx to find that for $n > m > 0$,

$$(19)\quad \Big\|\frac{F(2^mx)}{8^m}-\frac{F(2^n2^mx)}{8^{n+m}}\Big\| \leq \frac{1}{8^{m+1}}\sum_{i=0}^{n-1}\frac{1}{8^i}\Big[\frac{9\delta}{4}+\frac{13\phi(2^{m+i}x,0)}{12} +\frac{7[\phi(-2^{m+i}x,0)+\phi(0,2^{m+i}x)]}{12}\Big].$$

Sine the right-hand side of the inequality (19) tends to 0 as $m\to\infty$, $\{\frac{F(2^nx)}{8^n}\}$ is Cauchy sequence. Therefore, we may define a function $C : X\to Y$ by $C(x) := \lim_{n\to\infty}\frac{F(2^nx)}{8^n}$ for all $x\in X$. By letting $n\to\infty$ in (18), we arrive at the formula (10).

Now we show that C satisfies the functional equation (2) for all $x, y\in X$ with $x\perp y$: If $x\perp y$, then by (O3) $2^nx\perp 2^ny$. Let us replace x and y by 2^nx and 2^ny in (9) and divide by 8^n. Then it follows that

$$DC(x,y)=\lim_{n\to\infty}\frac{\|DF(2^nx,2^ny)\|}{8^n}\leq \lim_{n\to\infty}\frac{\delta+\phi(2^nx,2^ny)}{8^n}=0.$$

Hence we obtain the desired result. Since $C(0) = 0$, the lemma 2.1 implies that C is an orthogonally cubic.

It only remains to claim that C is unique: Let us assume that there exists an orthogonally cubic function T which satisfies (2) and the inequality (10). It is clear that $C(2^nx) = 8^nC(x)$ and $T(2^nx) = 8^nT(x)$

for all $x \in X$ and $n \in \mathbb{N}$. Hence it follows from (10) that

$$\begin{aligned}\|C(x) - T(x)\| &= \frac{\|C(2^n x) - T(2^n x)\|}{8^n} \\ &\le \frac{1}{8^n}\Big[\|C(2^n x) - f(2^n x)\| + \|f(2^n x) - T(2^n x)\|\Big] \\ &\le \frac{1}{8^n}\Big\{\frac{1}{4}\Big[\sum_{i=0}^{\infty}\frac{1}{8^i}\Big(\frac{9\delta}{4} + \frac{13\phi(2^i x, 0)}{12} + \frac{7[\phi(-2^i x, 0) + \phi(0, 2^i x)]}{12}\Big)\Big] \\ &\quad +2\|f(0)\|\Big\}.\end{aligned}$$

By letting $n \to \infty$, then we have $C(x) = T(x)$ for all $x \in X$, which completes the proof of the theorem. □

Corollary 2.3. *Let $p, q, \delta, \varepsilon_1$ and ε_2 be nonnegative real numbers with $p < 3$ and $q < 3$. Suppose that $f : X \to Y$ is a mapping such that*

$$\|Df(x, y)\| \le \delta + \varepsilon_1\|x\|^p + \varepsilon_2\|y\|^q$$

for all $x, y \in X$ with $x \perp y$. Then there exists a unique orthogonally cubic function $C : X \to Y$ satisfying the inequality

$$\|f(x) - C(x)\| \le \frac{9\delta}{28} + \frac{1}{8 - 2^p}\Big[\frac{5}{3}\varepsilon_1\|x\|^p + \frac{7}{12}\varepsilon_2\|x\|^q\Big]$$

for all $x \in X$.

Proof. Considering $\phi(x, y) = \varepsilon_1\|x\|^p + \varepsilon_2\|y\|^q$ in the theorem 2.2, we arrive at the conclusion of the corollary. □

References

[1] J. Aczél and J. Dhombres, *Functional Equations in Several Variables*, Cambridge Univ. Press, 1989.

[2] I.-S. Chang, K.-W. Jun and Y.-S. Jung, *The modified Hyers-Ulam-Rassias stability of a cubic type functional equation*, Math. Ineq. Appl. 8(4) (2005), 675-683.

[3] P. W. Cholewa, *Remarks on the stability of functional equations*, Aequationes Math. 27 (1984), 76-86.

[4] S. Czerwik, *On the stability of the quadratic mapping in normed spaces*, Abh. Math. Sem. Univ. Hamburg 62 (1992), 59-64.

[5] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. 27 (1941), 222-224.

[6] K.-W. Jun and H.-M. Kim, *The generalized Hyers-Ulam-Rassias stability of a cubic functional equation*, J. Math. Anal. Appl. 274(2) (2002), 867-878.

[7] M. S. Moslehian, *On stability of the orthogonal pexiderized Cauchy equation*, J. Math. Anal. Appl. 318 (1) (2006), 211-223.

[8] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. 72 (1978), 297-300.

[9] J. Rätz, *On orthogonally additive mappings*, Aequationes Math. 28 (1985), 35-49.
[10] F. Skof, *Proprietà locali e approssimazione di operatori*, Rend. Sem. Mat. Fis. Milano 53 (1983), 113-129.
[11] S. M. Ulam, *Problems in Modern Mathematics*, (1960) Chap. VI, Science ed., Wiley, New York.

*
Department of Mathematics
Mokwon University
Taejon 305-729, Republic of Korea
E-mail: ischang@mokwon.ac.kr