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APPROXIMATE RING HOMOMORPHISMS OVER p-ADIC FIELDS

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Abstract. In this paper, we prove the generalized Hyers–Ulam stability of ring homomorphisms over the p-adic field $\mathbb{Q}_p$ associated with the Cauchy functional equation $f(x+y) = f(x)+f(y)$ and the Cauchy–Jensen functional equation $2f(\frac{x+y}{2} + z) = f(x) + f(y) + 2f(z)$.

1. Introduction and preliminaries

In [9], Hensel introduced the concept of p-adic numbers as a tool for solving problems in algebra and number theory. His idea was to extend the analogies between the ring of integers $\mathbb{Z}$ and the field of rational numbers $\mathbb{Q}$ to the field of rational functions and Laurent series. The way this was accomplished was by expressing any rational number $x \in \mathbb{Q}$ as the sum

$$x = \sum_{n \geq n_0}^{\infty} a_n p^n,$$

where p is a prime number and $n_0, a_n \in \mathbb{Z} \quad (a_n \leq p-1)$. For a fixed value of p, we denote by $\mathbb{Q}_p$ the complete field of p-adic numbers (see [8]).

In 1940, S.M. Ulam [41] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms.

We are given a group G and a metric group G' with metric $\rho(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : G \to G'$ satisfies

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$\rho(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, then a homomorphism $h : G \to G'$ exists with $\rho(f(x), h(x)) < \epsilon$ for all $x \in G$?

By now an affirmative answer has been given in several cases, and some interesting variations of the problem have also been investigated. We shall call such an $f : G \to G'$ an *approximate homomorphism.*

In 1941, D.H. Hyers [10] considered the case of approximately additive mappings $f : E \to E'$, where E and E' are Banach spaces and f satisfies *Hyers inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon$$

for all $x, y \in E$. It was shown that the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and that $L : E \to E'$ is the unique additive mapping satisfying

$$\|f(x) - L(x)\| \le \epsilon.$$

No continuity conditions are required for this result, but if $f(tx)$ is continuous in the real variable t for each fixed $x \in E$, then L is linear, and if f is continuous at a single point of E then $L : E \to E'$ is also continuous.

In 1978, Th.M. Rassias [32] provided a generalization of Hyers' Theorem which allows the *Cauchy difference to be unbounded.*

Theorem 1.1. (Th.M. Rassias). *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon(\|x\|^p + \|y\|^p) \tag{1.1}$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \le \frac{2\epsilon}{2 - 2^p}\|x\|^p \tag{1.2}$$

for all $x \in E$. If $p < 0$ then inequality (1.1) holds for $x, y \neq 0$ and (1.2) for $x \neq 0$.

In 1990, Th.M. Rassias [33] during the 27^{th} International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geq 1$. In 1991, Z. Gajda [6] following the same approach as in Th.M. Rassias [32], gave an affirmative solution to this question for $p > 1$. It was shown by Z. Gajda [6], as well as by Th.M. Rassias and P. Šemrl [38] that one cannot prove a Th.M. Rassias' type Theorem when $p = 1$. The counterexamples of Z. Gajda [6], as well as of Th.M. Rassias and P. Šemrl [38] have stimulated several mathematicians to invent new definitions of *approximately additive* or *approximately linear* mappings, cf. P. Găvruta [7], S. Czerwik [3], S. Jung [17], who among others studied the Hyers–Ulam stability of functional equations. The inequality (1.1) that was introduced for the first time by Th.M. Rassias [32] provided a lot of influence in the development of a generalization of the Hyers–Ulam stability concept. This new concept is known as *generalized Hyers–Ulam stability* of functional equations (cf. the books of P. Czerwik [4], D.H. Hyers, G. Isac and Th.M. Rassias [11], S. Jung [18]).

Beginning around the year 1980 the topic of approximate homomorphisms and their stability theory in the field of functional equations and inequalities was taken up by several mathematicians (cf. D.H. Hyers and Th.M. Rassias [13], Th.M. Rassias [36] and the references therein).

J.M. Rassias [28] following the spirit of the innovative approach of Th.M. Rassias [32] for the unbounded Cauchy difference proved a similar stability theorem in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p \cdot \|y\|^q$ for $p, q \in \mathbb{R}$ with $p + q \neq 1$ (see also [29] for a number of other new results).

P. Găvruta [7] provided a further generalization of Th.M. Rassias' Theorem. In 1996, G. Isac and Th.M. Rassias [14] applied the generalized Hyers–Ulam stability theory to prove fixed point theorems and study some new applications in Nonlinear Analysis. In [12], D.H. Hyers, G. Isac and Th.M. Rassias studied the asymptoticity aspect of Hyers–Ulam stability of mappings. In [26], the author introduced the Cauchy–Jensen functional equation and proved the generalized Hyers–Ulam stability of the Cauchy–Jensen functional equation in Banach spaces. Several papers have been published on various generalizations and applications of Hyers–Ulam stability and generalized Hyers–Ulam stability to a number of functional equations and mappings, for example: quadratic functional equation, invariant means, multiplicative mappings - superstability, bounded nth differences, convex functions, generalized orthogonality functional equation, Euler–Lagrange functional equation, Navier–Stokes equations. Several mathematician have contributed works on these subjects; we mention a few: M. Amyari and M.S. Moslehian [1], L.M. Arriola and W.A. Beyer [2], K. Jun and H. Kim [15, 16], C. Park [22], C. Park, J. Park and J. Shin [27], F. Skof [40].

Everett and Ulam [5] presented results on generalizing Lorentz groups over p-adic fields. p-adic fields have become of considerable interest to physicists. A key property of p-adic fields is that they do not satisfy the Archimedean axiom; for all $a, b > 0$, there exists an integer n such that $a < nb$. This property has been found to be useful in theoretical physics. In quantum mechanics [20, 21], it has been recognized that fundamental limitations on measuring conjugate quantities such as position-momentum or energy-time exist because of the Heisenberg uncertainty principle. For example, any attempt at taking gravitational measurements at sub-Planck domains, say, of the order of $l = 10^{-35}m$, would change the underlying geometry and introduce distortions to l. Introducing a p-adic space-time could provide a means of quantifying the non-localization affects.

We recall some definitions and results that will be needed later.

Definition 1.2. (Non-Archimedean Valuation) Let $\mathbb{K}$ denote a scalar field, and $|\cdot|$ denote the usual absolute value, where $|\cdot| : \mathbb{K} \to \mathbb{R}$. A non-Archimedean valuation is a function $|\cdot|_p$ that satisfies the strong triangle inequality; namely,

$$|x + y|_p \leq \max\{|x|_p, |y|_p\} \leq |x|_p + |y|_p$$

for all $x, y \in \mathbb{K}$. The associated field $\mathbb{K}$ is referred to as a non-Archimedean field.

Lemma 1.3. [8] *For any nonzero rational number x, there exists a unique integer $n \in \mathbb{Z}$ such that $x = \frac{a}{b}p^n$, where a and b are integers not divisible by p. The p-adic valuation is defined by $|x|_p := p^{-n}$.*

Definition 1.4. (p-adic Field) For each prime p, define the p-adic field $\mathbb{Q}_p$ to be the set of all p-adic expansions $\mathbb{Q}_p := \{x \mid x = \sum_{k \geq n_0}^{\infty} a_k p^k\}$, where $a_k \leq p - 1$ are integers.

Throughout this paper, assume that B is a real Banach algebra with norm $\|\cdot\|$.

In this paper, we prove the generalized Hyers–Ulam stability of ring homomorphisms over the p-adic fields $\mathbb{Q}_p$ associated with the Cauchy functional equation and the Cauchy–Jensen functional equation.

2. Stability of ring homomorphisms over the p-adic field $\mathbb{Q}_p$ associated with the Cauchy functional equation

In this section, we prove the generalized Hyers–Ulam stability of ring homomorphisms over the p-adic field $\mathbb{Q}_p$ associated with the Cauchy functional equation.

THEOREM 2.1. *Let $r < 1$ be a nonnegative real number and $f : \mathbb{Q}_p \to B$ a mapping such that*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(|x|_p^r + |y|_p^r), \tag{2.1}$$

$$\|f(xy) - f(x)f(y)\| \leq \theta(|x|_p^r + |y|_p^r) \tag{2.2}$$

for all $x, y \in \mathbb{Q}_p$. Then there exists a unique ring homomorphism $H : \mathbb{Q}_p \to B$ such that

$$\|f(x) - H(x)\| \leq \frac{2\theta}{2-2^r}|x|_p^r \tag{2.3}$$

for all $x \in \mathbb{Q}_p$.

Proof. Letting $y = x$ in (2.1), we get

$$\|f(2x) - 2f(x)\| \leq 2\theta|x|_p^r$$

for all $x \in \mathbb{Q}_p$. So

$$\|f(x) - \frac{1}{2}f(2x)\| \leq \theta|x|_p^r$$

for all $x \in \mathbb{Q}_p$. Hence

$$\|\frac{1}{2^l}f(2^l x) - \frac{1}{2^m}f(2^m x)\| \leq \sum_{j=l}^{m-1} \frac{2^{rj}\theta}{2^j}|x|_p^r \tag{2.4}$$

for all nonnegative integers m and l with $m > l$ and all $x \in \mathbb{Q}_p$. It follows from (2.4) that the sequence $\{\frac{1}{2^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in \mathbb{Q}_p$. Since B is complete, the sequence $\{\frac{1}{2^n}f(2^n x)\}$ converges. So one can define the mapping $H : \mathbb{Q}_p \to B$ by

$$H(x) := \lim_{n\to\infty} \frac{1}{2^n}f(2^n x)$$

for all $x \in \mathbb{Q}_p$.

By (2.1),

$$\begin{aligned}\|H(x+y) - H(x) - H(y)\| &= \lim_{n\to\infty} \frac{1}{2^n}\|f(2^n x + 2^n y) - f(2^n x) - f(2^n y)\| \\ &\leq \lim_{n\to\infty} \frac{2^{nr}}{2^n}\theta(|x|_p^r + |y|_p^r) = 0\end{aligned}$$

for all $x, y \in \mathbb{Q}_p$. So

$$H(x+y) = H(x) + H(y)$$

for all $x, y \in \mathbb{Q}_p$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.4), we get (2.3).

Now, let $T : \mathbb{Q}_p \to B$ be another Cauchy additive mapping satisfying (2.3). Then we have

$$\begin{aligned} \|H(x) - T(x)\| &= \frac{1}{2^n}\|H(2^n x) - T(2^n x)\| \\ &\leq \frac{1}{2^n}(\|H(2^n x) - f(2^n x)\| + \|T(2^n x) - f(2^n x)\|) \\ &\leq \frac{4 \cdot 2^{nr}\theta}{(2-2^r)2^n}|x|_p^r, \end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in \mathbb{Q}_p$. So we can conclude that $H(x) = T(x)$ for all $x \in \mathbb{Q}_p$. This proves the uniqueness of H.

It follows from (2.2) that

$$\begin{aligned} \|H(xy) - H(x)H(y)\| &= \lim_{n\to\infty} \frac{1}{4^n}\|f(4^n xy) - f(2^n x)f(2^n y)\| \\ &\leq \lim_{n\to\infty} \frac{2^{nr}}{4^n}\theta(|x|_p^r + |y|_p^r) = 0 \end{aligned}$$

for all $x, y \in \mathbb{Q}_p$.

Therefore, there exists a unique ring homomorphism $H : \mathbb{Q}_p \to B$ satisfying (2.3), as desired. □

Theorem 2.2. *Let $r < \frac{1}{2}$ be a nonnegative real number and $f : \mathbb{Q}_p \to B$ a mapping such that*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta \cdot |x|_p^r \cdot |y|_p^r, \tag{2.5}$$

$$\|f(xy) - f(x)f(y)\| \leq \theta \cdot |x|_p^r \cdot |y|_p^r \tag{2.6}$$

for all $x, y \in \mathbb{Q}_p$. Then there exists a unique ring homomorphism $H : \mathbb{Q}_p \to B$ such that

$$\|f(x) - H(x)\| \leq \frac{\theta}{2-4^r}|x|_p^{2r} \tag{2.7}$$

for all $x \in \mathbb{Q}_p$.

Proof. Letting $y = x$ in (2.5), we get

$$\|f(2x) - 2f(x)\| \leq \theta|x|_p^{2r}$$

for all $x \in \mathbb{Q}_p$. So

$$\|f(x) - \frac{1}{2}f(2x)\| \leq \frac{\theta}{2}|x|_p^{2r}$$

for all $x \in \mathbb{Q}_p$. Hence

$$\|\frac{1}{2^l}f(2^l x) - \frac{1}{2^m}f(2^m x)\| \leq \sum_{j=l}^{m-1} \frac{4^{rj}\theta}{2^{j+1}}|x|_p^{2r} \tag{2.8}$$

for all nonnegative integers m and l with $m > l$ and all $x \in \mathbb{Q}_p$. It follows from (2.8) that the sequence $\{\frac{1}{2^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in \mathbb{Q}_p$. Since B is complete, the sequence $\{\frac{1}{2^n}f(2^n x)\}$ converges. So one can define the mapping $H : \mathbb{Q}_p \to B$ by

$$H(x) := \lim_{n\to\infty} \frac{1}{2^n}f(2^n x)$$

for all $x \in \mathbb{Q}_p$.

By (2.5),

$$\begin{aligned}\|H(x+y) - H(x) - H(y)\| &= \lim_{n\to\infty} \frac{1}{2^n}\|f(2^n x + 2^n y) - f(2^n x) - f(2^n y)\| \\ &\leq \lim_{n\to\infty} \frac{4^{nr}}{2^n}\theta \cdot |x|_p^r \cdot |y|_p^r = 0\end{aligned}$$

for all $x, y \in \mathbb{Q}_p$. So

$$H(x+y) = H(x) + H(y)$$

for all $x, y \in \mathbb{Q}_p$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.8), we get (2.7).

Now, let $T : \mathbb{Q}_p \to B$ be another Cauchy additive mapping satisfying (2.7). Then we have

$$\begin{aligned}\|H(x) - T(x)\| &= \frac{1}{2^n}\|H(2^n x) - T(2^n x)\| \\ &\leq \frac{1}{2^n}(\|H(2^n x) - f(2^n x)\| + \|T(2^n x) - f(2^n x)\|) \\ &\leq \frac{2 \cdot 4^{nr}\theta}{(2-4^r)2^n}|x|_p^{2r},\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in \mathbb{Q}_p$. So we can conclude that $H(x) = T(x)$ for all $x \in \mathbb{Q}_p$. This proves the uniqueness of H.

It follows from (2.6) that

$$\begin{aligned}\|H(xy) - H(x)H(y)\| &= \lim_{n\to\infty} \frac{1}{4^n}\|f(4^n xy) - f(2^n x)f(2^n y)\| \\ &\leq \lim_{n\to\infty} \frac{4^{nr}}{4^n}\theta \cdot |x|_p^r \cdot |y|_p^r) = 0\end{aligned}$$

for all $x, y \in \mathbb{Q}_p$.

Therefore, there exists a unique ring homomorphism $H : \mathbb{Q}_p \to B$ satisfying (2.7), as desired. □

THEOREM 2.3. *Let $r > 2$ be a real number and $f : B \to \mathbb{Q}_p$ a mapping such that*

$$|f(x+y) - f(x) - f(y)|_p \leq \theta(\|x\|^r + \|y\|^r), \tag{2.9}$$

$$|f(xy) - f(x)f(y)|_p \leq \theta(\|x\|^r + \|y\|^r) \tag{2.10}$$

for all $x, y \in B$. Then there exists a unique ring homomorphism $H : B \to \mathbb{Q}_p$ such that

$$|f(x) - H(x)|_p \leq \frac{2\theta}{2^r - 2}\|x\|^r \tag{2.11}$$

for all $x \in B$.

Proof. Letting $y = x$ in (2.9), we get

$$|f(2x) - 2f(x)|_p \leq 2\theta\|x\|^r$$

for all $x \in B$. So

$$|f(x) - 2f(\frac{x}{2})|_p \leq \frac{2\theta}{2^r}\|x\|^r$$

for all $x \in B$. Hence

$$|2^l f(\frac{x}{2^l}) - 2^m f(\frac{x}{2^m})|_p \leq \sum_{j=l}^{m-1} \frac{2^{j+1}\theta}{2^{rj+r}}\|x\|^r \tag{2.12}$$

for all nonnegative integers m and l with $m > l$ and all $x \in B$. It follows from (2.12) that the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in B$. Since $\mathbb{Q}_p$ is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $H : B \to \mathbb{Q}_p$ by

$$H(x) := \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$$

for all $x \in B$.

By (2.9),

$$\begin{aligned}|H(x+y) - H(x) - H(y)|_p &= \lim_{n\to\infty} |2^n(f(\frac{x}{2^n} + \frac{y}{2^n}) - f(\frac{x}{2^n}) - f(\frac{y}{2^n}))|_p \\ &\leq \lim_{n\to\infty} \frac{2^n}{2^{nr}}\theta(\|x\|^r + \|y\|^r) = 0\end{aligned}$$

for all $x, y \in B$. So

$$H(x+y) = H(x) + H(y)$$

for all $x, y \in B$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.12), we get (2.11).

By the same method as in the proof of Theorem 2.1, one can prove the uniqueness of H.

It follows from (2.10) that

$$\begin{aligned}|H(xy) - H(x)H(y)|_p &= \lim_{n\to\infty} |4^n(f(\frac{xy}{4^n}) - f(\frac{x}{2^n})f(\frac{y}{2^n}))|_p \\ &\leq \lim_{n\to\infty} \frac{4^n}{2^{nr}}\theta(\|x\|^r + \|y\|^r) = 0\end{aligned}$$

for all $x, y \in B$.

Therefore, there exists a unique ring homomorphism $H : B \to \mathbb{Q}_p$ satisfying (2.11), as desired. □

THEOREM 2.4. *Let $r > 1$ be a real number and $f : B \to \mathbb{Q}_p$ a mapping such that*

$$|f(x+y) - f(x) - f(y)|_p \leq \theta \cdot \|x\|^r \cdot \|y\|^r, \tag{2.13}$$

$$|f(xy) - f(x)f(y)|_p \leq \theta \cdot \|x\|^r \cdot \|y\|^r \tag{2.14}$$

for all $x, y \in B$. Then there exists a unique ring homomorphism $H : B \to \mathbb{Q}_p$ such that

$$|f(x) - H(x)|_p \leq \frac{\theta}{4^r - 2}\|x\|^{2r} \tag{2.15}$$

for all $x \in B$.

Proof. Letting $y = x$ in (2.13), we get

$$|f(2x) - 2f(x)|_p \leq \theta\|x\|^{2r}$$

for all $x \in B$. So

$$|f(x) - 2f(\frac{x}{2})|_p \leq \frac{\theta}{4^r}\|x\|^{2r}$$

for all $x \in B$. Hence

$$|2^l f(\frac{x}{2^l}) - 2^m f(\frac{x}{2^m})|_p \leq \sum_{j=l}^{m-1} \frac{2^j\theta}{4^{rj+r}}\|x\|^{2r} \tag{2.16}$$

for all nonnegative integers m and l with $m > l$ and all $x \in B$. It follows from (2.16) that the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in B$. Since $\mathbb{Q}_p$ is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $H : B \to \mathbb{Q}_p$ by

$$H(x) := \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$$

for all $x \in B$.

By (2.13),

$$\begin{aligned}|H(x+y)-H(x)-H(y)|_p &= \lim_{n\to\infty}|2^n(f(\frac{x}{2^n}+\frac{y}{2^n})-f(\frac{x}{2^n})-f(\frac{y}{2^n}))|_p \\ &\le \lim_{n\to\infty}\frac{2^n}{4^{nr}}\theta\cdot\|x\|^r\cdot\|y\|^r = 0\end{aligned}$$

for all $x, y \in B$. So

$$H(x+y) = H(x) + H(y)$$

for all $x, y \in B$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.16), we get (2.15).

By the same method as in the proof of Theorem 2.2, one can prove the uniqueness of H.

It follows from (2.14) that

$$\begin{aligned}|H(xy)-H(x)H(y)|_p &= \lim_{n\to\infty}|4^n(f(\frac{xy}{4^n})-f(\frac{x}{2^n})f(\frac{y}{2^n}))|_p \\ &\le \lim_{n\to\infty}\frac{4^n}{4^{nr}}\theta\cdot\|x\|^r\cdot\|y\|^r = 0\end{aligned}$$

for all $x, y \in B$.

Therefore, there exists a unique ring homomorphism $H : B \to \mathbb{Q}_p$ satisfying (2.15). □

3. Stability of ring homomorphisms over the p-adic field $\mathbb{Q}_p$ associated with the Cauchy–Jensen functional equation

In this section, we prove the generalized Hyers–Ulam stability of ring homomorphisms over the p-adic field $\mathbb{Q}_p$ associated with the Cauchy–Jensen functional equation.

THEOREM 3.1. *Let $r < 1$ be a nonnegative real number and $f : \mathbb{Q}_p \to B$ a mapping satisfying* (2.2) *such that*

$$\|2f(\frac{x+y}{2}+z)-f(x)-f(y)-2f(z)\| \le \theta(|x|_p^r+|y|_p^r+|z|_p^r) \tag{3.1}$$

for all $x, y, z \in \mathbb{Q}_p$. Then there exists a unique ring homomorphism $H : \mathbb{Q}_p \to B$ such that

$$\|f(x)-H(x)\| \le \frac{3\theta}{2(2-2^r)}|x|_p^r \tag{3.2}$$

for all $x \in \mathbb{Q}_p$.

Proof. Letting $y = z = x$ in (3.1), we get

$$\|2f(2x) - 4f(x)\| \leq 3\theta |x|_p^r$$

for all $x \in \mathbb{Q}_p$. So

$$\|f(x) - \frac{1}{2}f(2x)\| \leq \frac{3\theta}{4}|x|_p^r$$

for all $x \in \mathbb{Q}_p$. Hence

$$\|\frac{1}{2^l}f(2^l x) - \frac{1}{2^m}f(2^m x)\| \leq \sum_{j=l}^{m-1}\frac{3 \cdot 2^{rj}\theta}{4 \cdot 2^j}|x|_p^r \tag{3.3}$$

for all nonnegative integers m and l with $m > l$ and all $x \in \mathbb{Q}_p$. It follows from (3.3) that the sequence $\{\frac{1}{2^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in \mathbb{Q}_p$. Since B is complete, the sequence $\{\frac{1}{2^n}f(2^n x)\}$ converges. So one can define the mapping $H : \mathbb{Q}_p \to B$ by

$$H(x) := \lim_{n\to\infty}\frac{1}{2^n}f(2^n x)$$

for all $x \in \mathbb{Q}_p$.

By (3.1),

$$\begin{aligned}
&\|2H(\frac{x+y}{2} + z) - H(x) - H(y) - 2H(z)\| \\
&= \lim_{n\to\infty}\frac{1}{2^n}\|2f(\frac{2^n x + 2^n y}{2} + 2^n z) - f(2^n x) - f(2^n y) - 2f(2^n z)\| \\
&\leq \lim_{n\to\infty}\frac{2^{nr}}{2^n}\theta(|x|_p^r + |y|_p^r + |z|_p^r) = 0
\end{aligned}$$

for all $x, y, z \in \mathbb{Q}_p$. So

$$2H(\frac{x+y}{2} + z) = H(x) + H(y) + 2H(z)$$

for all $x, y, z \in \mathbb{Q}_p$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.3), we get (3.2).

The rest of the proof is similar to the proof of Theorem 2.1. □

THEOREM 3.2. *Let $r < \frac{1}{3}$ be a nonnegative real number and $f : \mathbb{Q}_p \to B$ a mapping satisfying* (2.6) *such that*

$$\|2f(\frac{x+y}{2} + z) - f(x) - f(y) - 2f(z)\| \leq \theta \cdot |x|_p^r \cdot |y|_p^r \cdot |z|_p^r \tag{3.4}$$

for all $x, y, z \in \mathbb{Q}_p$. Then there exists a unique ring homomorphism $H : \mathbb{Q}_p \to B$ such that

$$\|f(x) - H(x)\| \leq \frac{\theta}{2(2 - 8^r)}|x|_p^{3r} \tag{3.5}$$

for all $x \in \mathbb{Q}_p$.

Proof. Letting $y = z = x$ in (3.4), we get

$$\|2f(2x) - 4f(x)\| \leq \theta |x|_p^{3r}$$

for all $x \in \mathbb{Q}_p$. So

$$\|f(x) - \frac{1}{2}f(2x)\| \leq \frac{\theta}{4}|x|_p^{3r}$$

for all $x \in \mathbb{Q}_p$. Hence

$$\|\frac{1}{2^l}f(2^l x) - \frac{1}{2^m}f(2^m x)\| \leq \sum_{j=l}^{m-1} \frac{8^{rj}\theta}{2^{j+2}}|x|_p^{3r} \tag{3.6}$$

for all nonnegative integers m and l with $m > l$ and all $x \in \mathbb{Q}_p$. It follows from (3.6) that the sequence $\{\frac{1}{2^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in \mathbb{Q}_p$. Since B is complete, the sequence $\{\frac{1}{2^n}f(2^n x)\}$ converges. So one can define the mapping $H : \mathbb{Q}_p \to B$ by

$$H(x) := \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$$

for all $x \in \mathbb{Q}_p$.

By (3.4),

$$\begin{aligned}
&\|2H(\frac{x+y}{2} + z) - H(x) - H(y) - 2H(z)\| \\
&= \lim_{n\to\infty} \frac{1}{2^n}\|2f(\frac{2^n x + 2^n y}{2} + 2^n z) - f(2^n x) - f(2^n y) - 2f(2^n z)\| \\
&\leq \lim_{n\to\infty} \frac{8^{nr}}{2^n}\theta \cdot |x|_p^r \cdot |y|_p^r \cdot |z|_p^r = 0
\end{aligned}$$

for all $x, y, z \in \mathbb{Q}_p$. So

$$2H(\frac{x+y}{2} + z) = H(x) + H(y) + 2H(z)$$

for all $x, y, z \in \mathbb{Q}_p$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.6), we get (3.5).

The rest of the proof is similar to the proof of Theorem 2.2. $\square$

Theorem 3.3. *Let* $r > 2$ *be a real number and* $f : B \to \mathbb{Q}_p$ *a mapping satisfying* (2.10) *such that*

$$|2f(\frac{x+y}{2} + z) - f(x) - f(y) - 2f(z)|_p \leq \theta(\|x\|^r + \|y\|^r + \|z\|^r) \tag{3.7}$$

for all $x, y, z \in B$. *Then there exists a unique ring homomorphism* $H : B \to \mathbb{Q}_p$ *such that*

$$|2f(x) - H(x)|_p \leq \frac{3\theta}{2^r - 2}\|x\|^r \tag{3.8}$$

for all $x \in B$.

Proof. Letting $y = z = x$ in (3.7), we get

$$|2f(2x) - 4f(x)|_p \leq 3\theta\|x\|^r$$

for all $x \in B$. So

$$|2f(x) - 4f(\frac{x}{2})|_p \leq \frac{3\theta}{2^r}\|x\|^r$$

for all $x \in B$. Hence

$$|2^l \cdot 2f(\frac{x}{2^l}) - 2^m \cdot 2f(\frac{x}{2^m})|_p \leq \sum_{j=l}^{m-1} \frac{3 \cdot 2^j\theta}{2^r \cdot 2^{rj}}\|x\|^r \tag{3.9}$$

for all nonnegative integers m and l with $m > l$ and all $x \in B$. It follows from (3.9) that the sequence $\{2^n \cdot 2f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in B$. Since $\mathbb{Q}_p$ is complete, the sequence $\{2^n \cdot 2f(\frac{x}{2^n})\}$ converges. So one can define the mapping $H : B \to \mathbb{Q}_p$ by

$$H(x) := \lim_{n\to\infty} 2^n \cdot 2f(\frac{x}{2^n})$$

for all $x \in B$.

By (3.7),

$$\begin{aligned}
&|2H(\frac{x+y}{2} + z) - H(x) - H(y) - 2H(z)|_p \\
&= \lim_{n\to\infty} |2^n(4f(\frac{x}{2^{n+1}} + \frac{y}{2^{n+1}} + \frac{z}{2^n}) - 2f(\frac{x}{2^n}) - 2f(\frac{y}{2^n}) - 4f(\frac{z}{2^n}))|_p \\
&\leq \lim_{n\to\infty} \frac{2 \cdot 2^r}{2^{nr}}\theta(\|x\|^r + \|y\|^r + \|z\|^r) = 0
\end{aligned}$$

for all $x, y, z \in B$. So

$$2H(\frac{x+y}{2} + z) = H(x) + H(y) + 2H(z)$$

for all $x, y, z \in B$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.9), we get (3.8).

The rest of the proof is similar to the proof of Theorem 2.3. □

THEOREM 3.4. *Let* $r > 1$ *be a real number and* $f : B \to \mathbb{Q}_p$ *a mapping satisfying* (2.14) *such that*

$$|2f(\frac{x+y}{2} + z) - f(x) - f(y) - 2f(z)|_p \leq \theta \cdot \|x\|^r \cdot \|y\|^r \cdot \|z\|^r \tag{3.10}$$

for all $x, y, z \in B$. Then there exists a unique Cauchy–Jensen additive mapping $H : B \to \mathbb{Q}_p$ such that

$$|2f(x) - H(x)|_p \leq \frac{\theta}{(8^r - 2)} \|x\|^{3r} \tag{3.11}$$

for all $x \in B$.

Proof. Letting $y = z = x$ in (3.10), we get

$$|2f(2x) - 4f(x)|_p \leq \theta \|x\|^{3r}$$

for all $x \in B$. So

$$|2f(x) - 4f(\frac{x}{2})|_p \leq \frac{\theta}{8^r} \|x\|^{3r}$$

for all $x \in B$. Hence

$$|2^l \cdot 2f(\frac{x}{2^l}) - 2^m \cdot 2f(\frac{x}{2^m})|_p \leq \sum_{j=l}^{m-1} \frac{2^j \theta}{8^{rj+1}} \|x\|^{3r} \tag{3.12}$$

for all nonnegative integers m and l with $m > l$ and all $x \in B$. It follows from (3.12) that the sequence $\{2^n \cdot 2f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in B$. Since $\mathbb{Q}_p$ is complete, the sequence $\{2^n \cdot 2f(\frac{x}{2^n})\}$ converges. So one can define the mapping $H : B \to \mathbb{Q}_p$ by

$$H(x) := \lim_{n\to\infty} 2^n \cdot 2f(\frac{x}{2^n})$$

for all $x \in B$.

By (3.10),

$$\begin{aligned}
&|2H(\frac{x+y}{2} + z) - H(x) - H(y) - 2H(z)|_p \\
&= \lim_{n\to\infty} |2^n(4f(\frac{x}{2^{n+1}} + \frac{y}{2^{n+1}} + \frac{z}{2^n}) - 2f(\frac{x}{2^n}) - 2f(\frac{y}{2^n}) - 4f(\frac{z}{2^n}))|_p \\
&\leq \lim_{n\to\infty} \frac{2 \cdot 8^{nr}}{2^n} \theta \cdot \|x\|^r \cdot \|y\|^r \cdot \|z\|^r = 0
\end{aligned}$$

for all $x, y, z \in B$. So

$$2H(\frac{x+y}{2} + z) = H(x) + H(y) + 2H(z)$$

for all $x, y, z \in B$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.12), we get (3.11).

The rest of the proof is similar to the proof of Theorem 2.4. □

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