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Two types of uniform spaces

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Abstract

In strictly two-sided, commutative biquantale, we introduce the notion of Hutton $(L, \otimes)$-uniform spaces and $(L, \odot)$-uniform spaces and investigate the properties of them.

Key words : Hutton $(L, \otimes)$-uniform spaces, $(L, \odot)$-uniform spaces

1. Introduction

Uniformities in fuzzy sets, have the entourage approach of Lowen [17] and Höhle [7-8] based on powersets of the form $L^{X\times X}$, the uniform covering approach of Kotzé [15] and the uniform operator approach of Rodabaugh [19] as generalization of Hutton [13] based on powersets of the form $(L^X)^{(L^X)}$. For a fixed basis L, algebraic structures in L (cqm-lattices, quantales, MV-algebras) are extended for a completely distributive lattice L [13,16,22,23] or the unit interval [17,20] or t-norms [7-8]. Recently, Gutiérrez García et al.[5] introduced L-valued Hutton unifomity where a quadruple $(L, \leq, \otimes, *)$ is defined by a GL-monoid $(L, *)$ dominated by $\otimes$, a cl-quasi-monoid $(L, \leq, \otimes)$.

In this paper, as a somewhat different aspect in [5], we introduce the notion of Hutton $(L, \otimes)$-uniformities as a view point of the approach using uniform operators defined by Rodabaugh [19] and $(L, \odot)$-uniformities in a sense Lowen [17] and Höhle [7-8] based on powersets of the form $L^{X\times X}$. We investigate the relationship between Hutton $(L, \otimes)$-uniformities and $(L, \odot)$-uniformities.

For general background for a fuzzy logic, we refer to [6,9-12,18,19,24].

2. Preliminaries

Definition 2.1 [14,21] A triple $(L, \leq, \odot)$ is called a *strictly two-sided, commutative biquantale* (stsc-biquantale, for short) iff it satisfies the following properties:

(L1) $L = (L, \leq, \vee, \wedge, \top, \bot)$ is a completely distributive lattice where $\top$ is the universal upper bound and $\bot$ denotes the universal lower bound;

(L2) $(L, \odot)$ is a commutative semigroup;

(L3) $a = a \odot \top$, for each $a \in L$;

(L4) $\odot$ is distributive over arbitrary joins, i.e.

$$(\bigvee_{i\in\Gamma} a_i) \odot b = \bigvee_{i\in\Gamma} (a_i \odot b).$$

(L5) $\odot$ is distributive over arbitrary meets, i.e.

$$(\bigwedge_{i\in\Gamma} a_i) \odot b = \bigwedge_{i\in\Gamma} (a_i \odot b).$$

Remark 2.2 [12-14, 23](1) A completely distributive lattice (ref. [16]) is a stsc-biquantale. In particular, the unit interval $([0,1], \leq, \vee, \wedge, 0, 1)$ is a stsc-biquantale.

(2) The unit interval with a continuous t-norm t, $([0,1], \leq, t)$, is a stsc-biquantale.

(3) Let $(L, \leq, \odot)$ be a stsc-biquantale. For each $x, y \in L$, we define

$$x \to y = \bigvee \{z \in L \mid x \odot z \leq y\}.$$

Then it satisfies Galois correspondence, that is,
$(x \odot y) \leq z$ iff $x \leq (y \to z)$.

In this paper, we always assume that $(L, \leq, \odot, ^*)$ is a stsc-biquantale with strong negation * where $a^* = a \to 0$ unless otherwise specified.

Let X be a nonempty set. All algebraic operations on L can be extended pointwisely to the set L^X as follows: for all $x \in X$, $f, g \in L^X$ and $\alpha \in L$,

(1) $f \leq g$ iff $f(x) \leq g(x)$;

(2) $(f \odot g)(x) = f(x) \odot g(x)$;

(3) $1_X(x) = \top$, $\alpha \odot 1_X(x) = \alpha$ and $1_\emptyset(x) = \bot$;

(4) $(\alpha \to \lambda)(x) = \alpha \to \lambda(x)$ and $(\lambda \to \alpha)(x) = \lambda(x) \to \alpha$;

(5) $(\alpha \odot \lambda)(x) = \alpha \odot \lambda(x)$.

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Lemma 2.3 [6,10,24] For each $x, y, z \in L$, $\{y_i \mid i \in \Gamma\} \subset L$, we have the following properties.

(1) If $y \leq z$, $(x \odot y) \leq (x \odot z)$.
(2) $x \odot y \leq x \wedge y$.
(3) $x \odot (x \to y) \leq y$.
(4) $x \odot (y \to z) \leq y \to x \odot z$.
(5) $x \odot (x \odot y \to z) \leq y \to z$.
(6) $(x \to y) \odot (z \to w) \leq (x \odot z) \to (y \odot w)$.
(7) $x \to y \leq (y \to z) \to (x \to z)$.
(8) $x \to y = y^* \to x^*$.
(9) $\bigwedge_{i\in\Gamma} y_i^* = (\bigvee_{i\in\Gamma} y_i)^*$ and $\bigvee_{i\in\Gamma} y_i^* = (\bigwedge_{i\in\Gamma} y_i)^*$.
(10) $(x \odot y) \to z = x \to (y \to z)$.

3. Two types of quasi-uniform spaces

Definition 3.1 Let $\Omega(X)$ be a subset of $(L^X)^{(L^X)}$ such that

(O1) $\lambda \leq \phi(\lambda)$, for every $\lambda \in L^X$,
(O2) $\phi(\bigvee_{i\in\Gamma} \lambda_i) = \bigvee_{i\in\Gamma} \phi(\lambda_i)$, for $\{\lambda_i\}_{i\in\Gamma} \subset L^X$.
(O3) $\alpha \odot \phi(\lambda) = \phi(\alpha \odot \lambda)$, for $\lambda \in L^X$.

Example 3.2 Let $([0,1], \odot)$ be a biquantale such that $x \odot y = (x+y-1) \wedge 1$ and $X = \{x, y\}$. For $\rho(x) = 0.7, \rho(y) = 0.5$, define $\phi_\rho \in (L^X)^{(L^X)}$ as follows:

$$\phi_\rho(\lambda) = \begin{cases} 1_\emptyset & \text{if } \lambda = 1_\emptyset, \\ \rho & \text{if } 1_\emptyset \neq \lambda \leq \rho, \\ 1_X & \text{if } \lambda \not\leq \rho. \end{cases}$$

ϕ_ρ satisfies (O1) and (O2) but not (O3) because

$$\rho = \phi_\rho(0.3 \odot 1_{\{x\}}) \neq 0.3 \odot \phi_\rho(1_{\{x\}}) = 0.3 \odot 1_X.$$

Lemma 3.3 *For $\phi, \phi_1, \phi_2 \in \Omega(X)$, we define, for all $\lambda \in L^X$,*

$$\phi^{-1}(\lambda) = \bigwedge\{\rho \in L^X \mid \phi(\rho^*) \leq \lambda^*\},$$

$$\phi_1 \circ \phi_2(\lambda) = \phi_1(\phi_2(\lambda)),$$

$$\phi_1 \otimes \phi_2(\lambda) = \bigwedge\{\phi_1(\lambda_1) \odot \phi_2(\lambda_2) \mid \lambda = \lambda_1 \odot \lambda_2\}.$$

For $\phi_1, \phi_2, \phi_3 \in \Omega(X)$, the following properties hold:

(1) *If $\phi(1_{\{x\}}) = \rho_x$ for all $x \in X$, then $\phi(\lambda) = \bigvee_{z\in X} \lambda(z) \odot \rho_z$.*
(2) *If $\phi_1(1_{\{x\}}) = \phi_2(1_{\{x\}})$ for all $x \in X$, then $\phi_1 = \phi_2$.*
(3) *$\phi^{-1} \in \Omega(X)$ and $\phi_1 \circ \phi_2 \in \Omega(X)$.*
(4) *If $\phi_1 \leq \phi_2$, then $\phi_1^{-1} \leq \phi_2^{-1}$.*
(5) *$\phi_1 \otimes \phi_2 \in \Omega(X)$.*
(6) *$\phi_1 \otimes \phi_2 \leq \phi_1$ and $\phi_1 \otimes \phi_2 \leq \phi_2$.*
(7) *$(\phi_1 \otimes \phi_2) \otimes \phi_3 = \phi_1 \otimes (\phi_2 \otimes \phi_3)$,*
(8) *$(\phi_1 \otimes \phi_2) \circ (\phi_1 \otimes \phi_2) \leq (\phi_1 \circ \phi_1) \otimes (\phi_2 \circ \phi_2)$.*
(9) *Define $\phi_\top \in \Omega(X)$ as $\phi_\top(1_{\{x\}}) = 1_X, \forall x \in X$. Then $\phi \leq \phi_\top$ for all $\phi \in \Omega(X)$.*

Proof. (1) For all $\lambda \in L^X$, we write $\lambda = \bigvee_{z\in X} \lambda(z) \odot 1_{\{z\}}$. Thus,

$$\begin{aligned} \phi(\lambda) &= \phi(\textstyle\bigvee_{z\in X} \lambda(z) \odot 1_{\{z\}}) \\ &= \textstyle\bigvee_{z\in X} \lambda(z) \odot \phi(1_{\{z\}}) \\ &= \textstyle\bigvee_{z\in X} \lambda(z) \odot \rho_z. \end{aligned}$$

(2) For $\lambda = \bigvee_{z\in X} \lambda(z) \odot 1_{\{z\}}$, we have

$$\begin{aligned} \phi_1(\lambda) &= \textstyle\bigvee_{z\in X} \lambda(z) \odot \phi_1(1_{\{z\}}) \\ &= \textstyle\bigvee_{z\in X} \lambda(z) \odot \phi_2(1_{\{z\}}) \\ &= \phi_2(\lambda). \end{aligned}$$

(3) We only show (O3) $\alpha \odot \phi^{-1}(\lambda) = \phi^{-1}(\alpha \odot \lambda)$, for $\lambda \in L^X$.

$$\begin{aligned} \phi^{-1}(\alpha \odot \lambda) &= \textstyle\bigwedge\{\rho \in L^X \mid \phi(\rho^*) \leq (\alpha \odot \lambda)^*\} \\ &= \textstyle\bigwedge\{\rho \in L^X \mid \alpha \odot \phi(\rho^*) \leq \lambda^*\} \\ &= \textstyle\bigwedge\{\rho \in L^X \mid \phi(\alpha \odot \rho^*) \leq \lambda^*\} \\ \alpha \odot \phi^{-1}(\lambda) &= \alpha \odot \textstyle\bigwedge\{\mu \in L^X \mid \phi(\mu^*) \leq \lambda^*\} \\ &= \textstyle\bigwedge\{\alpha \odot \mu \mid \phi(\mu^*) \leq \lambda^*\} \end{aligned}$$

Let $\alpha \odot \mu \in L^X$ such that $\phi(\mu^*) \leq \lambda^*$. Since $\alpha \odot (\alpha \odot \mu)^* \leq \mu^*$ from Lemma 2.3(5) and $\phi\big(\alpha \odot (\alpha \odot \mu)^*\big) \leq \phi(\mu^*) \leq \lambda^*$. Hence $\phi^{-1}(\alpha \odot \lambda) \leq \alpha \odot \phi^{-1}(\lambda)$.

Let $\rho \in L^X$ with $\phi(\alpha \odot \rho^*) \leq \lambda^*$. Put $u^* = \alpha \odot \rho^*$. By Lemma 2.3(10),

$$u = u^{**} = (\alpha \odot \rho^*)^* = \alpha \to \rho^{**} = \alpha \to \rho.$$

Hence $\alpha \odot \mu = \alpha \odot (\alpha \to \rho) \leq \rho$. Thus, $\phi^{-1}(\alpha \odot \lambda) \geq \alpha \odot \phi^{-1}(\lambda)$.

Simiarly, $\phi_1 \circ \phi_2 \in \Omega(X)$ is easily proved.

(4) Since $\phi_1(\rho^*) \leq \phi_2(\rho^*) \leq \lambda^*$, it easily proved.

(5) (O3) We show $(\phi_1 \otimes \phi_2)(\bigvee_{i\in\Gamma} \mu_i) \leq \bigvee_{i\in\Gamma}(\phi_1 \otimes \phi_2)(\mu_i)$. Suppose

$$(\phi_1 \otimes \phi_2)(\bigvee_{i\in\Gamma} \mu_i) \not\leq \bigvee_{i\in\Gamma} (\phi_1 \otimes \phi_2)(\mu_i)$$

$$= \bigvee_{i\in\Gamma} \Big(\bigwedge\{\phi_1(\lambda_i) \odot \phi_2(\rho_i) \mid \lambda_i \odot \rho_i = \mu_i\}\Big).$$

Since L is a completely distributive lattice, by the definition of $(\phi_1 \otimes \phi_2)(\mu_i)$, for each $i \in \Gamma$, there exist λ_i, ρ_i with $\mu_i = \lambda_i \odot \rho_i$ such that

$$(\phi_1 \otimes \phi_2)(\bigvee_{i\in\Gamma} \mu_i) \not\leq \bigvee_{i\in\Gamma} \{\phi_1(\lambda_i) \odot \phi_2(\rho_i)\}.$$

On the other hand, since $\bigvee_{i\in\Gamma} \mu_i = (\bigvee_{i\in\Gamma} \lambda_i) \odot (\bigvee_{i\in\Gamma} \rho_i)$ from (L4),

$$\begin{aligned} (\phi_1 \otimes \phi_2)(\textstyle\bigvee_{i\in\Gamma} \mu_i) &\leq \phi_1(\textstyle\bigvee_{i\in\Gamma} \lambda_i) \odot \phi_2(\textstyle\bigvee_{i\in\Gamma} \rho_i) \\ &= \Big(\textstyle\bigvee_{i\in\Gamma} \phi_1(\lambda_i)\Big) \odot \Big(\textstyle\bigvee_{i\in\Gamma} \phi_2(\rho_i)\Big) \\ &= \textstyle\bigvee_{i\in\Gamma} \{\phi_1(\lambda_i) \odot \phi_2(\rho_i)\}. \end{aligned}$$

It is a contradiction. Hence the result holds. (O1) and (O3) are easily proved. So, $\phi_1 \otimes \phi_2 \in \Omega(X)$.

(6) For $\mu = \mu \odot 1_X$, we have $\phi_1(\mu) = \phi_1(\mu) \odot \phi_1(1_X) \geq (\phi_1 \otimes \phi_2)(\mu)$.

(7) Suppose there exists $\mu \in L^X$ with $(\phi_1 \otimes (\phi_2 \otimes \phi_3))(\mu) \not\leq ((\phi_1 \otimes \phi_2) \otimes \phi_3)(\mu)$. Then there exist μ_i with $\mu = \mu_1 \odot \mu_2$ such that

$$(\phi_1 \otimes (\phi_2 \otimes \phi_3))(\mu) \not\leq (\phi_1 \otimes \phi_2)(\mu_1) \odot \phi_3(\mu_2).$$

By (L5), there exist ρ_1 and ρ_2 with $\mu_1 = \rho_1 \odot \rho_2$ such that

$$(\phi_1 \otimes (\phi_2 \otimes \phi_3))(\mu) \not\leq (\phi_1(\rho_1) \odot \phi_2(\rho_2)) \odot \phi_3(\mu_2).$$

On the other hand, since $(\rho_1 \odot \rho_2) \odot \mu_2 = \rho_1 \odot (\rho_2 \odot \mu_2)$

$$(\phi_1 \otimes (\phi_2 \otimes \phi_3))(\mu) \leq \phi_1(\rho_1) \odot (\phi_2(\rho_2) \odot \phi_3(\mu_2)).$$

It is a contradiction. Thus, $\phi_1 \otimes (\phi_2 \otimes \phi_3) \leq (\phi_1 \otimes \phi_2) \otimes \phi_3$.

Similarly, $\phi_1 \otimes (\phi_2 \otimes \phi_3) \geq (\phi_1 \otimes \phi_2) \otimes \phi_3$.

(8) Suppose there exists $\mu \in L^X$ with $(\phi_1 \otimes \phi_2) \circ (\phi_1 \otimes \phi_2)(\mu) \not\leq (\phi_1 \circ \phi_1) \otimes (\phi_2 \circ \phi_2)(\mu)$. Then there exist μ_i with $\mu = \mu_1 \odot \mu_2$ such that

$$(\phi_1 \otimes \phi_2) \circ (\phi_1 \otimes \phi_2)(\mu) \not\leq (\phi_1 \circ \phi_1)(\mu_1) \odot (\phi_2 \circ \phi_2)(\mu_2).$$

But

$$\begin{aligned}&(\phi_1 \otimes \phi_2) \circ (\phi_1 \otimes \phi_2)(\mu)\\ &\leq (\phi_1 \otimes \phi_2)(\phi_1(\mu_1) \odot \phi_2(\mu_2))\\ &\leq \phi_1(\phi_1(\mu_1)) \odot \phi_2(\phi_2(\mu_2)).\end{aligned}$$

It is a contradiction. Thus, $(\phi_1 \otimes \phi_2) \circ (\phi_1 \otimes \phi_2) \leq (\phi_1 \circ \phi_1) \otimes (\phi_2 \circ \phi_2)$.

(9) Since $\phi(1_{\{x\}}) \leq \phi_\top(1_{\{x\}}) = 1_X, \forall x \in X$, we have $\phi \leq \phi_\top$ for all $\phi \in \Omega(X)$.

We define a somewhat different aspect in [5], we introduce the notion of $(L, \otimes)$-uniformities as a view point of the approach using uniform operators defined by Rodabaugh [19].

Definition 3.4 A nonempty subset $\mathbf{U}$ of $\Omega(X)$ is called a Hutton $(L, \otimes)$-*quasi-uniformity* on X if it satisfies the following conditions:

(QU1) If $\phi \leq \psi$ with $\phi \in \mathbf{U}$ and $\psi \in \Omega(X)$, then $\psi \in \mathbf{U}$.

(QU2) For each $\phi, \psi \in \mathbf{U}$, $\phi \otimes \psi \in \mathbf{U}$.

(QU3) For each $\phi \in \mathbf{U}$, there exists $\psi \in \mathbf{U}$ such that $\psi \circ \psi \leq \phi$.

The pair $(X, \mathbf{U})$ is said to be a *Hutton* $(L, \otimes)$- *quasi-uniform space*.

A Hutton $(L, \otimes)$- quasi-uniform space is said to be a *Hutton* $(L, \otimes)$-*uniform space* if it satisfies

(U) For each $\phi \in \mathbf{U}$, there exists $\phi^{-1} \in \mathbf{U}$.

Example 3.5 Let $X = \{x, y, z\}$ be a set and $([0,1], \odot)$ an biquantale defined by $x \odot y = \max\{0, x+y-1\}$ (ref.[6,10-12,18,24]).

(1) Define $\phi \in \Omega(X)$ as follows:

$$\phi(1_{\{x\}}) = \phi(1_{\{y\}}) = 1_{\{x,y\}}, \quad \phi(1_{\{z\}}) = \phi(1_{\{z\}})$$

Since

$$\phi \otimes \phi(1_{\{x\}}) = \phi \otimes \phi(1_{\{y\}}) = 1_{\{x,y\}}, \phi \otimes \phi(1_{\{z\}}) = 1_{\{z\}},$$

by Lemma 3.3(2), $\phi \otimes \phi = \phi$. We have $\phi \circ \phi = \phi$ because

$$\phi \circ \phi(1_{\{x\}}) = \phi \circ \phi(1_{\{y\}}) = 1_{\{x,y\}}, \phi \circ \phi((1_{\{z\}})) = 1_{\{z\}}.$$

Since

$$\phi^{-1}(1_{\{x\}}) = \phi^{-1}(1_{\{y\}}) = 1_{\{x,y\}}, \phi^{-1}(1_{\{z\}}) = 1_{\{z\}},$$

Hence $\phi^{-1} = \phi$.

(2) Define $\mathbf{U} = \{\psi \in \Omega(X) \mid \phi \leq \psi\}$. Then $\mathbf{U}$ is a Hutton $(L, \otimes)$-uniformity on X from (1).

We define an $(L, \odot)$-uniformity in a sense Lowen [17] and Höhle [7-8] based on powersets of the form $L^{X \times X}$.

Definition 3.6 Let $E(X \times X) = \{u \in L^{X \times X} \mid u(x,x) = 1\}$ be a subset of $L^{X \times X}$. A nonempty subset $\mathbf{D}$ of $E(X \times X)$ is called an $(L, \odot)$-*quasi-uniformity* on X if it satisfies the following conditions:

(QD1) If $u \leq v$ with $u \in \mathbf{D}$ and $v \in E(X \times X)$, then $v \in \mathbf{D}$.

(QD2) For each $u, v \in \mathbf{D}$, $u \odot v \in \mathbf{D}$.

(QD3) For each $u \in \mathbf{D}$, there exists $v \in \mathbf{D}$ such that $v \circ v \leq u$ where

$$v \circ v(x,y) = \bigvee_{z \in X} (v(x,z) \odot v(z,y)).$$

The pair $(X, \mathbf{D})$ is said to be an $(L, \odot)$- *quasi-uniform space*.

An $(L, \odot)$-quasi-uniform space is said to be an $(L, \odot)$-*uniform space* if it satisfies

(D) For each $u \in \mathbf{D}$, there exists $u^s \in \mathbf{U}$ where $u^s(x,y) = u(y,x)$.

Definition 3.7 A function $u : X \times X \to L$ is called an $\odot$-*quasi-equivalence relation* iff it satisfies the following properties

(E1) $u(x,x) = 1$ for all $x \in X$.

(E2) $u(x,y) \odot u(y,z) \leq u(x,z)$.

An $\odot$-quasi-equivalence relation is called an $\odot$-equivalence relation on X if it satisfies

(E) $u(x,y) = u(y,x)$.

We denote $u^2 = u \odot u$ and $u^{n+1} = u^n \odot u$ for each $u \in L^{X \times X}$.

Theorem 3.8 *Let $u : X \times X \to L$ be an $\odot$-equivalence relation. We define a mapping $\mathbf{D}_u$ as follows:*

$$\mathbf{D}_u = \{v \in E(X \times X) \mid \exists n \in N, u^n \leq v\}.$$

Then $\mathbf{D}_u$ is an $(L, \odot)$-uniformity on X.

Proof. (QD1) Obvious. (QD2) Let $v_i \in \mathbf{D}_u$ for $i = 1, 2$. There exist $n_i \in N$ such that $u^{n_i} \leq v_i$. Hence $n_1+n_2 \in N$ such that $u^{n_1+n_2} \leq v_1 \odot v_2$.

(QD3). For $v \in \mathbf{D}_u$, there exists $n \in N$ such that $u^n \leq v$. Since u is an $\odot$-equivalence relation, $u \circ u \leq u$. We have $(u \circ u)^2 \geq u^2 \circ u^2$ because

$$\begin{aligned}&(u \circ u)^2(x,y)\\ &= \bigvee_{z\in X}\Big(u(x,z)\odot u(z,y)\Big)\odot\bigvee_{w\in X}\Big(u(x,w)\odot u(w,y)\Big)\\ &= \bigvee_{z\in X}\bigvee_{w\in X}\Big(u(x,z)\odot u(z,y)\Big)\odot\Big(u(x,w)\odot u(w,y)\Big)\\ &\geq \bigvee_{z\in X}\Big(u(x,z)\odot u(z,y)\odot u(x,z)\odot u(z,y)\Big)\\ &\geq \bigvee_{z\in X}\Big(u^2(x,z)\odot u^2(z,y)\Big)\\ &= u^2\circ u^2(x,y).\end{aligned}$$

We obtain $u^n \circ u^n \leq (u \circ u)^n \leq u^n \leq v$ and $u^n \in \mathbf{D}_u$.

(D) For $v \in \mathbf{D}_u$, there exists $n \in N$ such that $u^n \leq v$. Since u is an $\odot$-equivalence relation, $u^s = u$. Then $u^n = (u^s)^n = (u^n)^s \leq v^s$ implis $v^s \in \mathbf{D}_u$.

Example 3.9 Let X and $(L = [0,1], \odot)$ be defined as in Example 3.5. Let $u \in E(X \times X)$ be an $\odot$-fuzzy quasi-equivalence relation on X as

$$u(x,x) = u(y,y) = u(z,z) = u(x,y) = 1,$$
$$u(y,x) = 0.7, u(y,z) = u(z,y) = 0.6,$$
$$u(x,z) = u(z,x) = 0.5.$$

Then

$$u^3(x,x) = u^3(y,y) = u^3(z,z) = u^3(x,y) = 1,$$
$$u^3(y,x) = u^3(y,z) = u^3(z,y) = u^3(x,z) = u^3(z,x) = 0.$$

Define $\mathbf{D} = \{v \in E(X \times X) \mid u^3 \leq v\}$. Then $\mathbf{D}$ is an $(L,\odot)$-quasi-uniformity on X but not an $(L,\odot)$-uniformity on X because $(u^3)^s \notin \mathbf{D}$.

Theorem 3.10 *We define a mapping $\Gamma : E(X \times X) \to \Omega(X)$ as follows:*

$$\Gamma(u)(\lambda)(y) = \bigvee_{x\in X} \lambda(x) \odot u(x,y).$$

Then we have the following properties:

(1) *For $u \in E(X \times X)$, $\Gamma(u) \in \Omega(X)$ and $\Gamma(u)$ has a right adjoint mapping $\Gamma^{\leftarrow}(u)$ defined by*

$$\Gamma(u)^{\leftarrow}(\lambda) = \bigvee\{\rho \in L^X \mid \Gamma(u)(\rho) \leq \lambda\}.$$

(2) *Γ is injective and join preserving map.*

(3) *Γ has a right adjoint mapping $\Lambda : \Omega(X) \to E(X \times X)$ as follows:*

$$\Lambda(\phi)(x,y) = \phi(1_{\{x\}})(y).$$

(4) *$\Gamma \circ \Lambda = 1_{\Omega(X)}$ and $\Lambda \circ \Gamma = E(X \times X)$.*

Proof. (1) Γ is well-defined because $\Gamma(u) \in \Omega(X)$ from the following statements:

(O1) $\Gamma(u)(\lambda)(x) = \bigvee_{z\in X} \lambda(z) \odot u(z,x) \geq \lambda(x) \odot u(x,x) = \lambda(x)$ for all $\lambda \in L^X$.

(O2) $\Gamma(u)$ is join preserving because

$$\begin{aligned}\Gamma(u)(\textstyle\bigvee_i \lambda_i)(y) &= \textstyle\bigvee_{x\in X}(\bigvee_i \lambda_i(x)) \odot u(x,y)\\ &= \textstyle\bigvee_i(\bigvee_{x\in X} \lambda_i(x) \odot u(x,y))\\ &= \textstyle\bigvee_i \Gamma(u)(\lambda_i)(y)\end{aligned}$$

(O3) is easily proved.

By (O2), $\Gamma(u)$ has a right adjoint mapping $\Gamma(u)^{\leftarrow}$.

(2) Γ is injective because, for each $1_{\{z\}}$, $\Gamma(u_1)(1_{\{z\}})(y) = u_1(z,y) = u_2(z,y) = \Gamma(u_2)(1_{\{z\}})(y)$.

Γ is join preserving because

$$\begin{aligned}\Gamma(\textstyle\bigvee_i u_i)(\lambda)(y) &= \textstyle\bigvee_{x\in X} \lambda(x) \odot \bigvee_i u_i(x,y)\\ &= \textstyle\bigvee_i(\bigvee_{x\in X} \lambda(x) \odot u_i(x,y))\\ &= \textstyle\bigvee_i \Gamma(u_i)(\lambda)(y)\end{aligned}$$

(3)

$$\begin{aligned}&\Lambda(\phi)(x,y)\\ &= \bigvee\{u(x,y) \mid \Gamma(u)(\lambda(x)\odot 1_{\{x\}})(y) \leq \phi(\lambda(x)\odot 1_{\{x\}})(y)\}\\ &= \bigvee\{u(x,y) \mid \lambda(x)\odot u(x,y) \leq \phi(\lambda(x)\odot 1_{\{x\}})(y)\}\\ &\quad(\text{ put } \lambda(x) = \alpha)\\ &= \bigvee\{u(x,y) \mid u(x,y) \leq \textstyle\bigwedge_\alpha(\alpha \to \alpha\odot\phi(1_{\{x\}})(y))\}\\ &= \textstyle\bigwedge_\alpha\Big(\alpha \to \alpha\odot\phi(1_{\{x\}})(y)\Big)\end{aligned}$$

Since

$$\bigwedge_\alpha\Big(\alpha \to \alpha\odot\phi(1_{\{x\}})(y)\Big) \leq \top \to \top\odot\phi(1_{\{x\}})(y) = \phi(1_{\{x\}})(y),$$

we have $\Lambda(\phi)(x,y) \leq \phi(1_{\{x\}})(y)$. Since $\alpha\odot\phi(1_{\{x\}}) \leq \alpha\odot\phi(1_{\{x\}})$, we have

$$\phi(1_{\{x\}})(y) \leq \bigwedge_\alpha\Big(\alpha \to \alpha\odot\phi(1_{\{x\}})(y)\Big) = \Lambda(\phi)(x,y).$$

Hence $\Lambda(\phi)(x,y) = \phi(1_{\{x\}})(y)$.

Furthermore, $\Lambda(\phi) \in E(X \times X)$ from:

$$\Lambda(\phi)(x,x) = \phi(1_{\{x\}})(x) = \top.$$

(4)

$$\begin{aligned}\Gamma(\Lambda(\phi))(\lambda)(y) &= \textstyle\bigvee_{x\in X} \lambda(x)\odot\Lambda(\phi)(x,y)\\ &= \textstyle\bigvee_{x\in X}\Big(\lambda(x)\odot\phi(1_{\{x\}})(y)\Big)\\ &= \textstyle\bigvee_{x\in X}\phi(\lambda(x)\odot 1_{\{x\}})(y)\\ &= \phi(\lambda)(y).\end{aligned}$$

Hence $\Gamma \circ \Lambda = 1_{\Omega(X)}$.

We have $\Lambda \circ \Gamma = 1_{E(X\times X)}$ from:

$$\Lambda(\Gamma(u))(x,y) = \Gamma(u)(1_{\{x\}})(y)$$
$$= \bigvee_{z\in X}(1_{\{x\}}(z)\odot u(z,y)) = u(x,y)$$

Example 3.11 Let X and $(L = [0,1], \odot)$ be defined as in Example 3.5. Then $x \to y = \min\{1, 1-x+y\}$ for each $x, y \in L$. Let $u \in E(X \times X)$ be an $\odot$-fuzzy quasi-equivalence relation on X as

$$u(x,x) = u(y,y) = u(z,z) = u(x,y) = 1,$$
$$u(y,x) = 0.7, u(y,z) = u(z,y) = 0.6,$$
$$u(x,z) = u(z,x) = 0.5.$$

Then

$$\Gamma(u)(1_x) = \rho_x, \quad \rho_x(x) = 1, \rho_x(y) = u(x,y) = 1,$$
$$\rho_x(z) = 0.5,$$
$$\Gamma(u)(1_y) = \rho_y, \quad \rho_y(x) = 0.7, \rho_y(y) = 1, \rho_y(z) = 0.6,$$
$$\Gamma(u)(1_z) = \rho_z, \quad \rho_z(x) = 0.5, \rho_z(y) = 0.6, \rho_z(z) = 1.$$

Since

$$\Lambda(\Gamma(u))(z,x) = \Gamma(u)(1_{\{z\}})(x) = u(z,x),$$

by a similar method, we obtain $\Lambda \circ \Gamma(u) = u$ for all $u \in E(X \times X)$.

Theorem 3.12 *Let $u, u_1, u_2 \in E(X \times X)$. Then we have the following properties:*

(1) *If $u_1 \leq u_2$, $\Gamma(u_1) \leq \Gamma(u_2)$.*
(2) $\Gamma(u_1 \odot u_2) \leq \Gamma(u_1) \otimes \Gamma(u_2)$.
(3) $\Gamma(1_\Delta) = 1_{L^X}$.
(4) $\Gamma(u)^{-1} = \Gamma(u^s)$.
(5) $\Gamma(u)^{-1}(\lambda \to \bot) = \Gamma(u)^{\leftarrow}(\lambda) \to \bot$, *for all* $\lambda \in L^X$.
(6) $\Gamma(u_1 \circ u_2) = \Gamma(u_2) \circ \Gamma(u_1)$.
(7) $\Gamma(\alpha \odot u) = \alpha \odot \Gamma(u)$.
(8) *If u is an $\odot$-equivalence relation on X, then*

$$(\Gamma(u))^{-1} = \Gamma(u^s) = \Gamma(u), \quad \Gamma(u) \circ \Gamma(u) = \Gamma(u).$$

Proof. (1) It is easy from the definition of Γ.
(2) $\Gamma(u_1 \odot u_2) \leq \Gamma(u_1) \otimes \Gamma(u_2)$ from the following: for all $\lambda = \lambda_1 \odot \lambda_2$,

$$\begin{aligned}
&\Gamma(u_1 \odot u_2)(\lambda_1 \odot \lambda_2)(y) \\
&= \bigvee_{x\in X}(\lambda_1 \odot \lambda_2)(x) \odot (u_1 \odot u_2)(x,y) \\
&\leq (\bigvee_{x\in X} \lambda_1(x) \odot u_1(x,y)) \odot (\bigvee_{z\in X} \lambda_2(z) \odot u_2(z,y)) \\
&= \Gamma(u_1)(\lambda_1)(y) \odot \Gamma(u_2)(\lambda_2)(y).
\end{aligned}$$

(3) $\Gamma(1_\Delta)(\lambda)(y) = \bigvee_{x\in X}(\lambda(x) \odot 1_\Delta(x,y)) = \lambda(y)$ for all $y \in X$.
(4) $(\Gamma(u))^{-1} \leq \Gamma(u^s)$ from:

$$\begin{aligned}
&\Gamma(u)\Big(\Gamma(u^s)(\lambda) \to \bot\Big)(y) \\
&= \bigvee_{x\in X} \Big(\Gamma(u^s)(\lambda) \to \bot\Big)(x) \odot u(x,y) \\
&= \bigvee_{x\in X} \Big(\bigvee_{z\in X} \lambda(z) \odot u(-,z)) \to \bot\Big)(x) \odot u(x,y) \\
&= \bigvee_{x\in X} \bigwedge_{z\in X} \Big(\lambda(z) \odot u(x,z) \to \bot\Big) \odot u(x,y) \\
&\leq \bigvee_{x\in X} \Big(\lambda(y) \odot u(x,y) \to \bot\Big) \odot u(x,y) \\
&\leq \lambda(y) \to \bot
\end{aligned}$$

$(\Gamma(u))^{-1} \geq \Gamma(u^s)$ from:

$$\begin{aligned}
&\Gamma(u)(\rho \to \bot)(y) = \bigvee_{x\in X}(\rho \to \bot)(x) \odot u(x,y) \leq \lambda(y) \to \bot \\
&\Leftrightarrow (\rho \to \bot)(x) \odot u(x,y) \leq \lambda(y) \to \bot \\
&\Leftrightarrow (\rho \to \bot)(x) \leq u(x,y) \to (\lambda(y) \to \bot) \\
&\Leftrightarrow \rho(x) \geq \lambda(y) \odot u(x,y) \\
&\Leftrightarrow \rho(x) \geq \bigvee_{y\in X} \lambda(y) \odot u^s(y,x)
\end{aligned}$$

(5) For all $\lambda \in L^X$, $\Gamma(u)^{-1}(\lambda \to \bot) = \Gamma(u)^{\leftarrow}(\lambda) \to \bot$ from:

$$\begin{aligned}
\Gamma(u)^{-1}(\lambda \to \bot) &= \bigwedge\{\rho \in L^X \mid \Gamma(u)(\rho^*) \leq \lambda\} \\
&= \bigvee\{\rho^* \in L^X \mid \Gamma(u)(\rho^*) \leq \lambda\} \to \bot \\
&= \Gamma(u)^{\leftarrow}(\lambda) \to \bot.
\end{aligned}$$

(6)

$$\begin{aligned}
&\Gamma(u_2)(\Gamma(u_1)(\lambda))(y) \\
&= \bigvee_{x\in X} \Gamma(u_1)(\lambda)(x) \odot u_2(x,y) \\
&= \bigvee_{x\in X} \Big(\bigvee_{z\in X} \lambda(z) \odot u_1(z,x)\Big) \odot u_2(x,y) \\
&= \bigvee_{x\in X} \bigvee_{z\in X} \Big(\lambda(z) \odot (u_1(z,x) \odot u_2(x,y))\Big) \\
&= \bigvee_{z\in X} \Big(\lambda(z) \odot \bigvee_{x\in X}(u_1(z,x) \odot u_2(x,y))\Big) \\
&= \bigvee_{z\in X} \Big(\lambda(z) \odot (u_1 \circ u_2)(z,y)\Big) \\
&= \Gamma(u_1 \circ u_2)(\lambda)(y).
\end{aligned}$$

(7) Obvious.
(8) Since $u \circ u \leq u$ and $u^s = u$, it is easily proved.

Example 3.13 Let X, $(L = [0,1], \odot)$ and $u \in E(X \times X)$ be defined as in Example 3.11. Since $\Gamma(u)^{-1}(1_x) = \bigwedge\{\rho^* \mid \Gamma(u)(\rho) \leq 1_{\{y,z\}}\}$, then

$$\Gamma(u)^{-1}(1_x) =. \Big(0.3 \odot 1_{\{y\}} \vee 0.5 \odot 1_{\{z\}}\Big) \to \bot$$

$$\Gamma(u^s)(1_{\{x\}}) = \bigvee_{z\in X} 1_{\{x\}}(z) \odot u^s(z,-) = u^s(x,-) = u(-,x).$$

It follows $\Gamma(u)^{-1}(1_{\{x\}}) = \Gamma(u^s)(1_{\{x\}})$. Similarly, we have $\Gamma(u)^{-1}(1_{\{z\}}) = \Gamma(u^s)(1_{\{z\}})$ for all $z \in X$. Hence $\Gamma(u)^{-1} = \Gamma(u^s)$.

Theorem 3.14 *Let $u : X \times X \to L$ be an $\odot$-equivalence relation. We define a mapping $\mathbf{U}_u$ as follows:*

$$\mathbf{U}_u = \{\phi \in \Omega(X) \mid \exists n \in N, \ \Gamma(u^n) \leq \phi\}.$$

Then $\mathbf{U}_u$ is a Hutton $(L, \otimes)$-uniformity on X.

Proof. (QU1) Obvious.

(QU2) If $\phi, \psi \in \mathbf{U}_u$, there exist $m, n \in N$ such that $\Gamma(u^m) \leq \phi$ and $\Gamma(u^n) \leq \psi$. Then $\Gamma(u^{m+n}) \leq \Gamma(u^m) \otimes \Gamma(u^n) \leq \phi \otimes \psi$. Hence $\phi \otimes \psi \in \mathbf{U}_u$.

(QU3) For $\phi \in \mathbf{U}_u$, there exists $n \in N$ such that $\Gamma(u^n) \leq \phi$. Since $u^n \circ u^n \leq (u \circ u)^n \leq u^n$, there exists $\Gamma(u^n) \in \mathbf{U}_u$ such that

$$\Gamma(u^n) \circ \Gamma(u^n) = \Gamma(u^n \circ u^n) \leq \Gamma((u \circ u)^n) \leq \Gamma(u^n) \leq \phi.$$

(U) For $\phi \in \mathbf{U}_u$, there exists $n \in N$ such that $\Gamma(u^n) \leq \phi$. Then $\phi^{-1} \in \mathbf{U}_u$ because

$$\Gamma(u^n) = \Gamma((u^s)^n) = \Gamma((u^n)^s) = \Gamma(u^n)^{-1} \leq \phi^{-1}.$$

Theorem 3.15 *Let $\phi, \phi_1, \phi_2 \in \Omega(X)$. Then we have the following properties:*

(1) *If $\phi_1 \leq \phi_2$, then $\Lambda(\phi_1) \leq \Lambda(\phi_2)$.*

(2) $\Lambda(\phi_1) \odot \Lambda(\phi_2) = \Lambda(\phi_1 \otimes \phi_2)$.

(3) $\Lambda(1_{L^X}) = 1_\Delta$.

(4) $\Lambda(\phi)^s = \Lambda(\phi^{-1})$.

(5) $\Lambda(\phi_1) \circ \Lambda(\phi_2) = \Lambda(\phi_2 \circ \phi_1)$.

(6) $\Lambda(\alpha \odot \phi) = \alpha \odot \Lambda(\phi)$.

(7) *If $\phi \circ \phi = \phi$ and $\phi = \phi^{-1}$, then $\Lambda(\phi)$ is an $\odot$-equivalence relation.*

Proof. (1) It is easy from the definition of Λ.

(2) We have $\Lambda(\phi_1) \odot \Lambda(\phi_1) = \Lambda(\phi_1 \otimes \phi_2)$ because

$$(\Lambda(\phi_1) \odot \Lambda(\phi_2))(x, y) = \phi_1(1_{\{x\}})(y) \odot \phi_2(1_{\{x\}})(y)$$
$$= (\phi_1 \otimes \phi_2)(1_{\{x\}})(y).$$

(3) $\Lambda(1_{L^X})(x, y) = 1_{L^X}(1_{\{x\}})(y) = 1_{\{x\}}(y) = 1_\Delta(x, y)$.

(4) Suppose $\Lambda(\phi)^s(x, y) \not\leq \Lambda(\phi^{-1})(x, y)$. Since

$$\Lambda(\phi)^s(x, y) = \Lambda(\phi)(y, x) = \phi(1_{\{y\}})(x)$$
$$\Lambda(\phi^{-1})(x, y) = \phi^{-1}(1_{\{x\}})(y),$$

by the definition of $\Lambda(\phi^{-1})$, there exists λ with $\phi(\lambda^*) \leq 1_{\{x\}} \to \perp$ such that

$$\Lambda(\phi)^s(x, y) \not\leq \lambda(y).$$

Since $\phi(\lambda^*) \leq 1_{\{x\}} \to \perp$ implies $\phi(\lambda^*)(x) = \perp$, we have

$$\begin{aligned} &\Lambda(\phi)^s(x, y) \not\leq \lambda(y) \\ &\Rightarrow \Lambda(\phi)^s(x, y) \not\leq (\phi(\lambda^*)(x) \to \perp) \to \lambda(y) \\ &\Leftrightarrow \Lambda(\phi)^s(x, y) \not\leq \lambda^*(y) \to \phi(\lambda^*)(x) \\ &\text{(by Lemma 2.3(8))} \end{aligned}$$

Since $\lambda^* = \bigvee_{z \in X} \lambda^*(z) \odot 1_{\{z\}}$, we have

$$\begin{aligned} &\lambda^*(y) \to \phi(\lambda^*)(x) \\ &= \lambda^*(y) \to \textstyle\bigvee_{z \in X} (\phi(\lambda^*(z) \odot 1_{\{z\}})(x) \\ &\geq \lambda^*(y) \to \phi(\lambda^*(y) \odot 1_{\{y\}})(x) \\ &\geq \phi(1_{\{y\}})(x). \end{aligned}$$

Thus,

$$\Lambda(\phi)^s(x, y) \not\leq \phi(1_{\{y\}})(x).$$

It is a contradiction. Hence $\Lambda(\phi)^s \leq \Lambda(\phi^{-1})$.

Suppose $\Lambda(\phi)^s(x, y) \not\geq \Lambda(\phi^{-1})(x, y)$. By the definition of $\Lambda(\phi)^s(x, y)$, there exists $\alpha \in L$ such that

$$\alpha \to \alpha \odot \phi(1_{\{y\}})(x) \not\geq \Lambda(\phi^{-1})(x, y)$$

Since $\alpha \odot \phi(1_{\{y\}}) = \phi(\alpha \odot 1_{\{y\}})$, we have $\phi(1_{\{y\}}) \leq \alpha \to \alpha \odot \phi(1_{\{y\}})$. Put $\rho = \Big(\alpha \to \alpha \odot \phi(1_{\{y\}})\Big) \to \perp$, then $\phi^{-1}(\rho) \leq 1^*_{\{y\}}$. Since $\rho = \bigvee_{z \in X} \rho(z) \odot 1_{\{z\}}$. Put $\beta = \rho(x)$. Then

$$\beta \to \beta \odot \phi^{-1}(1_{\{x\}})(y) \leq \beta \to 1^*_{\{y\}}(y) = \beta^*$$

It implies

$$\begin{aligned} \Lambda(\phi^{-1})(x, y) &= \textstyle\bigwedge_\alpha \Big(\alpha \to \alpha \odot \phi^{-1}(1_{\{x\}})(y)\Big) \\ &\leq \beta \to \beta \odot \phi^{-1}(1_{\{x\}})(y) \\ &\leq \beta \to 1^*_{\{y\}}(y) \\ &= \beta^* = \alpha \to \alpha \odot \phi(1_{\{y\}})(x). \end{aligned}$$

It is a contradiction. Hence $\Lambda(\phi)^s \geq \Lambda(\phi^{-1})$.

(5) $\Lambda(\phi_1) \circ \Lambda(\phi_2) \leq \Lambda(\phi_1 \circ \phi_2)$ from:

$$\begin{aligned} &(\Lambda(\phi_1) \circ \Lambda(\phi_2))(x, y) \\ &= \textstyle\bigvee_z \Big\{\phi_1(1_{\{x\}})(z) \odot \phi_2(1_{\{z\}})(y)\Big\} \\ &= \textstyle\bigvee_z \Big(\phi_2(\phi_1(1_{\{x\}})(z) \odot 1_{\{z\}})\Big)(y) \\ &= \phi_2\Big(\textstyle\bigvee_z \phi_1(1_{\{x\}})(z) \odot 1_{\{z\}}\Big)(y) \\ &= \phi_2 \circ \phi_1(1_{\{x\}})(y) \end{aligned}$$

(6) $\lambda(\alpha \odot \phi)(x, y) = (\alpha \odot \phi(1_{\{x\}}))(y) = \alpha \odot \phi(1_{\{x\}})(y) = \alpha \odot \lambda(\phi)(x, y)$.

(7) Since $\phi \circ \phi = \phi$ and $\phi = \phi^{-1}$, it is proved from

$$\Lambda(\phi) \circ \Lambda(\phi) = \Lambda(\phi), \Lambda(\phi^{-1}) = \Lambda(\phi)^s = \Lambda(\phi).$$

Theorem 3.16 *Let $\mathbf{D}$ be an $(L, \odot)$-uniform space. We define a mapping $\mathbf{U_D} \subset \Omega(X)$ as follows:*

$$\mathbf{U_D} = \{\phi \in \Omega(X) \mid \exists u \in \mathbf{D}, \Gamma(u) \leq \phi\}.$$

Then $\mathbf{U_D}$ is a Hutton $(L, \otimes)$- uniformity on X.

Proof. (QU1) Obvious. (QU2) Let $\phi_i \in \mathbf{U_D}$ for $i = 1, 2$. There exists $u_i \in \mathbf{D}$ such that $\Gamma(u_i) \leq \phi_i$. Since

$$\Gamma(u_1 \odot u_2) \leq \Gamma(u_1) \otimes \Gamma(u_2) \leq \phi_1 \otimes \phi_2,$$

we have $\phi_1 \otimes \phi_2 \in \mathbf{U_D}$.

(QU3) Let $\phi \in \mathbf{U_D}$. There exists $u \in \mathbf{D}$ such that $\Gamma(u) \leq \phi$. Since $\mathbf{D}$ is an $(L, \odot)$-uniformity, for $u \in \mathbf{D}$, there exists $v \in \mathbf{D}$ such that $v \circ v \leq u$. Since

$$\Gamma(v) \circ \Gamma(v) = \Gamma(v \circ v) \leq \Gamma(u) \leq \phi,$$

there exists $\Gamma(v) \in \Omega(X)$.

(U) Let $\phi \in \mathbf{U_D}$. There exists $u \in \mathbf{D}$ such that $\Gamma(u) \leq \phi$. Since $\mathbf{D}$ is an $(L, \odot)$-uniformity, for $u \in \mathbf{D}$, there exists $u^s \in \mathbf{D}$ such that $\Gamma(u^s) = \Gamma(u)^{-1} \leq \phi^{-1}$ from Lemma 3.3(4). Hence $\phi^{-1} \in \mathbf{U_D}$.

Theorem 3.17 *Let* $\mathbf{U}$ *be a Hutton* $(L, \otimes)$*-uniformity on* X. *We define a mapping* $\mathbf{D_U} \subset E(X \times X)$ *as follows:*

$$\mathbf{D_U} = \{u \in E(X \times X) \mid \exists \phi \in \mathbf{U}, \Lambda(\phi) \leq u\}.$$

Then:

(1) $\mathbf{D_U}$ *is an* $(L, \odot)$*-uniformity on* X.

(2) $\mathbf{D_{U_D}} = \mathbf{D}$ *and* $\mathbf{U_{D_U}} = \mathbf{U}$.

Proof. (1) (QD2) If $u_i \in \mathbf{D_U}$ for $i = 1, 2$, then there exist $\phi_i \in \mathbf{U}$ such that $\Lambda(\phi_i) \leq u_i$. Since

$$\Lambda(\phi_1 \otimes \phi_2) = \Lambda(\phi_1) \odot \Lambda(\phi_2) \leq u_1 \odot u_2,$$

then $u_1 \odot u_2 \in \mathbf{D_U}$.

(QD3) If $u \in \mathbf{D_U}$, then there exists $\phi \in \mathbf{U}$ such that $\Lambda(\phi) \leq u$. Since $\mathbf{U}$ is a Hutton $(L, \odot)$-uniformity, for $\phi \in \mathbf{U}$, there exists $\psi \in \mathbf{U}$ such that $\psi \circ \psi \leq \phi$. Since

$$\Lambda(\psi) \circ \Lambda(\psi) = \Lambda(\psi \circ \psi) \leq \Lambda(\phi) \leq u$$

then $\Lambda(\psi) \in \mathbf{D_U}$.

(D) If $u \in \mathbf{D_U}$, then there exists $\phi \in \mathbf{U}$ such that $\Lambda(\phi) \leq u$. Since $\mathbf{U}$ is a Hutton $(L, \odot)$-uniformity, for $\phi \in \mathbf{U}$, there exists $\phi^{-1} \in \mathbf{U}$ such that $\Lambda(\phi^{-1}) = \Lambda(\phi)^s \leq u^s$. Thus, $u^s \in \mathbf{D_U}$.

(2) Let $u \in \mathbf{D_{U_D}}$. Then there exists $\phi \in \mathbf{U_D}$ such that $\Lambda(\phi) \leq u$. Since $\phi \in \mathbf{U_D}$, there exists $v \in \mathbf{D}$ such that $\Gamma(v) \leq \phi$. Since $v = \Lambda(\Gamma(v)) \leq u$, then $u \in \mathbf{D}$. Hence $\mathbf{D_{U_D}} \subset \mathbf{D}$. Let $u \in \mathbf{D}$. Then $\Gamma(u) \in \mathbf{U_D}$. Also, $u = \Lambda(\Gamma(u)) \in \mathbf{D_{U_D}}$. Similarly, we prove $\mathbf{U_{D_U}} = \mathbf{U}$.

Example 3.18 Let $\mathbf{U} = \{\psi \in \Omega(X) \mid \phi \leq \psi\}$ be defined as in Theorem 3.5. We obtain $\mathbf{D_U} = \{u \in E(X \times X) \mid \Lambda(\phi) \leq u\}$. Since $\phi \circ \phi = \phi$ and $\phi^{-1} = \phi$, by Theorem 3.15(7), $\Lambda(\phi)$ is an $\odot$-equivalence relation such that

$$\Lambda(\phi)(x, y) = \phi(1_{\{x\}})(y) = 1_{\{x,y\}}(y) = 1,$$

$$\Lambda(\phi)(x, x) = 1,\ \Lambda(\phi)(x, z) = 0$$

$$\Lambda(\phi)(y, x) = 1,\ \Lambda(\phi)(y, y) = 1,\ \Lambda(\phi)(y, z) = 0$$

$$\Lambda(\phi)(z, x) = 0,\ \Lambda(\phi)(z, y) = 0,\ \Lambda(\phi)(z, z) = 1$$

Furthermore, $\Lambda(\phi) \circ \Lambda(\phi) = \Lambda(\phi)$, $\Lambda(\phi^{-1}) = \Lambda(\phi)^s = \Lambda(\phi)$ and $\Lambda(\phi) \odot \Lambda(\phi) = \Lambda(\phi \otimes \phi) = \Lambda(\phi)$. Hence $\mathbf{D_U}$ is an $(L, \odot)$-uniformity on X and $\mathbf{U_{D_U}} = \mathbf{U}$.

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