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EXISTENCE AND EXPONENTIAL STABILITY OF ALMOST PERIODIC SOLUTIONS FOR CELLULAR NEURAL NETWORKS WITH CONTINUOUSLY DISTRIBUTED DELAYS

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ABSTRACT. In this paper cellular neural networks with continuously distributed delays are considered. Sufficient conditions for the existence and exponential stability of the almost periodic solutions are established by using fixed point theorem, Lyapunov functional method and differential inequality technique. The results of this paper are new and they complement previously known results.

1. Introduction

Consider the following models for cellular neural networks(CNNs) with continuously distributed delays

$$
\begin{aligned}
x_i'(t) = &- c_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t)f_j(x_j(t)) \\
&+ \sum_{j=1}^{n} b_{ij}(t) \int_0^\infty K_{ij}(u)f_j(x_j(t-u))du + I_i(t),
\end{aligned}
\tag{1.1}
$$

in which n corresponds to the number of units in a neural network, $x_i(t)$ corresponds to the state vector of the ith unit at the time t, $c_i(t) > 0$ represents the rate with which the ith unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs at the time t. $a_{ij}(t)$, $b_{ij}(t)$ are the connection weights

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at the time t, and $I_i(t)$ denote the external inputs at time t. f_j ($j = 1, 2, \dots, n$) are signal transmission functions.

Throughout this paper, it will be assumed that $c_i, I_i, a_{ij}, b_{ij} : R \to R$ are almost periodic functions, where $i,\ j = 1, 2, \dots, n$. We suppose that $\tilde{c_i}$, $\overline{a_{ij}}$, $\overline{b_{ij}}$ and $\overline{I_i}$ are constants such that

$$\begin{aligned} &0 < \tilde{c_i} = \inf_{t\in R} |c_i(t)|, \quad \sup_{t\in R} |a_{ij}(t)| = \overline{a_{ij}}, \\ &\sup_{t\in R} |b_{ij}(t)| = \overline{b_{ij}}, \quad \sup_{t\in R} |I_i(t)| = \overline{I_i}, \end{aligned} \tag{1.2}$$

where $i,\ j = 1,\ 2, \dots, n$.

We also assume that the following conditions (T_0), (T_1) and (T_2) hold.

(T_0) For $j \in \{1,\ 2, \dots, n\}$, $f_j : R \to R$ are Lipschitz continuous with Lipschitz constants L_j, that is,

$$|f_j(u_j) - f_j(v_j)| \le L_j|u_j - v_j|, \forall\ u_j,\ v_j \in R.$$

(T_1) For $i, j \in \{1,\ 2, \dots, n\}$, the delay kernels $K_{ij} : [0, \infty) \to R$ are continuous, integrable and satisfy

$$\int_0^\infty |K_{ij}(s)| ds \le k_{ij}.$$

(T_2) For $i, j \in \{1,\ 2, \dots, n\}$, there exists a constant $\lambda_0 > 0$ such that

$$\int_0^\infty |K_{ij}(s)| e^{\lambda_0 s} ds < +\infty.$$

The initial conditions associated with system (1.1) are of the form

$$x_i(s) = \varphi_i(s), s \in (-\infty,\ 0],\ i = 1, 2, \dots, n, \tag{1.3}$$

where $\varphi = (\varphi_1(t), \varphi_2(t), \dots, \varphi_n(t))^T$, $\varphi_i(t) : R \to R,\ \ i = 1,\ 2, \dots, n$, are almost periodic functions.

DEFINITION 1.1. (see [7, 11]) Let $u(t) : R \longrightarrow R^n$ be continuous in t. $u(t)$ is said to be almost periodic on R if, for any $\varepsilon > 0$, the set $T(u, \varepsilon) = \{\delta : |u(t + \delta) - u(t)| < \varepsilon,\ \forall t \in R\}$ is relatively dense, i.e., for $\forall \varepsilon > 0$, it is possible to find a real number $l = l(\varepsilon) > 0$, for any interval with length $l(\varepsilon)$, there exists a number $\delta = \delta(\varepsilon)$ in this interval such that $|u(t + \delta) - u(t)| < \varepsilon$, for $\forall t \in R$.

DEFINITION 1.2. Let $Z^*(t) = (x_1^*(t), x_2^*(t), \dots, x_n^*(t))^T$ be an almost periodic solution of system (1.1) with initial value $\varphi^* = (\varphi_1^*(t), \varphi_2^*(t), \dots, \varphi_n^*(t))^T$. If there exist constants $\lambda > 0$ and $M_\varphi > 1$ such that for every solution $Z(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$ of system (1.1) with any initial value $\varphi = (\varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t))^T$,

$$|x_i(t) - x_i^*(t)| \leq M_\varphi \|\varphi - \varphi^*\| e^{-\lambda t}, \forall t > 0, \quad i = 1,\ 2, \ldots,\ n,$$

where $\varphi_i(t), \varphi_i^*(t) : R \to R,\ \ i = 1,\ 2, \ldots, n$, are almost periodic functions, $\|\varphi - \varphi^*\| = \sup\limits_{-\infty \leq s \leq 0} \max\limits_{1 \leq i \leq n} |\varphi_i(s) - \varphi_i^*(s)|$. Then $Z^*(t)$ is said to be global exponential stable.

It is well known that the delayed cellular neural networks (DCNNs) have been successfully applied to signal and image processing, pattern recognition and optimization. Hence, they have been the object of intensive analysis by numerous authors in recent years. In particular, there have been extensive results on the problem of the existence and stability of periodic solutions of system (1.1) in the literature. We refer the reader to [2, 3, 6, 8, 9, 12, 16] and the references cited therein. However, there exist few results on the existence and exponential stability of the almost periodic solutions of system (1.1). We only find that Liu [15] and Chen [4] studied the existence and exponential stability of the almost periodic solutions of CNNs with continuously distributed delays. In [15], we also find that the conditions (T_2) is a sufficient condition for the functions $\mathcal{G}_i(\mu)$ and $\mathcal{F}_j(\mu)$ of (4.2) to be continuous functions. Therefore, we can introduce the main results of [15] and [4] as follows.

THEOREM A. *Suppose that (T_0) and (T_1) hold, $K_{ij}(s) \geq 0$ for all $s \in [0, \infty)$, $M[c_i] > 0$, and there exist constants $\lambda_0 > 0$ and $r < 1$ such that*

$$(1.4) \quad \int_0^\infty K_{ij}(s) e^{\lambda_0 s} ds < +\infty, \ r = \max_{1 \leq i \leq n} \Big\{ \sum_{j=1}^n \tilde{c_i}^{-1} (\overline{a_{ij}} + \overline{b_{ij}} k_{ij}) L_j \Big\},$$

where $i, j = 1, 2, \ldots, n$, $M[c_i] = \lim\limits_{T \to +\infty} \frac{1}{T} \int_t^{t+T} c_i(s) ds$. Then system (1.1) has exactly one almost periodic solution. Moreover, the almost periodic solution is globally exponentially stable.

The main purpose of this paper is to give the conditions for the existence and exponentially stability of the almost periodic solutions for system (1.1). By applying fixed point theorem and differential inequality technique, we derive some new sufficient conditions ensuring the existence, uniqueness and exponential stability of the almost periodic solution, which impose a less restrictive constraint than those in [15] and [4], and improve and extend some previous result. Moreover, an example is also provided to illustrate the effectiveness of the new results.

For convenience, we introduce some notations. We will use $x = (x_1, x_2, \ldots, x_n)^T \in R^n$ to denote a column vector, in which the symbol $(^T)$ denotes the transpose of a vector. For matrix $D = (d_{ij})_{n\times n}$, D^T denotes the transpose of D, and E_n denotes the identity matrix of size n. A matrix or vector $D \geq 0$ means that all entries of D are greater than or equal to zero. $D > 0$ can be defined similarly. For matrices or vectors D and E, $D \geq E$ (resp. $D > E$) means that $D - E \geq 0$ (resp. $D - E > 0$).

The following lemmas and definitions will be useful to prove our main results in Section 2.

Definition 1.3. (see [7, 11]) Let $x \in R^n$ and $Q(t)$ be an $n \times n$ continuous matrix defined on R. The linear system

$$x'(t) = Q(t)x(t) \tag{1.5}$$

is said to admit an exponential dichotomy on R if there exist positive constants k, α, projection P and the fundamental solution matrix $X(t)$ of (1.5) satisfying

$$\|X(t)PZ^{-1}(s)\| \leq ke^{-\alpha(t-s)} \quad \text{for } t \geq s,$$
$$\|X(t)(I-P)X^{-1}(s)\| \leq ke^{-\alpha(s-t)} \quad \text{for } t \leq s.$$

Lemma 1.1. (see [7, 11]) *If the linear system* (1.5) *admits an exponential dichotomy, then almost periodic system*

$$x'(t) = Q(t)x + g(t) \tag{1.6}$$

has a unique almost periodic solution $x(t)$, and

$$x(t) = \int_{-\infty}^{t} X(t)PZ^{-1}(s)g(s)ds - \int_{t}^{+\infty} X(t)(I-P)X^{-1}(s)g(s)ds. \tag{1.7}$$

Lemma 1.2. (see [7, 11]) *Let $c_i(t)$ be an almost periodic function on R and*

$$M[c_i] = \lim_{T\to+\infty} \frac{1}{T}\int_{t}^{t+T} c_i(s)ds > 0, \quad i = 1, 2, \ldots, n.$$

Then the linear system

$$x'(t) = diag(-c_1(t), -c_2(t), \ldots, -c_n(t))x(t)$$

admits an exponential dichotomy on R.

Definition 1.4. (see [1, 14]) A real $n \times n$ matrix $W = (w_{ij})_{n\times n}$ is said to be an M-matrix if $w_{ij} \leq 0$, $i, j = 1, 2, \ldots, n$, $i \neq j$, and $W^{-1} \geq 0$, where W^{-1} denotes the inverse of W.

LEMMA 1.3. (see [1, 14]) *Let $W = (w_{ij})_{n\times n}$ with $w_{ij} \leq 0,\ i,j = 1,\ 2,\dots,n,\ i \neq j$. Then the following statements are equivalent.*

(1) *W is an M-matrix.*
(2) *There exists a vector $\eta = (\eta_1, \eta_2, \dots, \eta_n) > (0,0,\dots,0)$ such that $\eta W > 0$.*
(3) *There exists a vector $\xi = (\xi_1, \xi_2, \dots, \xi_n)^T > (0,0,\dots,0)^T$ such that $W\xi > 0$.*

LEMMA 1.4. (see [1, 14]) *Let $A \geq 0$ be an $n\times n$ matrix and $\rho(A) < 1$. Then $(E_n - A)^{-1} \geq 0$, where $\rho(A)$ denotes the spectral radius of A.*

2. Existence of almost periodic solutions

THEOREM 2.1. *Suppose that (T_0) and (T_1) hold, and $\rho(D^{-1}\overline{E}L) < 1$, where*

$$D = diag(\tilde{c_1},\ \tilde{c_2},\ \dots,\tilde{c_n}),\ \bar{A} = (\overline{a_{ij}})_{n\times n},$$
$$\bar{B} = (\overline{b_{ij}}k_{ij})_{n\times n},\quad L = diag(L_1,\ L_2,\dots,L_n),$$
$$I = (\overline{I_1},\ \overline{I_2},\dots,\ \overline{I_n}),\ \ \overline{E} = (\overline{a_{ij}} + \overline{b_{ij}}k_{ij})_{n\times n} = \bar{A} + \bar{B}.$$

Then, there exists at least one almost periodic solution of system (1.1).

Proof. Let

$$X = \{\phi|\phi = (\phi_1(t),\ \phi_2(t),\dots,\phi_n(t))^T\},$$

where $\phi_i : R \to R$ is an almost periodic function, $i = 1,2,\dots,n$. Then, X is a Banach space with the norm defined by $\|\phi\|_X = \sup\limits_{t\in R}\max\limits_{1\leq i\leq n} |\phi_i(t)|$.

To proceed further, we need to introduce an auxiliary equation

$$\begin{aligned} x_i'(t) =& -c_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t)f_j(\phi_j(t)) \\ &+ \sum_{j=1}^{n} b_{ij}(t)\int_0^\infty K_{ij}(u)f_j(\phi_j(t-u))du + I_i(t), \end{aligned} \tag{2.1}$$

where $i = 1,\ 2,\dots,n,\ \phi(t) = (\phi_1(t),\ \phi_2(t),\dots,\phi_n(t))^T \in X$. Notice that $M[c_i] > 0,\ i = 1,\ 2,\dots,n$, it follows from Lemma 1.2 that the linear system

$$x'(t) = -c_i(t)x(t), i = 1,\ 2,\dots,n, \tag{2.2}$$

admits an exponential dichotomy on R. Thus, by Lemma 1.1, we obtain that the system (2.1) has exactly one almost periodic solution:

$$
\begin{aligned}
x^{\phi}(t) &= (x_1^{\phi}(t),\ x_2^{\phi}(t), \ldots, x_n^{\phi}(t))^T \\
&= \Big(\int_{-\infty}^{t} e^{-\int_s^t c_1(u)du} [\sum_{j=1}^{n} a_{1j}(s) f_j(\phi_j(s)) \\
&\quad + \sum_{j=1}^{n} b_{1j}(s) \int_0^{\infty} K_{1j}(u) f_j(\phi_j(s-u))du \\
&\quad + I_1(s)]ds, \ldots, \quad \int_{-\infty}^{t} e^{-\int_s^t c_n(u)du} [\sum_{j=1}^{n} a_{nj}(s) f_j(\phi_j(s)) \\
&\quad + \sum_{j=1}^{n} b_{nj}(s) \cdot \int_0^{\infty} K_{nj}(u) f_j(\phi_j(s-u))du + I_n(s)]ds \Big)^T.
\end{aligned} \tag{2.3}
$$

Define a mapping $\Phi : X \longrightarrow X$ by setting

$$\Phi(\phi(t)) = x^{\phi}(t),\ \forall\ \phi \in X.$$

Let $\phi, \psi \in X$. Then, by (T_0) and (T_1), we have

$$
\begin{aligned}
&|\Phi(\phi(t)) - \Phi(\psi(t))| \\
&= (|(\Phi(\phi(t)) - \Phi(\psi(t)))_1|, \ldots,\ |(\Phi(\phi(t)) - \Phi(\psi(t)))_n|)^T \\
&= \Big(\Big| \int_{-\infty}^{t} e^{-\int_s^t c_1(u)du} (\sum_{j=1}^{n} a_{1j}(s)(f_j(\phi_j(s)) - f_j(\psi_j(s))) \\
&\quad + \sum_{j=1}^{n} b_{1j}(s) \int_0^{\infty} K_{1j}(u) \cdot (f_j(\phi_j(s-u)) - f_j(\psi_j(s-u)))du)ds \Big|, \\
&\quad \ldots, \Big| \int_{-\infty}^{t} e^{-\int_s^t c_n(u)du} (\sum_{j=1}^{n} a_{nj}(s)(f_j(\phi_j(s)) - f_j(\psi_j(s))) \\
&\quad + \sum_{j=1}^{n} b_{nj}(s) \int_0^{\infty} K_{nj}(u)(f_j(\phi_j(s-u)) - f_j(\psi_j(s-u)))du)ds \Big| \Big)^T \\
&\le \Big(\int_{-\infty}^{t} e^{-\bar{c_1}(t-s)} (\sum_{j=1}^{n} \overline{a_{1j}} L_j |\phi_j(s) - \psi_j(s)|
\end{aligned}
$$

$$+\sum_{j=1}^{n}\overline{b_{1j}}\int_0^{\infty}|K_{1j}(u)|L_j|\phi_j(s-u)-\psi_j(s-u)|du)ds,$$

$$\ldots,\int_{-\infty}^{t}e^{-\tilde{c_n}(t-s)}\bigg(\sum_{j=1}^{n}\overline{a_{nj}}L_j|\phi_j(s)-\psi_j(s)|+\sum_{j=1}^{n}\overline{b_{nj}}$$

$$\cdot\int_0^{\infty}|K_{nj}(u)|L_j|\phi_j(s-u)-\psi_j(s-u)|du)ds\bigg)^T$$

$$\leq\bigg(\int_{-\infty}^{t}e^{-\tilde{c_1}(t-s)}\bigg(\sum_{j=1}^{n}\overline{a_{1j}}L_j\sup_{t\in R}|\phi_j(t)-\psi_j(t)|$$

$$+\sum_{j=1}^{n}\overline{b_{1j}}\int_0^{\infty}|K_{1j}(u)|L_j\sup_{t\in R}|\phi_j(t)-\psi_j(t)|du)ds,$$

$$\ldots,\int_{-\infty}^{t}e^{-\tilde{c_n}(t-s)}\bigg(\sum_{j=1}^{n}\overline{a_{nj}}L_j\sup_{t\in R}|\phi_j(t)-\psi_j(t)|+\sum_{j=1}^{n}\overline{b_{nj}}$$

$$\cdot\int_0^{\infty}|K_{nj}(u)|L_j\sup_{t\in R}|\phi_j(t)-\psi_j(t)|du)ds\bigg)^T$$

$$\leq\bigg(\sum_{j=1}^{n}\tilde{c_1}^{-1}(\overline{a_{1j}}+\overline{b_{1j}}k_{1j})L_j\sup_{t\in R}|\phi_j(t)-\psi_j(t)|,$$

$$\ldots,\sum_{j=1}^{n}\tilde{c_1}^{-1}(\overline{a_{nj}}+\overline{b_{nj}}k_{nj})L_j\cdot\sup_{t\in R}|\phi_j(t)-\psi_j(t)|\bigg)^T,$$

which implies that

(2.4)

$$\bigg(\sup_{t\in R}|(\Phi(\phi(t))-\Phi(\psi(t)))_1|,\ldots,\sup_{t\in R}|(\Phi(\phi(t))-\Phi(\psi(t)))_n|\bigg)^T$$

$$\leq\bigg(\sum_{j=1}^{n}\tilde{c_1}^{-1}(\overline{a_{1j}}+\overline{b_{1j}}k_{1j})L_j\sup_{t\in R}|\phi_j(t)-\psi_j(t)|,$$

$$\ldots,\sum_{j=1}^{n}\tilde{c_n}^{-1}(\overline{a_{nj}}+\overline{b_{nj}}k_{nj})L_j\cdot\sup_{t\in R}|\phi_j(t)-\psi_j(t)|\bigg)^T$$

$$\leq F\bigg(\sup_{t\in R}|\phi_1(t)-\psi_1(t)|,\ldots,\sup_{t\in R}|\phi_n(t)-\psi_n(t)|\bigg)^T$$

$$=F\bigg(\sup_{t\in R}|(\phi(t)-\psi(t))_1|,\ldots,\sup_{t\in R}|(\phi(t)-\psi(t))_n|\bigg)^T,$$

where $F = D^{-1}\overline{E}L$. Let m be a positive integer. Then, from (2.4), we get

(2.5)

$$\begin{aligned}
&\left(\sup_{t\in R}|(\Phi^m(\phi(t)) - \Phi^m(\psi(t)))_1|, \dots, \sup_{t\in R}|(\Phi^m(\phi(t)) - \Phi^m(\psi(t)))_n|\right)^T \\
&= \Big(\sup_{t\in R}|(\Phi(\Phi^{m-1}(\phi(t))) - \Phi(\Phi^{m-1}(\psi(t))))_1|, \dots, \sup_{t\in R}|(\Phi(\Phi^{m-1}(\phi(t))) \\
&\quad - \Phi(\Phi^{m-1}(\psi(t))))_n|\Big)^T \\
&\le F\Big(\sup_{t\in R}|(\Phi^{m-1}(\phi(t)) - \Phi^{m-1}(\psi(t)))_1|, \\
&\quad \dots, \sup_{t\in R}|(\Phi^{m-1}(\phi(t)) - \Phi^{m-1}(\psi(t)))_n|\Big)^T \\
&\quad \vdots \\
&\le F^m\left(\sup_{t\in R}|(\phi(t) - \psi(t))_1|, \dots, \sup_{t\in R}|(\phi(t) - \psi(t))_n|\right)^T \\
&= F^m\left(\sup_{t\in R}|\phi_1(t) - \psi_1(t)|, \dots, \sup_{t\in R}|\phi_n(t) - \psi_n(t)|\right)^T.
\end{aligned}$$

Since $\rho(F) < 1$, we obtain

$$\lim_{m\longrightarrow+\infty} F^m = 0,$$

which implies that there exist a positive integer N and a positive constant $r < 1$ such that

$$F^N = (D^{-1}EL)^N = (h_{ij})_{n\times n}, \quad \text{and} \quad \sum_{j=1}^{n} h_{ij} \le r, i = 1,\ 2, \dots, n. \tag{2.6}$$

In view of (2.5) and (2.6), we have

$$\begin{aligned}
|(\Phi^N(\phi(t)) - \Phi^N(\psi(t)))_i| &\le \sup_{t\in R}|(\Phi^N(\phi(t)) - \Phi^N(\psi(t)))_i| \\
&\le \sum_{j=1}^{n} h_{ij} \sup_{t\in R}|\phi_j(t) - \psi_j(t)|
\end{aligned}$$

$$\leq (\sup_{t\in R}\max_{1\leq j\leq n}|\phi_j(t)-\psi_j(t)|)\sum_{j=1}^{n}h_{ij}$$
$$\leq r\|\phi(t)-\psi(t)\|_X,$$

for all $t\in R,\ i=1,\ 2,\dots,n$. It follows that

$$\begin{aligned}\|\Phi^N(\phi(t))-\Phi^N(\psi(t))\|_X &= \sup_{t\in R}\max_{1\leq i\leq n}|(\Phi^N(\phi(t))-\Phi^N(\psi(t)))_i| \\ &\leq r\|\phi(t)-\psi(t)\|_X.\end{aligned} \tag{2.7}$$

This implies that the mapping $\Phi^N : X \longrightarrow X$ is a contraction mapping.

By the fixed point theorem of Banach space, Φ possesses a unique fixed point Z^* in X such that $\Phi Z^* = Z^*$. We know from (2.1) and (2.3) that Z^* satisfies system (1.1), and therefore, it is an almost periodic solution of system (1.1). The proof of Theorem 2.1 is now complete.

3. Uniqueness and exponential stability

In this section, we establish some results for the uniqueness and exponential stability of the almost periodic solution of (1.1).

THEOREM 3.1. *Let* (T_2) *hold. Suppose that all the conditions of Theorem 2.1 are satisfied. Then system* (1.1) *has exactly one almost periodic solution* $Z^*(t)$. *Moreover,* $Z^*(t)$ *is globally exponentially stable.*

Proof. From Theorem 2.1, system (1.1) has at least one almost periodic solution $Z^*(t)=(x_1^*(t),x_2^*(t),\dots,x_n^*(t))^T$ with initial value $\varphi^*=(\varphi_1^*(t),\ \varphi_2^*(t),\dots,\varphi_n^*(t))^T$. Let $Z(t)=(x_1(t),x_2(t),\dots,x_n(t))^T$ be an arbitrary solution of system (1.1) with initial value $\varphi=(\varphi_1(t),\varphi_2(t),\dots,\varphi_n(t))^T$, Set $y(t)=(y_1(t),y_2(t),\dots,y_n(t))^T=Z(t)-Z^*(t)$. Then

$$\begin{aligned}y_i'(t) =& -c_i(t)y_i(t)+\sum_{j=1}^{n}a_{ij}(t)f_j^*(y_j(t)) \\ &+\sum_{j=1}^{n}b_{ij}(t)\int_0^{\infty}K_{ij}(u)f_j^*(y_j(t-u))du,\end{aligned} \tag{3.1}$$

where $i=1,2,\dots,n$,

$$f_j^*(y_j(t))=f_j(y_j(t)+x_j^*(t))-f_j(x_j^*(t)),$$
$$\begin{aligned}\int_0^{\infty}K_{ij}(u)f_j^*(y_j(t-u))du =& \int_0^{\infty}K_{ij}(u)(f_j(y_j(t-u)+x_j^*(t-u)) \\ &-f_j(x_j^*(t-u)))du.\end{aligned}$$

Since $\rho(F) = \rho(D^{-1}\overline{E}L) < 1$, it follows from Lemma 1.4 that $E_n - D^{-1}\overline{E}L$ is an M-matrix. In view of Lemma 1.3, there exists a constant vector $\bar{\xi} = (\bar{\xi}_1,\ \bar{\xi}_2, \ldots, \bar{\xi}_n)^T > (0, 0, \ldots, 0)^T$ such that

$$(E_n - D^{-1}\overline{E}L)\bar{\xi} > (0, 0, \ldots, 0)^T.$$

Then,

$$-\tilde{c}_i\bar{\xi}_i + \sum_{j=1}^{n} \bar{\xi}_j(\overline{a_{ij}} + \overline{b_{ij}}k_{ij})L_j < 0, \quad i = 1, 2, \ldots, n.$$

Therefore, we can choose a constant $d > 1$ such that

$$\xi_i = d\bar{\xi}_i > \sup_{-\infty < t \leq 0} |y_i(t)|,\ i = 1,\ 2,\ \ldots, n, \tag{3.2}$$

and

$$\begin{aligned} &-\tilde{c}_i\xi_i + \sum_{j=1}^{n} \xi_j(\overline{a_{ij}} + \overline{b_{ij}}k_{ij})L_j \\ &= [-\tilde{c}_i\bar{\xi}_i + \sum_{j=1}^{n} \bar{\xi}_j(\overline{a_{ij}} + \overline{b_{ij}}k_{ij})L_j]d < 0,\ i = 1, 2, \ldots, n. \end{aligned} \tag{3.3}$$

Set

$$\begin{aligned} \Gamma_i(\omega) = \omega\xi_i - \tilde{c}_i\xi_i + \sum_{j=1}^{n} \xi_j(\overline{a_{ij}} + \overline{b_{ij}} \int_0^\infty |K_{ij}(s)|e^{\omega s}ds)L_j, \\ i = 1, 2, \ldots, n. \end{aligned} \tag{3.4}$$

Clearly, $\Gamma_i(\omega), i = 1, 2, \ldots, n$, are continuous functions on $[0,\ \lambda_0]$. Since

$$\begin{aligned} \Gamma_i(0) &= -\tilde{c}_i\xi_i + \sum_{j=1}^{n} \xi_j(\overline{a_{ij}} + \overline{b_{ij}} \int_0^\infty |K_{ij}(s)|ds)L_j \\ &\leq -\tilde{c}_i\xi_i + \sum_{j=1}^{n} \xi_j(\overline{a_{ij}} + \overline{b_{ij}}k_{ij})L_j < 0,\ i = 1, 2, \ldots, n, \end{aligned}$$

we can choose a positive constant $\lambda \in [0,\ \lambda_0]$ such that

$$\begin{aligned} \Gamma_i(\lambda) = (\lambda - \tilde{c}_i)\xi_i + \sum_{j=1}^{n} \xi_j(\overline{a_{ij}} + \overline{b_{ij}} \int_0^\infty |K_{ij}(s)|e^{\lambda s}ds)L_j \\ < 0, \quad i = 1, 2, \ldots, n. \end{aligned} \tag{3.5}$$

We consider the Lyapunov functional

$$V_i(t) = |y_i(t)|e^{\lambda t}, \quad i = 1, 2, \ldots, n. \tag{3.6}$$

Obviously, for any $y_i(t) \neq 0$, $V_i(t) > 0$. Calculating the upper right derivative of $V_i(t)$ along the solution $y(t) = (y_1(t), y_2(t), \dots, y_n(t))^T$ of system (3.1) with the initial value $\bar{\varphi} = \varphi - \varphi^*$, we have

(3.7)
$$\begin{aligned} D^+(V_i(t)) \leq & - \tilde{c_i}|y_i(t)|e^{\lambda t} + \sum_{j=1}^{n} \overline{a_{ij}} L_j |y_j(t)| e^{\lambda t} \\ & + \sum_{j=1}^{n} \overline{b_{ij}} L_j e^{\lambda t} \int_0^\infty |K_{ij}(u)||y_j(t-u)|du + \lambda |y_i(t)| e^{\lambda t} \\ = & [(\lambda - \tilde{c_i})|y_i(t)| + \sum_{j=1}^{n} \overline{a_{ij}} L_j |y_j(t)| \\ & + \sum_{j=1}^{n} \overline{b_{ij}} L_j \int_0^\infty |K_{ij}(u)||y_j(t-u)|du] e^{\lambda t}, \end{aligned}$$

where $i = 1, 2, \dots, n$.

We claim that

$$V_i(t) = |y_i(t)|e^{\lambda t} < \xi_i, \quad t > 0, \quad i = 1, 2, \dots, n. \tag{3.8}$$

Contrarily, there must exist $i \in \{1, 2, \dots, n\}$ and $t_i > 0$ such that

$$V_i(t_i) = \xi_i \quad \text{and} \quad V_j(t) < \xi_j, \forall\, t \in (-\infty,\ t_i),\ j = 1, 2, \dots, n, \tag{3.9}$$

which implies that

(3.10)
$$V_i(t_i) - \xi_i = 0 \quad \text{and} \quad V_j(t) - \xi_j < 0, \forall\, t \in (-\infty,\ t_i),\ j = 1, 2, \dots, n.$$

Together with (3.7) and (3.10), we obtain

$$\begin{aligned} 0 \leq & D^+(V_i(t_i) - \xi_i) \\ = & D^+(V_i(t_i)) \\ \leq & [(\lambda - \tilde{c_i})|y_i(t_i)| + \sum_{j=1}^{n} \overline{a_{ij}} L_j |y_j(t_i)| \\ & + \sum_{j=1}^{n} \overline{b_{ij}} L_j \int_0^\infty |K_{ij}(u)||y_j(t_i-u)|du] e^{\lambda t_i} \end{aligned} \tag{3.11}$$

$$= (\lambda - \tilde{c}_i)\xi_i + \sum_{j=1}^{n} \overline{a_{ij}} L_j |y_j(t_i)| e^{\lambda t_i}$$
$$+ \sum_{j=1}^{n} \overline{b_{ij}} L_j \int_0^\infty e^{\lambda u} |K_{ij}(u)| |y_j(t_i - u)| e^{\lambda(t_i - u)} du$$
$$\leq (\lambda - \tilde{c}_i)\xi_i + \sum_{j=1}^{n} \xi_j (\overline{a_{ij}} + \overline{b_{ij}} \int_0^\infty |K_{ij}(s)| e^{\lambda s} ds) L_j.$$

Thus,

$$0 \leq (\lambda - \tilde{c}_i)\xi_i + \sum_{j=1}^{n} \xi_j (\overline{a_{ij}} + \overline{b_{ij}} \int_0^\infty |K_{ij}(s)| e^{\lambda s} ds) L_j,$$

which contradicts (3.5). Hence, (3.8) holds. It follows that

$$|y_i(t)| < \max_{1 \leq i \leq n} \{\xi_i\} e^{-\lambda t}, \quad t > 0, \quad i = 1, 2, \ldots, n. \tag{3.12}$$

Letting $\|\bar{\varphi}\| = \|\varphi - \varphi^*\| > 0$, it follows from (3.12) that we can choose a constant $M_\varphi > 1$ such that

$$\max_{1 \leq i \leq n} \{\xi_i\} \leq M_\varphi \|\varphi - \varphi^*\|, \quad i = 1, 2, \ldots, n. \tag{3.13}$$

In view of (3.11) and (3.12), we get

$$|x_i(t) - x_i^*(t)| = |y_i(t)| \leq \max_{1 \leq i \leq n} \{\xi_i\} e^{-\lambda t} \leq M_\varphi \|\varphi - \varphi^*\| e^{-\lambda t},$$

where $i = 1, 2, \ldots, n,\ t > 0$. This completes the proof.

REMARK 3.1. Theorem A has been obtained under the assumption that the row norm of matrix $D^{-1}\overline{E}L$ is less than 1. Clearly, this implies that $\rho(D^{-1}\overline{E}L) < 1$. Therefore, the existing results in [15] and [4] are direct corollaries of Theorem 3.1 of this paper.

COROLLARY 3.1. *Let (T_0), (T_1), and (T_2) hold. Suppose that $E_n - D^{-1}\overline{E}L$ is an M-matrix. Then system (1.1) has at least exactly one almost periodic solution $Z^*(t)$. Moreover, $Z^*(t)$ is globally exponentially stable.*

Proof. Noticing that $E_n - D^{-1}\overline{E}L$ is an M-matrix, it follows that there exists a vector $\eta = (\eta_1, \eta_2, \ldots, \eta_n)^T > 0$ such that

$$(E_n - D^{-1}\overline{E}L)\eta > 0, \tag{3.13}$$

that is,

$$(3.14)\qquad -c_i\eta_i + \sum_{j=1}^{n}(\overline{a_{ij}} + \overline{b_{ij}})L_j\eta_j < 0, \qquad i = 1, 2, \ldots, n.$$

For any matrix norm $\|\cdot\|$ and nonsingular matrix B, $\|A\|_B = \|B^{-1}AB\|$ also defines a matrix norm. Let $B = diag(\eta_1, \eta_2, \ldots, \eta_n)$. Then (3.14) implies that the row norm of matrix $B^{-1}(D^{-1}\overline{E}L)B$ is less than 1. Therefore, $\rho(D^{-1}\overline{E}L) < 1$. Corollary 3.1 follows immediately from Theorem 3.1. □

4. An example

In this section, we give an example to demonstrate the results obtained in previous sections.

EXAMPLE 4.1. Consider the following CNNs with continuously distributed delays:

$$(4.1)\qquad \begin{cases} x_1'(t) = -x_1(t) + \frac{1}{4}(\sin t)f_1(x_1(t)) + \frac{1}{36}(\cos t)f_2(x_2(t)) \\ \qquad +\frac{1}{4}(\sin t)\int_0^\infty(\sin u)e^{-u}\cdot f_1(x_1(t-u))du \\ \qquad +\frac{1}{36}(\cos t)\int_0^\infty(\sin u)e^{-u}f_1(x_2(t-u))du + I_1(t), \\ x_2'(t) = -x_2(t) + (\sin 2t)f_1(x_1(t)) + \frac{1}{4}(\cos 4t)f_2(x_2(t)) \\ \qquad +(\sin 2t)\int_0^\infty(\sin u)e^{-u}\cdot f_1(x_1(t-u))du \\ \qquad +\frac{1}{4}(\cos 4t)\int_0^\infty(\sin u)e^{-u}f_2(x_2(t-u))du + I_2(t), \\ x_3'(t) = -x_3(t) + \frac{1}{4}(\cos 4t)f_3(x_3(t)) \\ \qquad +\frac{1}{4}(\sin 2t)\int_0^\infty(\sin u)e^{-u}f_3(x_1(t-u))du + I_3(t), \end{cases}$$

where$f_1(x) = f_2(x) = f_3(x) = \frac{1}{2}(|x+1| - |x-1|)$, $I_1(t)$, $I_2(t)$ and $I_3(t)$ are almost periodic functions on R.

Notice that $\tilde{c_1} = \tilde{c_2} = \tilde{c_3} = L_1 = L_2 = L_3 = 1, \overline{a_{11}} = \overline{b_{11}} = \frac{1}{4}, \overline{a_{12}} = \overline{b_{12}} = \frac{1}{36}$, $\overline{a_{13}} = \overline{b_{13}} = 0, \overline{a_{21}} = \overline{b_{21}} = 1$, $\overline{a_{22}} = \overline{b_{22}} = \frac{1}{4}, \overline{a_{23}} = \overline{b_{23}} = \overline{a_{31}} = \overline{b_{31}} = \overline{a_{32}} = \overline{b_{32}} = 0$, $\overline{a_{33}} = \overline{b_{33}} = \frac{1}{4}, k_{11} = k_{12} = k_{21} = k_{22} = k_{33} = 1$. Then, we get

$$D^{-1}\overline{E}L = D^{-1}(\bar{A} + \bar{B})L = \begin{pmatrix} \frac{1}{2} & \frac{1}{18} & 0 \\ 2 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

Hence, we have

$$\rho(D^{-1}\overline{E}L) = \rho(D^{-1}(\bar{A} + \bar{B})L) = \frac{5}{6} < 1.$$

Therefore, it follows from Lemma 3.1 that system (4.1) has at least exactly one almost periodic solution. Moreover, The almost periodic solution is globally exponentially stable.

REMARK. (4.1) is a very simple form of DCNNs equations. One can observe that $\|D^{-1}(\bar{A}+\bar{B})L\|_1 = \frac{5}{2}$, where $\|\cdot\|_1$ is the row norm of matrix. Therefore, all the results in [2-6, 8, 13, 15, 16] and the references therein can not be applicable to system (4.1). This implies that the results of this paper are essentially new.

5. Conclusion

In this paper, cellular neural networks with continuously distributed delays have been studied. Some sufficient conditions for the existence and exponential stability of the almost periodic solutions have been established. These obtained results are new and they complement previously known results. Moreover, an example is given to illustrate the effectiveness of the new results.

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